STRONG CONVERGENCE OF A MODIFIED SP-ITERATION PROCESS FOR GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract. In this paper, we establish strong convergence theorems of the modified SP-iteration generalized asymptotically quasi-nonexpansive mapping in CAT(0) spaces which extend and improve the recent ones announced by Phuengrattana and Suantai (J. Comput. Appl. Math. 235, 3006-3014, 2011), Sahin and Basarir (Journal of Inequalities and Applications, 2013), Nanjaras and Panyanak (Fixed point Theory and Application, 2010) and some others.

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1. Introduction

Let $(X, d)$ be a metric space and $x, y \in X$ with $d(x, y) = l$. A geodesic path from $x$ to $y$ is a isometry $c : [0, l] \rightarrow X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic part is called a geodesic segment. A metric space $X$ is a (uniquely) geodesic space, if every two point of $X$ are joined by only one geodesic segment. We will use $[x, y]$ to denote a geodesic segment joining $x$ and $y$. A subset $C$ of a geodesic space is said to be convex if $[x, y] \in C$ for any $x, y \in C$.

A metric space $X$ is a CAT(0) space if it is geodesically connected and if every geodesic triangle in $X$ is at least as thin as its comparison triangle in the Euclidean plane.

In 1976, Lim [3] introduced the concept of $\Delta$-convergence in a general metric space. In 2008, Kirk and Panyanak [4] specialized Lim’s concept to CAT(0) spaces and proved that it is very similar to weak convergence in the Banach space setting. Every nonexpansive (single-valued) mapping defined on closed bounded convex subset of complete CAT(0) space always has a fixed point. Since then the fixed point theory in CAT(0) space has been rapidly developed and many paper has appeared [8-16].

The Man iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n Tx_n + (1 - \alpha_n) x_n, \quad n \geq 1, \end{cases}$$

(1.1)

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

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The Ishikawa iteration process is defined by the sequence \( \{ x_n \} \),

\[
\begin{align*}
x_1 &\in C, \\
x_{n+1} &= \alpha_n Tx_n + (1 - \alpha_n) x_n, \\
y_n &= \beta_n Tx_n + (1 - \beta_n) x_n, \quad n \geq 1,
\end{align*}
\]

(1.2)

where \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) is a sequence in \((0,1)\).

The Noor iteration process is defined by the sequence \( \{ x_n \} \),

\[
\begin{align*}
x_1 &\in C, \\
z_n &= \gamma_n Tx_n + (1 - \gamma_n) x_n, \\
y_n &= \beta_n Tz_n + (1 - \beta_n) z_n, \\
x_{n+1} &= \alpha_n Ty_n + (1 - \alpha_n) y_n, \quad n \geq 1,
\end{align*}
\]

(1.3)

where \( \{ \alpha_n \}, \{ \beta_n \} \) and \( \{ \gamma_n \} \) is a sequence in \([0,1]\).

Recently, Phuengrattana and Suantai [17] introduced the SP-iteration process is defined by the sequence \( \{ x_n \} \),

\[
\begin{align*}
x_1 &\in C, \\
z_n &= \gamma_n Tx_n + (1 - \gamma_n) x_n, \\
y_n &= \beta_n Tz_n + (1 - \beta_n) z_n, \\
x_{n+1} &= \alpha_n Ty_n + (1 - \alpha_n) y_n, \quad n \geq 1,
\end{align*}
\]

(1.4)

where \( \{ \alpha_n \}, \{ \beta_n \} \) and \( \{ \gamma_n \} \) is a sequence in \([0,1]\).

2. Preliminaries

Complete CAT(0) spaces are often called Hadamard spaces (see [1]). If \( x, y_1, y_2 \) are points of a CAT(0) space and \( y_0 \) is the midpoint of the segment \([y_1, y_2]\), which we will denote by \((y_1 \oplus y_2)/2\), then the CAT(0) inequality implies

\[
d^2 \left( x, \frac{y_1 \oplus y_2}{2} \right) \leq \frac{1}{2} d^2 (x, y_1) + \frac{1}{2} d^2 (x, y_2) - \frac{1}{4} d^2 (y_1, y_2) \quad \text{(2.1)}
\]

The inequality (2.1) is the (CN) inequality of Bruhat and Titz [18].

A geodesic metric spaces is a CAT(0) space if and only if it satisfies the (CN) inequality.

A subset \( K \) of a CAT(0) space \( X \) is convex if for any \( x, y \in K \), we have \([x, y] \subset K\).

**Lemma 2.1** ([9]). Let \( X \) be a CAT(0) space.

1. For any \( x, y, z \in X \) and \( t \in [0, 1] \), has

\[
d((1 - t) x \oplus ty, z) \leq (1 - t) d(x, z) + t d(y, z) \quad \text{(2.2)}
\]

2. For any \( x, y, z \in X \) and \( t \in [0, 1] \), has

\[
d^2 \left( (1 - t) x \oplus ty, z \right) \leq (1 - t) d^2 (x, z) + t d^2 (y, z) - t(1 - t) d^2 (x, y) \quad \text{(2.3)}
\]

Let \( C \) be nonempty subset of a CAT(0) space. We denote that the set of fixed points of \( T \) by \( F(T) = \{ x \in C : Tx = x \} \).

**Definition 2.2** ([21]). A mapping \( T : C \rightarrow C \) called:

1. Nonexpansive if \( d(Tx, Ty) \leq d(x, y) \) for all \( x, y \in C \).
2. Quasi-nonexpansive if \( d(Tx, p) \leq d(x, p) \) for all \( x \in X \) and for all \( p \in F(T) \).
3. Asymptotically nonexpansive if there exists \( k_n \in [0, 1] \) for all \( n \geq 1 \) with \( \lim_{n \to \infty} k_n = 0 \).
0 such that \(d(T^nx, T^ny) \leq (1 + k_n)d(x, y)\) for all \(x, y \in C\).

(4) Asymptotically quasi-nonexpansive if there exists \(k_n \in [0, 1)\) for all \(n \geq 1\) with \(\lim_{n \to \infty} k_n = 0\) such that \(d(T^nx, p) \leq (1 + k_n)d(x, p)\) for all \(x \in C\), for all \(p \in F(T)\).

(5) Generalized asymptotically nonexpansive if \(F(T) \neq \emptyset\) and there exist two sequences of real numbers \(\{u_n\}\) with \(\lim_{n \to \infty} u_n = 0\) such that \(d(T^nx, p) \leq d(x, p) + (1 + u_n)d(x, p)\) for all \(x \in C, p \in F(T)\) and \(n \geq 1\).

(6) Generalized asymptotically quasi-nonexpansive if \(F(T) \neq \emptyset\) and there exist two sequences of real numbers \(\{u_n\}\) and \(\{c_n\}\) with \(\lim_{n \to \infty} u_n = 0 = \lim_{n \to \infty} c_n\) such that \(d(T^nx, p) \leq d(x, p) + (1 + u_n)d(x, p) + c_n\) for all \(x \in C, p \in F(T)\) and \(n \geq 1\).

(7) Uniformly \(L\)-Lipschitzian if for some \(L > 0\), \(d(T^nx, T^ny) \leq Ld(x, y)\) for all \(x, y \in C\) and \(n \geq 1\).

(8) Semi-compact if for any bounded sequence \(\{x_n\}\) in \(C\) with \(d(x_n, T^nx_n) \to 0\) as \(n \to \infty\), there is a convergent subsequence of \(\{x_n\}\).

Let \(\{x_n\}\) be a sequence in a metric space \((X, d)\), and let \(C\) be a subset of \(X\). We say that \(\{x_n\}\), (1) is of monotone type \((A)\) with respect to \(C\) if for each \(p \in C\), there exist two sequences \(\{r_n\}\) and \(\{s_n\}\) of nonnegative real numbers such that \(\sum_{n=1}^{\infty} r_n < \infty\), \(\sum_{n=1}^{\infty} s_n < \infty\) and \(d(x_{n+1}, p) \leq (1 + r_n)d(x_n, p) + s_n\), (2) of monotone type \((B)\) with respect to \(C\) if there exist sequence \(\{r_n\}\) and \(\{s_n\}\) of nonnegative real numbers such that \(d(x_{n+1}, C) \leq (1 + r_n)d(x_n, C) + s_n\).

A mapping \(T : C \to C\) with \(F(T) \neq \emptyset\) is said to satisfy condition \((I)\) if there exists a non-decreasing function \(f : [0, \infty) \to [0, \infty)\) with \(f(0) = 0\) and \(f(r) > 0\) for all \(r \in (0, \infty)\) such that \(d(Tx, Tx) \geq f(d(x, F(T)))\), for all \(x \in C\).

Let \(\{x_n\}\) be a bounded sequence in \(CAT(0)\) space \(X\). For \(x \in X\), set
\[
    r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n). \tag{2.4}
\]

The asymptotic radius \(r(\{x_n\})\) of \(\{x_n\}\) is given by
\[
    r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}, \tag{2.5}
\]

and the asymptotic center \(A(\{x_n\})\) of \(\{x_n\}\) is the set
\[
    A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\} \tag{2.6}
\]

**Lemma 2.3** ([19]). If \(C\) is a closed convex subset of a complete \(CAT(0)\) space \(X\) and if \(\{x_n\}\) is a bounded sequence in \(C\), then the asymptotic center of \(\{x_n\}\) is in \(C\).

**Lemma 2.4** ([19]). Every bounded sequence in a complete \(CAT(0)\) space always has a \(\Delta\)-convergent subsequence.

**Lemma 2.5** ([9]). Let \(X\) be a complete \(CAT(0)\) space and \(\{x_n\}\) be a bounded sequence in \(X\) with \(A(\{x_n\}) = \{p\}\) and \(\{u_n\}\) be a subsequence of \(\{x_n\}\) with \(A(\{u_n\}) = \{u\}\) and the sequence \(\{d(x_n, u)\}\) converges, then \(p = u\).

**Lemma 2.6** ([20]). Let \(X\) be a \(CAT(0)\) space, \(x \in X\) be given point and \(\{t_n\}\) be a sequence in \([b, c]\) with \(b, c \in (0, 1)\) and \(0 < b(1 - c) \leq \frac{1}{2}\). Let \(\{x_n\}\) and \(\{y_n\}\) be any sequence in \(X\) such that \(\limsup_{n \to \infty} d(x_n, x) \leq r\), \(\limsup_{n \to \infty} d(y_n, x) \leq r\) and \(\lim_{x \to \infty} d((1 - t_n)x_n \oplus t_n y_n, x) = r\), for some \(r \geq 0\). Then \(\lim_{x \to \infty} d(x_n, y_n) = 0\).
Lemma 2.7 ([12]). Let \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) be three nonnegative sequences satisfying
\[
a_{n+1} \leq (1 + b_n) a_n + c_n, \quad n \geq 1.
\]
If \( \sum_{n=1}^{\infty} c_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \), then
\[
\begin{align*}
(1) \quad & \lim_{n \to \infty} a_n \text{ exists}, \\
(2) \quad & \text{If } \{a_n\} \text{ has a subsequence which converges strongly to zero, then } \lim_{n \to \infty} a_n = 0.
\end{align*}
\]

3. Main results

In this section, we establish some convergence results of SP-iterations to a fixed point for generalized asymptotically quasi-nonexpansive mappings in the general class of CAT(0) spaces.

**Theorem 3.1.** Let \((X, d)\) be a complete CAT(0) space and let \(C\) be a nonempty closed convex subset of \(X\). Let \(T : C \to C\) be a generalized asymptotically quasi-nonexpansive mapping with \(\{s_n\}, \{t_n\} \subset [0, \infty)\) such that \(\sum_{n=1}^{\infty} s_n < \infty\) and \(\sum_{n=1}^{\infty} t_n < \infty\). Suppose that \(F(T)\) is closed. For arbitrarily chosen \(x_1 \in C\), the sequence \(\{x_n\}\) be the SP-iteration defined as follows:
\[
\begin{align*}
z_n &= \gamma_n T^m x_n + (1 - \gamma_n) x_n, \\
y_n &= \beta_n T^m z_n + (1 - \beta_n) z_n, \\
x_{n+1} &= \alpha_n T^m y_n + (1 - \alpha_n) y_n,
\end{align*}
\]
(3.1)
where \(\{\gamma_n\}, \{\beta_n\}, \{\alpha_n\}\) are real sequence in \([0, 1]\). Then the sequence \(\{x_n\}\) is of monotone type (A) and monotone type (B) with respect to \(F(T)\). Moreover, \(\{x_n\}\) converges strongly to a fixed point \(q\) of the mapping \(T\) if and only if
\[
\lim_{n \to \infty} \inf_{q \in F(T)} d(x_n, F(T)) = 0,
\]
where \(d(x, F(T)) = \inf_{q \in F(T)} \{d(x, q)\}\).

**Proof** Following (2.2), Definition 2.2(6) and (3.1), we have
\[
d(z_n, q) = d(\gamma_n T^m x_n + (1 - \gamma_n) x_n, q) \\
\leq \gamma_n d(T^m x_n, q) + (1 - \gamma_n) d(x_n, q) \\
\leq \gamma_n [(1 + s_n) d(x_n, q) + t_n] + (1 - \gamma_n) d(x_n, q) \\
\leq (1 + s_n) [\gamma_n + 1 - \gamma_n] d(x_n, q) + \gamma_n t_n \\
= (1 + s_n) d(x_n, q) + \gamma_n t_n
\]
(3.2)
and
\[
d(y_n, q) = d(\beta_n T^m z_n + (1 - \beta_n) z_n, q) \\
\leq \beta_n d(T^m z_n, q) + (1 - \beta_n) d(z_n, q) \\
\leq \beta_n [(1 + s_n) d(z_n, q) + t_n] + (1 - \beta_n) d(z_n, q) \\
\leq (1 + s_n) [\beta_n + 1 - \beta_n] d(z_n, q) + \beta_n t_n \\
= (1 + s_n) d(z_n, q) + \beta_n t_n.
\]
(3.3)
Substituting (3.2) into (3.3) and combining, we have
\[
d(y_n, q) \leq (1 + s_n) [(1 + s_n) d(x_n, q) + \gamma_n t_n] + \beta_n t_n \\
\leq (1 + s_n)^2 d(x_n, q) + (1 + s_n) \gamma_n t_n + \beta_n t_n,
\]
(3.4)
and
\[
d(x_{n+1}, q) = d(\alpha_n T_n y_n + (1 - \alpha_n) y_n, q)
\leq \alpha_n d(T_n y_n, q) + (1 - \alpha_n) d(y_n, q)
\leq \alpha_n [(1 + s_n) d(y_n, q) + t_n] + (1 - \alpha_n) d(y_n, q)
\leq (1 + s_n) [\alpha_n + 1 - \alpha_n] d(y_n, q) + \alpha_n t_n
= (1 + s_n) d(y_n, q) + \alpha_n t_n.
\]

Substituting (3.4) into (3.5) and combining, we have
\[
d(x_{n+1}, q) \leq (1 + s_n) [(1 + s_n)^2 d(x_n, q) + (1 + s_n) \gamma_n t_n + \beta_n t_n] + \alpha_n t_n
\leq (1 + s_n)^3 d(x_n, q) + (1 + s_n)^2 \gamma_n t_n + \beta_n t_n + \alpha_n t_n
= (1 + \psi_n) d(x_n, q) + \varphi_n
\]
where \(\psi_n = 3s_n + 3s_n^2 + s_n^3\) and \(\varphi_n = (1 + s_n)^2 \gamma_n t_n + \beta_n t_n + \alpha_n t_n\). Since \(\sum_{n=1}^{\infty} s_n < \infty\) and \(\sum_{n=1}^{\infty} t_n < \infty\), it follows that \(\sum_{n=1}^{\infty} \psi_n < \infty\) and \(\sum_{n=1}^{\infty} \varphi_n < \infty\). Now, from (3.6), we get
\[
d(x_{n+1}, q) \leq (1 + \psi_n) d(x_n, q) + \varphi_n,
\]
These inequalities, respectively, we prove that \(\{x_n\}\) is a sequence of monotone type (A) and monotone type (B) with respect to \(F(T)\).

Now, we prove that \(\{x_n\}\) converges strongly to a fixed point of the mapping \(T\) if and only if \(\liminf_{n \to \infty} d(x_n, F(T)) = 0\). If \(x_n \to q \in F(T)\), then \(\lim_{n \to \infty} d(x_n, q) = 0\). Since \(0 \leq (x_n, F(T)) \leq d(x_n, q)\), we have \(\liminf_{n \to \infty} d(x_n, F(T)) = 0\).

Conversely, suppose that \(\liminf_{n \to \infty} d(x_n, F(T)) = 0\). From (3.8) using Lemma 2.7, we have that \(\liminf_{n \to \infty} d(x_n, F(T))\) exists. Further, by hypothesis \(\liminf_{n \to \infty} d(x_n, F(T)) = 0\), we conclude that \(\lim_{n \to \infty} d(x_n, F(T)) = 0\). Next, we show that \(\{x_n\}\) is a Cauchy sequence. Since \(1 + a \leq e^a\) for \(a \geq 0\), hence from (3.7), we have
\[
d(x_{n+m}, q) \leq e^{\psi_{n+m-1}} d(x_{n+m-1}, q) + \varphi_{n+m-1}
\leq e^{\psi_{n+m-1}} d(x_{n+m-1}, q) + \varphi_{n+m-1}
\leq e^{\psi_{n+m-1}} [e^{\psi_{n+m-1}} d(x_{n+m-2}, q) + \varphi_{n+m-2}] + \varphi_{n+m-1}
\leq e^{\sum_{k=n}^{n+m-1} \psi_k} d(x_n, q) + e^{\sum_{k=n}^{n+m-1} \psi_k} \left(\sum_{k=n}^{n+m-1} \varphi_k\right)
\leq Md(x_n, q) + M \left(\sum_{k=n}^{n+m-1} \varphi_k\right),
\]
where \(M = e^{\sum_{k=n}^{n+m-1} \psi_k}\), for \(M > 0\) and for the natural numbers \(m, n\) and \(q \in F(T)\).

Since \(\lim_{n \to \infty} d(x_n, F(T)) = 0\), therefore for any \(\varepsilon > 0\), there exists a natural number \(N_0\)
such that $d(x_n, F(T)) < \frac{\varepsilon}{8M}$ and $\sum_{k=n}^{n+m-1} \varphi_k < \frac{\varepsilon}{4M}$ for all $n > n_0$. And so, we can find $q^* \in F(T)$ such that $d(x_{n_0}, q^*) < \frac{\varepsilon}{4M}$, thus, for all $n > n_0$ and $m \geq 1$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, q^*) + d(x_n, q^*)$$

$$\leq Md(x_{n_0}, q^*) + M \sum_{k=n_0}^{\infty} \varphi_k + Md(x_{n_0}, q^*) + M \sum_{k=n_0}^{\infty} \varphi_k$$

$$= 2M \left( d(x_{n_0}, q^*) + \sum_{k=n_0}^{\infty} \varphi_k \right) \leq 2M \left( \frac{\varepsilon}{4M} + \frac{\varepsilon}{4M} \right) = \varepsilon$$

(3.10)

This proves that $\{x_n\}$ is a Cauchy sequence. Hence, By the completeness of $X$, we assume that $\lim_{n \to \infty} x_n = a$. Since $C$ is closed, therefore $a \in C$. Next, we show that $a \in F(T)$. Following two inequalities:

$$d(a, q) \leq d(a, x_n) + d(x_n, q) \quad \forall q \in F(T), \ n \geq 1,$n \geq 1,$

$$d(a, x_n) \leq d(a, q) + d(x_n, q) \quad \forall q \in F(T), \ n \geq 1,$$

(3.11)

give that

$$-d(a, x_n) \leq d(a, F(T)) - d(x_n, F(T)) \leq d(a, x_n), \quad n \geq 1.$$n \geq 1.$$

(3.12)

That is

$$|d(a, F(T)) - d(x_n, F(T))| \leq d(a, x_n), \quad n \geq 1.$$n \geq 1.$$

(3.13)

And $\lim_{n \to \infty} x_n = a$ and $\lim_{n \to \infty} d(x_n, F(T)) = 0$, we conclude that $a \in F(T)$. The proof is completed.

**Corollary 3.2.** Let $(X, d)$ be a complete CAT(0) space and let $C$ be a nonempty closed convex subset of $X$. Let $T : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping with $\{s_n\}, \{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F(T)$ is closed. For arbitrarily chosen $x_1 \in C$, let $\{x_n\}$ be the SP-iteration sequence defined by (3.1). Then the sequence $\{x_n\}$ converges strongly to a common fixed point $q$ of the mapping $T$ if and only if there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $q \in F(T)$.

**Corollary 3.3.** Let $(X, d)$ be a complete CAT(0) space and let $C$ be a nonempty closed convex subset of $X$. Let $T : C \to C$ be asymptotically quasi-nonexpansive mapping with $\{s_n\}, \{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F(T)$ is closed. For arbitrarily chosen $x_1 \in C$, let $\{x_n\}$ be the SP-iteration sequence defined by (3.1). Then the sequence $\{x_n\}$ is of monotone type (A) and monotone type (B) with respect to $F(T)$. Moreover, $\{x_n\}$ converges strongly to a fixed point $q$ of the mapping $T$ if and only if

$$\lim_{n \to \infty} d(x_n, F(T)) = 0.$$n \to \infty}$

**Proof** Follows from Theorem 3.1 with $t_n = 0$ for all $n \geq 1$.

**Corollary 3.4.** Let $X$ be Banach space and let $C$ be a nonempty closed convex subset of $X$. Let $T : C \to C$ be asymptotically quasi-nonexpansive mapping with $\{s_n\}, \{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F(T)$ is closed. For arbitrarily chosen $x_1 \in C$, let $\{x_n\}$ be the SP-iteration sequence defined by (3.1). Then the sequence $\{x_n\}$ converges strongly to a common fixed point $q$ of the mapping $T$ if and only if

$$\lim_{n \to \infty} d(x_n, F(T)) = 0.$$n \to \infty}
\[0, \infty)\) such that \(\sum_{n=1}^{\infty} s_n < \infty\) and \(\sum_{n=1}^{\infty} t_n < \infty\). Suppose that \(F(T)\) is closed. For arbitrarily chosen \(x_1 \in C\), let \(\{x_n\}\) be the SP-iteration sequence defined by (3.1). Then the sequence \(x_n\) is of monotone type (A) and monotone type (B) with respect to \(F(T)\). Moreover, \(\{x_n\}\) converges strongly to a fixed point \(q\) of the mapping \(T\) if and only if
\[
\liminf_{n \to \infty} d(x_n, F(T)) = 0.
\]

**Proof** Take \(\lambda x \oplus (1 - \lambda) y = \lambda x + (1 - \lambda) y\) in Corollary 3.2.

**Lemma 3.5.** Let \((X, d)\) be a complete \(\text{CAT}(0)\) space and let \(C\) be a nonempty closed convex subset of \(X\). Let \(T: C \to C\) be a uniformly continuous generalized asymptotically quasi-nonexpansive mapping with \(\{s_n\}, \{t_n\} \subset [0, \infty)\) such that \(\sum_{n=1}^{\infty} s_n < \infty\) and \(\sum_{n=1}^{\infty} t_n < \infty\). Suppose that \(F(T) \neq \emptyset\). Let \(\{x_n\}\) be the SP-iteration sequence defined by (3.1). Let \(\{\alpha_n\} \subset [\delta, 1 - \delta]\) and \(\{\beta_n\} \subset [\delta, 1 - \delta]\) for some \(\delta \in (0, 1)\). Then

1. \(\lim_{n \to \infty} d(T^n y_n, y_n) = 0\),
2. \(\lim_{n \to \infty} d(T^n z_n, z_n) = 0\),
3. \(\lim_{n \to \infty} d(T^n x_n, x_n) = 0\).

**Proof** Let \(q \in F(T)\). By Theorem 3.1, we have \(\lim d(x_n, q)\) exists and \(\{x_n\}\) is bounded. Without loss of generality. Let \(\lim_{n \to \infty} d(x_n, q) = b \geq 0\). Taking lim sup on both in inequality (3.4), we have
\[
\lim_{n \to \infty} \sup_{n \to \infty} d(y_n, q) \leq b. \tag{3.14}
\]

Since
\[
d(T^n y_n, q) \leq (1 + s_n) d(y_n, q) + t_n, \tag{3.15}
\]
we have
\[
\lim_{n \to \infty} \sup_{n \to \infty} d(T^n y_n, q) \leq b. \tag{3.16}
\]

On the other hand, since
\[
\lim_{n \to \infty} d(x_{n+1}, q) = \lim_{n \to \infty} d(\alpha_n T^n y_n \oplus (1 - \alpha_n) y_n, q) = b, \tag{3.17}
\]
by Lemma 2.6, we have
\[
\lim_{n \to \infty} d(T^n y_n, y_n) = 0. \tag{3.18}
\]

Hence assertion (1) of the lemma is proved.

In addition, since
\[
d(x_{n+1}, q) \leq d(x_{n+1}, T^n y_n) + d(T^n y_n, q)
\leq (1 - \alpha_n) d(y_n, T^n y_n) + (1 + s_n) d(y_n, q) + t_n, \tag{3.19}
\]
we have \(\liminf_{n \to \infty} d(y_n, q) \geq b\). By combined (3.17) and it yields that \(\lim_{n \to \infty} d(y_n, q) = b\). This implies
\[
\lim_{n \to \infty} d(\beta_n T^n z_n \oplus (1 - \beta_n) z_n, q) = b. \tag{3.20}
\]
Taking lim sup on both sides in inequality (3.3), we have
\[
\lim_{n \to \infty} \sup_{n \to \infty} d(z_n, q) \leq b. \tag{3.21}
\]
Since
\[ d(T^n z_n, q) \leq (1 + s_n) d(z_n, q) + t_n, \] (3.22)
we have
\[ \limsup_{n \to \infty} d(T^n z_n, q) \leq b. \] (3.23)
By Lemma 2.6, we have
\[ \lim_{n \to \infty} d(T^n z_n, z_n) = 0. \] (3.24)
Hence assertion (2) of the lemma is proved.
Thus, by the same method, we can prove that
\[ \lim_{n \to \infty} d(T^n x_n, x_n) = 0. \] (3.25)
Hence assertion (3) of the lemma is proved. The proof is completed.

**Lemma 3.6.** Let \((X, d)\) be a complete CAT(0) space and let \(C\) be a nonempty closed convex subset of \(X\). Let \(T : C \to C\) be a uniformly \(L\)-Lipschitzian generalized asymptotically quasi-nonexpansive mapping with \(\{s_n\}, \{t_n\} \subset [0, \infty)\) such that \(\sum_{n=1}^{\infty} s_n < \infty\) and \(\sum_{n=1}^{\infty} t_n < \infty\). Suppose that \(F(T) \neq \emptyset\). Let \(\{x_n\}\) be the SP-iteration sequence defined by (3.1). Let \(\{\alpha_n\} \subset [\delta, 1 - \delta]\) and \(\{\beta_n\} \subset [\delta, 1 - \delta]\) for some \(\delta \in (0, 1)\). Then \(\lim_{n \to \infty} d(T x_n, x_n) = 0\).

**Proof** From Lemma 3.5, we have
\[ \lim_{n \to \infty} d(T^n z_n, z_n) = 0, \quad \lim_{n \to \infty} d(T^n y_n, y_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(T^n x_n, x_n) = 0. \] (3.26)
Hence, we get
\[
d(x_{n+1}, y_n) = d(\alpha_n T^n y_n \oplus (1 - \alpha_n) y_n, y_n) \\
\leq \alpha_n d(T^n y_n, y_n) \to 0 \quad \text{as} \quad n \to \infty.
\] (3.27)
Similary, we have
\[
d(y_n, z_n) \leq \beta_n d(T^n z_n, z_n) \to 0 \quad \text{as} \quad n \to \infty.
\] (3.28)
and
\[
d(z_n, x_n) \leq \alpha_n d(T^n x_n, x_n) \to 0 \quad \text{as} \quad n \to \infty.
\] (3.29)
It follows that
\[
d(x_{n+1}, x_n) \leq d(x_{n+1}, y_n) + d(y_n, z_n) + d(z_n, x_n) \to 0 \quad \text{as} \quad n \to \infty.
\] (3.30)
Since \(T\) is uniformly \(L\)-Lipschitzian, we have
\[
d(T x_n, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) \\
+ d(T^{n+1} x_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, T x_n) \\
\leq (1 + L) d(x_{n+1}, x_n) + d(x_{n+1}, T^{n+1} x_{n+1}) + L d(T^n x_n, x_n) \\
\to 0 \quad \text{as} \quad n \to \infty,
\] (3.31)
which implies
\[ \lim_{n \to \infty} d(Tx_n, x_n) = 0. \] (3.32)

The proof is completed.

**Theorem 3.7.** Let \( X, C, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\} \) satisfy the hypotheses of Theorem 3.1. Then the sequence \( \{x_n\} \) \( \Delta \)-converges a fixed point of \( T \).

**Proof** By Lemma 3.6, we have \( \lim_{n \to \infty} d(Tx_n, x_n) = 0 \). In fact, \( \lim_{n \to \infty} d(x_n, q) \) exists for all \( q \in F(T) \). This implies that sequence \( \{x_n\} \) is bounded. Let \( W_{\Delta}(x_n) = \bigcup A(\{u_n\}) \subseteq F(T) \) and \( W_{\Delta}(x_n) \) consists exactly of one point. In fact, let \( u \in W_{\Delta}(x_n) \), then there exists subsequence \( \{u_n\} \) of \( \{x_n\} \) such that \( \bigcup A(\{u_n\}) = \{u\} \). By Lemma 2.4 and Lemma 2.3, there exists a subsequence \( \{r_n\} \) of \( \{u_n\} \) such that \( \Delta \lim_{n \to \infty} r_n = r \in C \). By Lemma 2.6, \( r \in F(T) \). By Theorem 3.1, \( \lim_{n \to \infty} d(x_n, r) \) exists. Assume that \( u \neq r \). By the uniqueness of asymptotic centers, we have
\[
\limsup_{n \to \infty} d(r_n, r) < \limsup_{n \to \infty} d(r_n, u) \\
\leq \limsup_{n \to \infty} d(u_n, u) \\
\leq \limsup_{n \to \infty} d(u_n, r) \\
= \limsup_{n \to \infty} d(x_n, r) \\
\leq \limsup_{n \to \infty} d(r_n, r) 
\] (3.33)

This is a contradiction. Hence \( u = r \in F(T) \). Finally, we prove \( \{x_n\} \) \( \Delta \)-converges a fixed point of \( T \). We claim that \( x = r \). If not, then the existence of \( \lim_{n \to \infty} d(x_n, r) \) and uniqueness of asymptotic centers imply that there exists a contradiction as (3.33) and therefore \( x = r \in F(T) \). Thus, \( W_{\Delta}(x_n) = \{x_n\} \). This shows that \( \{x_n\} \) \( \Delta \)-converges a fixed point of \( T \). The proof is completed.

**Theorem 3.8.** Let \( X, C, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\} \) satisfy the hypotheses of Theorem 3.1. Assume, in addition that \( T \) is semi-compact. Then the sequence \( \{x_n\} \) converges a strongly to a fixed point of \( T \).

**Proof** From Theorem 3.1, sequence \( \{x_n\} \) is bounded. By Lemma 3.6, we have \( \lim_{n \to \infty} d(Tx_n, x_n) = 0 \) and by the semi-compactness of \( T \), there exists a subsequence \( \{x_{n_k}\} \subset \{x_n\} \) such that \( \{x_{n_k}\} \) converges strongly to some point \( q \in C \). By uniformly continuity of \( T \), we have
\[ d(Tq, q) = \lim_{n \to \infty} d(Tx_{n_k}, x_{n_k}) = 0. \] (3.34)

This implies that \( q \in F(T) \). By Theorem 3.1, \( \lim_{n \to \infty} d(x_n, q) \) exists. Thus, \( q \) is the strong limit of sequence \( \{x_n\} \). The sequence \( \{x_n\} \) converges a strongly to a fixed point \( q \) of \( T \). The proof is completed.

**Theorem 3.9.** Let \( X, C, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\} \) satisfy the hypotheses of Theorem 3.1. Assume, in addition that \( T \) satisfies condition (I). Then the sequence \( \{x_n\} \) converges a strongly to a fixed point of \( T \).
Proof From Theorem 3.1, $$\lim_{n \to \infty} d(x_n, F(T))$$ exists. By condition (I) and Lemma 3.6, we have

$$\lim_{n \to \infty} f(d(x_n, F(T))) \leq \lim_{n \to \infty} d(x_n, Tx_n) = 0.$$  \hspace{1cm} (3.35)

This is, $$\lim_{n \to \infty} f(d(x_n, F(T))) = 0.$$ Since $$f$$ is a non-decreasing function satisfying $$f(0) = 0$$ and $$f(r) > 0$$, for all $$r \in (0, \infty)$$, we have $$\lim_{n \to \infty} d(x_n, F(T)) = 0.$$ By Theorem 3.1 implies that sequence $$\{x_n\}$$ converges a strongly to a fixed point $$q$$ of $$T$$. The proof is completed.

Competing interests
The authors declare that they have no competing interests.

Author's contributions
All authors read and approved the final manuscript.

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