ASYMPTOTIC BEHAVIOR AND A POSTERIORI ERROR ESTIMATES IN SOBOLEV SPACES FOR THE GENERALIZED OVERLAPPING DOMAIN DECOMPOSITION METHOD FOR EVOLUTIONARY HJB EQUATION

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Abstract. In this paper, a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions (DBC) on the interfaces for the discrete solutions on subdomains of evolutionary HJB equation with nonlinear source terms are established using the semi-implicit time scheme combined with a finite element spatial approximation, as well as the techniques of the residual a posteriori error analysis are used. Moreover, using Benssoussan-Lions’ algorithm, an asymptotic behavior in $H^1_0$-norm is deduced. Furthermore, the results of some numerical experiments are presented to support the theory.

1. Introduction

The paper deals with a posteriori error estimates in $H^1$-norm for the generalized overlapping domain decomposition method for the following evolutionary HJB equation: find $u^i(t, x) \in \left( L^2(0, T, D(\Omega)) \cap C^2(0, T, H^{-1}(\Omega)) \right)^M$

\[
\begin{cases}
\frac{\partial u^i}{\partial t} + \max_{i=1,\ldots,M} \left( A^i u - f^i(u) \right) = 0, \quad \text{in } \Sigma, \\
\quad u^i = 0 \text{ in } \Gamma, \quad u^i(x, 0) = u_{0i} \text{ in } \Omega,
\end{cases}
\]

(1.1)

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$, and $\Sigma$ is a set in $\mathbb{R} \times \mathbb{R}^2$ defined as $\Sigma = [0, T] \times \Omega$ with $T < +\infty$. $A^i, (i \leq 1, \ldots, M)$ are the elliptic operator defined as follows:

\[
A^i = -\Delta + a^i_0
\]

(1.2)

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and \( a_i^0 \in \left( L^2(0, T, L^\infty(\Omega)) \cap C^0(0, T, H^{-1}(\Omega)) \right)^M \), \( i \leq 1, \ldots, M \) are sufficiently smooth functions and satisfy the following condition:

(1.3) \( a_i^0(t, x) \geq \beta > 0 \), \( \beta \) is a constant,

with the right hand side \( f^1(\cdot), f^2(\cdot), \ldots, f^M(\cdot) \) are \( M \) nonlinear and Lipschitz functions with Lipschitz constant \( c \) and satisfying the following condition:

(1.4) \( f^i > 0 \) and also it is increasing,

\( c \leq \beta \).

\( K \) is an implicit convex set defined as follows

(1.5) \[
K = \left\{ \left( u^1, u^2, \ldots, u^M \right) \in \left( L^2(0, T, L^2(\Omega)) \cap C^1 \left( 0, T, H^{-1}(\Omega) \right) \right)^M, u^i(x) \leq l + u^{i+1}, u^i = 0 \text{ in } \Gamma, u^i(x, 0) = u_0^i \text{ in } \Omega. \right\}
\]

The symbol \( (\cdot, \cdot)_\Omega \) stands for the inner product in \( L^2(\Omega) \).

The Schwarz alternating method can be used to solve the elliptic boundary value problems on domains which consist of two or more overlapping subdomains. It has been invented by Herman Amandus Schwarz in 1890. This method has been used to solve the stationary and evolutionary boundary value and free boundary problems on domains which consists of two or more overlapping subdomains (see [1, 2-5, 7, 12-16, 27]). The solution of these qualitative problems are approximated by an infinite sequence of functions which results from solving a sequence of stationary or evolutionary boundary value problems in each of the subdomain. Extensive analysis of Schwarz alternating method for nonlinear elliptic boundary value and free boundary problems can be found in [12-14, 16, 20]. Also the effectiveness of Schwarz methods for these problems, especially those in fluid mechanics has been demonstrated in many papers [15, 21-26]. Moreover, the a priory estimate of the error for stationary problem is given in several papers, see for instance [22, 23] which a variational formulation of the classical Schwarz method is derived. In [7] a geometry related convergence results are obtained. In [16-18], the accelerated version of the GODDM has been treated. In addition, in [14] the convergence for a simple rectangular or circular geometries has been studied. However, these authors did not give a criterion to stop the iterative process. All these results can also be found in the recent books on domain decomposition methods [8-9]. Recently in [17, 18], an improved version of the Schwarz method for highly heterogeneous media has been presented. This method uses a new optimized interface conditions specially designed to take into account the heterogeneity between the subdomains on the interfaces. A recent overview of the current state of the art on domain decomposition methods can be found in two special issues of the computer methods in applied mechanics and engineering journal edited by [20-22].

In general, the a priory estimate for stationary problems is not suitable for assessing the quality of the approximate solution on subdomains since it depends mainly on the exact solution itself which is unknown. The alternative approach
is to use the approximate solution itself in order to find such an estimate. This approach, known as a posteriori estimate, became very popular in the nineties of the last century with finite element methods, see the monographs [22, 23]. In [23] an a posteriori estimate for a nonoverlapping domain decomposition algorithm for the elliptic case has been given and a posteriori error analysis has also been used by [22] to determine an optimal value of the penalty parameter for penalty domain decomposition methods to construct fast solvers.

Quite a few works on maximum norm error analysis of overlapping nonmatching grids methods for elliptic boundary value problems are known in the literature (cf., e.g., [8-11]). To prove the main result of this paper, we proceed similar to that in [7]. More precisely, we develop an approach which combines a geometrical convergence result due to [5] and a lemma which consists of estimating the error of the maximum norm between the continuous and discrete Schwarz iterates.

In [7], the authors derived a posteriori error estimates for the generalized overlapping domain decomposition method with Robin boundary conditions on the interfaces for second-order boundary value problems, they shown that the error estimate in the continuous case depends on the differences of the traces of the subdomain solutions on the interfaces after a discretization of the domain by a standard finite elements method. Also they obtained an posteriori error estimate for the discrete solutions on subdomains. A numerical study of stationary and evolutionary free boundary problems of the finite element combined with a finite difference methods has been archived [5-15] and using the domain decomposition method combined with finite element method has been treated in [3-6], [8-11]. Moreover, in recent research [3], we treated the overlapping domain decomposition method combined with a finite element approximation for the elliptic quasi-variational inequalities related with impulse control problem with respect to the mixed boundary conditions for a simple Laplace operator $\Delta$, where a maximum norm analysis of an overlapping Schwarz method on non-matching grids has been proved. Then, in [4] we extended the last result [3] for the parabolic quasi variational with the similar conditions and using the theta time scheme combined with a finite element spatial approximation and proved that the discretization on every subdomain converges in uniform norm. Furthermore a result of asymptotic behavior in uniform norm has been given.

Moreover, in [5], we concerned with the system of parabolic quasi-variational inequalities (PQVIs) related to HJB equation with non linear source terms, our goal is to show that evolutionary HJB equations can be properly approximated by a semi-implicit time scheme combined with a finite element spatial method which turns out to be quasi-optimally accurate in uniform norm. Also we established an asymptotic behavior in uniform norm which similar to that in, [28], which investigated the stationary HJB equation with linear source terms, and we gave the following estimate

$$
\|U_h^p - U^\infty\|_{\infty} = \max_{1 \leq i \leq M} \left\| u_i^{h,p} - u_i^{k,\infty} \right\|_{\infty} \leq C^* \left[ h^2 |\log h|^3 + \left( \frac{1 + k \epsilon}{1 + k \beta} \right)^p \right],
$$

with $C^*$ a constant independent of both $h$ and $k$, where $U_h^p = (u_1^h, \ldots, u_p^h)$, the discrete solution calculated at the moment-end $T = p \Delta t$ for an index of the time discretization $k = 1, \ldots, p$, and $U^\infty$, the asymptotic continuous solution with respect the right hand side condition.
In this paper we prove a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the interfaces for the discrete solutions on subdomains of evolutionary HJB equation with nonlinear source terms using the semi-implicit time scheme combined with a finite element spatial approximation similar to that in [7], which investigated a simple Laplace equation. Moreover, an asymptotic behavior in $H^1_0$-norm is deduced using Bensoussan-Lions' algorithms. Furthermore, the results of some numerical experiments are presented to support the theory.

The outline of the paper is as follows: In section 2, we introduce some necessary notations, then we give the variational formulation of our model. In section 3, a posteriori error estimate is proposed for the convergence of the discrete solution using the semi implicit-time scheme combined with a finite element method on subdomains. In section 4, we associate with the discrete introduced problem a fixed point mapping and we use that in proving the existence of a unique discrete solution. Then in section 5, an $H^1_0$-asymptotic behavior estimate for each subdomain is derived. Finally, in section 5 the results of some numerical experiments are presented to support the theory.

2. Semi-continuous system of parabolic quasi-varational inequalities

The problem (1.1) can be approximated by the following system of the continuous parabolic inequalities: find $(u^1, u^2, ..., u^M) \in K^M$ solution to

\[
\begin{align*}
\frac{\partial u^i}{\partial t} - \Delta u^i + a^i_0(t, x)u^i &\leq f^i(u^i) \quad \text{in } \Sigma, \\
u^i &\leq l + u^{i+1}, \quad u^{M+1} = u^1, \quad i = 1, ..., M, \\
\left(\frac{\partial u^i}{\partial t} + A^i u^i - f^i(u^i)\right) (u^i - (l + u^{i+1})) &= 0, \quad , \\
u^i(0, x) &= u^i_0 \quad \text{in } \Omega, \\
u^i &= 0 \quad \text{in } \Gamma,
\end{align*}
\]

(2.1)

which is similar to that in [28] which investigated the stationary Hamilton-Jacobi-Bellman equations.

2.1. The time discretization. We discretize the problem (2.1) with respect to time by using the semi-implicit scheme. Therefore, we search a sequence of elements $u^{i,k} \in (H^1_0(\Omega))^M$ which approach $u^i(x, t_k), \quad t_k = k\Delta t$, with initial data $u^{i,0} = u^i_0$.

Thus, we have for $k = 1, ..., n$
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\[ \frac{u^{i,k} - u^{i,k-1}}{\Delta t} - \Delta u^{i,k} + a_0^{i,k} u^i \leq f^{i,k} (u^{i,k}) \text{ in } \Sigma, \]

\[ u^{i,k} \leq l + u^{i+1,k}, \quad u^{M+1,k} = u^{1,k}, \quad i = 1, \ldots, M, \]

\[ u^{i,0} (x) = u_0^i \text{ in } \Omega, \quad u^i = 0 \text{ on } \partial \Omega, \]

\[ u^i = 0 \text{ in } \Gamma. \]

Firstly we define the following mapping

\( T : (H_0^1 (\Omega))^M \rightarrow (H_0^1 (\Omega))^M \)

\[ W \rightarrow TW = \xi^{i,k} = \left( \xi_1^{1,k}, \xi_2^{1,k}, \ldots, \xi_M^{1,k} \right) = \partial \left( F^{i,k} \left( u^i \right), l + w^{i+1} \right), \]

where \( \xi^{i,k}, \forall i = 1, \ldots, M \) is the solution of the following problem

\[ \frac{\xi^{i,k} - \xi^{i,k-1}}{\Delta t} - \Delta \xi^{i,k} + a_0^{i,k} \xi^{i,k} \leq f^i \left( \xi^{i,k} \right) \text{ in } \Sigma, \]

\[ \xi^{i,k} \leq l + w^{i+1}, \quad i = 1, \ldots, M, \]

\[ \xi^{M+1,k} = \xi^{1,k}. \]

2.2. Iterative semi-discrete algorithm. We choose \( u^{i,0} = u_0^i \) the solution of the following stationary equation

\[ A^{i,0} u^i = g^{i,0} \]

and \( g^{i,0} \) is an \( M \) regular function.

Now we give the following semi-discrete algorithm or

\[ U^k = TU^{k-1}, \quad k = 1, \ldots, n, \]

where \( U^k = (u_1^{1,k}, \ldots, u_M^{M,k}) \), the solution of the problem (2.2).

Remark 1. We denote by

\[ Q = \left\{ W \in (H_0^1 (\Omega))^M, \text{ such that } 0 \leq W \leq U^0 \right\}, \]

where \( U^0 = U_0 = (u_0^1, \ldots, u_0^M) \).

Since \( f^{i,k} (\cdot) \geq 0 \), and \( u_0^{i,0} = u_{h0}^i \geq 0 \), combining comparison results in variational inequalities with a simple induction, it follows that \( u^{i,k} \geq 0 \), i.e., \( U^k \geq 0 \), \( \forall k = 1, \ldots, M \) and \( TW \geq 0 \).

Furthermore, by (2.6), (2.7) we have

\[ U^1 = TU^0 \leq U^0. \]

Similar to that in previous work [6,28], the mapping \( T \) is a monotone increasing for the stationary HJB equation with non linear source term. Then it can be easily
verified that
\[ U^2 = T U^1 \leq T U^0 = U^1 \leq U^0, \]
thus, inductively
\[ U^{k+1} = T U^k \leq \ldots \leq U^0, \quad \forall k = 1, \ldots, n \]
and also it can be seen the sequence \((u^k)_k\) stays in \(Q\).

According the assumption (1.4), we have \(f^i(\cdot)\) is increasing and under the remark 1, we have for \(k = 1, \ldots, n\)
\[ f^i(u^{i,k}) \leq f^i(u^{i,k-1}), \]
then we can rewrite (2.2) as follow

\[ \begin{align*}
\frac{u^{i,k} - u^{i,k-1}}{\Delta t} - \Delta u^{i,k} + a^{i,k}_0 u^i & \leq f^i(u^{i,k-1}) \quad \text{in } \Sigma, \\
u^{i,k} & \leq l + u^{i+1,k}, \quad u^{M+1,k} = u^{1,k}, \quad i = 1, \ldots, I, \\
u^i & = 0 \quad \text{in } \Gamma.
\end{align*} \]

(2.8)

the problem (2.8) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities (EQVIs)

\[ \begin{align*}
b^i (u^{i,k}, v^i - u^i) & \geq (F(u^{k-1}), v^i - u^{i,k}) \quad \text{in } \Sigma, \\
u^{i,k} & \leq l + u^{i+1,k}, \quad u^{M+1,k} = u^{1,k}, \quad i = 1, \ldots, I, \\
u^i & = 0 \quad \text{on } \partial \Omega,
\end{align*} \]

(2.9)

where

\[ \begin{align*}
b^i (u^{i,k}, v^i - u^i) & = \lambda (u^{i,k}, v^i - u^{i,k}) + a^i (u^{i,k}, v^i - u^{i,k}), \\
F(u^{i,k-1}) & = f(u^{i,k-1}) + \lambda u^{i,k-1}, \\
\lambda & = \frac{1}{\Delta t} = \frac{1}{k} = \frac{T}{n}, \quad k = 1, \ldots, n
\end{align*} \]

(2.10)

and \(a^i(\cdot, \cdot)\) the bilinear forms associated with \(A^i\) defined in (1.2)

\[ a^i(u, u) = \int_{\Omega} \left( \sum_{j,k=1}^{2} \frac{\partial u^i}{\partial x_j} \frac{\partial u^i}{\partial x_k} + a^i_0(t, x) u \right) dx. \]

(2.11)
2.3. The space-continuous for generalized overlapping domain decomposition. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with a piecewise \( C^{1,1} \) boundary \( \partial \Omega \). We split the domain \( \Omega \) into two overlapping subdomains \( \Omega_1 \) and \( \Omega_2 \) such that

\[
\Omega_1 \cap \Omega_2 = \Omega_{12}, \quad \partial \Omega_s \cap \Omega_t = \Gamma_s, \quad s \neq t \quad \text{and} \quad s, t = 1, 2.
\]

We need the spaces

\[
V_s = H^1(\Omega) \cap H^1(\Omega_s) = \{ v \in H^1(\Omega) : v|_{\partial \Omega_s \cap \partial \Omega} = 0 \}
\]

and

\[
W_s = H_0^1(\Gamma_s) = \{ v|_{\Gamma_s} \, v \in V_s \quad \text{and} \quad v = 0 \text{ on } \partial \Omega_s \setminus \Gamma_s \},
\]

which is a subspace of

\[
H^1_0(\Gamma_s) = \{ \psi \in L^2(\Gamma_s) : \psi = \varphi|_{\Gamma_s} \quad \text{for } \varphi \in V_s, \quad s = 1, 2 \},
\]

equipped with the norm

\[
\| \varphi \|_{W_s} = \inf_{v \in V_s, v \equiv \varphi \text{ on } \Gamma_s} \| v \|_{1, \Omega}.
\]

We define the continuous counterparts of the continuous Schwarz sequences defined in (2.9), respectively by \( u_{1}^{i,k,m+1} \in (H^1_0(\Omega))^M \), \( m = 0, 1, 2, \ldots \), solution of

\[
\begin{cases}
 b^i\left( u_{1}^{i,k,m+1}, v^i - u_{1}^{i,k,m+1} \right) \geq \left( F\left( u_{1}^{i,k-1,m+1}, v^i - u_{1}^{i,k,m+1} \right) \right)_{\Omega_1}, \\
u_{1}^{i,k,m+1} = 0, \quad \text{on } \partial \Omega_1 \cap \partial \Omega = \partial \Omega_1 - \Gamma_1,
\end{cases}
\]

and \( u_{2}^{i,k,m+1} \in (H^1_0(\Omega))^M \) solution of

\[
\begin{cases}
 b^i\left( u_{2}^{i,k,m+1}, v^i - u_{2}^{i,k,m+1} \right) \geq \left( F\left( u_{2}^{i,k-1,m+1}, v^i - u_{2}^{i,k,m+1} \right) \right)_{\Omega_2}, \quad m = 0, 1, 2, \ldots,
\end{cases}
\]

\[
\begin{cases}
 u_{2}^{i,k,m+1} = 0, \quad \text{on } \partial \Omega_2 \cap \partial \Omega = \partial \Omega_2 - \Gamma_2,
\end{cases}
\]

where \( \eta_s \) is the exterior normal to \( \Omega_s \) and \( \alpha_s \) is a real parameter, \( s = 1, 2 \).

In the next section, our main interest is to obtain an a posteriori error estimate, we need for stopping the iterative process as soon as the required global precision is reached. Namely, by applying Green formula in (1.2) with the new boundary
conditions of generalized Schwarz alternating method, we get

\[
\begin{align*}
&
\left(-\Delta u_i^{i,k,m+1}, v_i^i - u_i^{i,k,m+1}\right)_{\Omega_1} = \left(\nabla u_i^{i,k,m+1}, \nabla \left(v_i^i - u_i^{i,k,m+1}\right)\right)_{\Omega_1} \\
&- \left(\frac{\partial u_i^{i,k,m+1}}{\partial \eta_1}, v_i^i - u_i^{i,k,m+1}\right)_{\partial \Omega_1 - \Gamma_1} + \left(\frac{\partial u_i^{i,k,m+1}}{\partial \eta_1}, v_i^i - u_i^{i,k,m+1}\right)_{\Gamma_1} \\
&= \left(\nabla u_i^{i,k,m+1}, \nabla \left(v_i^i - u_i^{i,k,m+1}\right)\right)_{\Omega_1} - \left(\frac{\partial u_i^{i,k,m+1}}{\partial \eta_1}, v_i^i - u_i^{i,k,m+1}\right)_{\Gamma_1}
\end{align*}
\]

thus we can deduce

\[
\begin{align*}
&
\left(-\Delta u_i^{i,k,m+1}, v_i^i - u_i^{i,k,m+1}\right)_{\Omega_1} = \left(\nabla u_i^{i,k,m+1}, \nabla \left(v_i^i - u_i^{i,k,m+1}\right)\right)_{\Omega_1} \\
&- \left(\frac{\partial u_i^{i,k,m+1}}{\partial \eta_1}, v_i^i - u_i^{i,k,m+1}\right)_{\partial \Omega_1 - \Gamma_1} + \left(\frac{\partial u_i^{i,k,m+1}}{\partial \eta_1}, v_i^i - u_i^{i,k,m+1}\right)_{\Gamma_1} \\
&= \left(\nabla u_i^{i,k,m+1}, \nabla \left(v_i^i - u_i^{i,k,m+1}\right)\right)_{\Omega_1} - \left(\frac{\partial u_i^{i,k,m+1}}{\partial \eta_1}, v_i^i - u_i^{i,k,m+1}\right)_{\Gamma_1}
\end{align*}
\]

thus the problem (2.14) equivalent to; find \( u_1^{i,k,m+1} \in (V_1)^M \) such that

(2.16)

\[
\begin{align*}
b_i^i\left(u_1^{i,k,m+1}, v_i^i - u_i^{i,k,m+1}\right) + \left(\alpha_1 u_1^{i,k,m+1}, v_i^i - u_i^{i,k,m+1}\right)_{\Gamma_1} \\
&\geq \left(F(u_i^{i,k-1,m+1}, v_i^i - u_i^{i,k,m+1})\right)_{\Omega_1} + \\
&\left(\frac{\partial u_2^{i,k,m}}{\partial \eta_1} + \alpha_1 u_2^{i,k,m}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}, \forall v_1 \in V_1
\end{align*}
\]

and for (2.15) \( u_2^{i,k,m+1} \in V_2 \), we have
The text content is as follows:

(2.17)
\[ b^i(u^{i,k,m+1}_2, v^i_2 - u^{i,k,m+1}_2) + \left( \alpha_2 u^{i,k,m+1}_2, v^i_2 - u^{i,k,m+1}_2 \right)_{\Gamma_2} \]
\[ \geq \left( F(u^{i,k-1,m+1}_2), v^i_2 - u^{i,k,m+1}_2 \right)_{\Omega_2} + \left( \frac{\partial u^{i,k,m}_1}{\partial \eta_2} + \alpha_2 u^{i,k,m}_1, v^i_2 - u^{i,k,m+1}_2 \right)_{\Gamma_2}. \]

3. A Posteriori Error Estimate in the Continuous Case

Since it is numerically easier to compare the subdomain solutions on the interfaces \( \Gamma_1 \) and \( \Gamma_2 \) rather than on the overlap \( \Omega_{12} \), thus we need to introduce two auxiliary problems defined on nonoverlapping subdomains of \( \Omega \). This idea allows us to obtain the a posteriori error estimate by following the steps of Otto and Lube [26]. We define these auxiliary problems by coupling each one of the problems (2.14) and (2.15) with another problem in a nonoverlapping way over \( \Omega \). These auxiliary problems are needed for the analysis and not for the computation section.

To define these auxiliary problems we need to split the domain \( \Omega \) into two sets of disjoint subdomains: \( \Omega_1 \cup \Omega_3 \) and \( \Omega_2 \cup \Omega_4 \) such that
\[ \Omega = \Omega_1 \cup \Omega_3, \] \[ \Omega_1 \cap \Omega_3 = \emptyset, \quad \Omega = \Omega_2 \cup \Omega_4, \quad \Omega_2 \cap \Omega_4 = \emptyset. \]

Let \( (u^{i,k,m}_1, u^{i,k,m}_2) \) be the solution of problems (2.14) and (2.15), we define the couple \( (u^{i,k,m}_1, u^{i,k,m}_3) \) over \( (\Omega_1, \Omega_3) \) to be the solution of the following nonoverlapping problems

(3.1)
\[
\begin{aligned}
\frac{u^{i,k,m+1}_1 - u^{i,k-1,m+1}_1}{\Delta t} - \Delta u^{i,k,m+1}_1 + a^{i,k}_0 u^{i,k,m+1}_1 \geq f^i \left( u^{i,k-1,m+1}_1 \right) \text{ in } \Omega_1, \\
u^{i,k,m+1}_1 = 0, \quad \text{on } \partial \Omega_1 \cap \partial \Omega, \quad k = 1, \ldots, n, \quad i = 1, \ldots, M \\
\frac{\partial u^{i,k,m+1}_1}{\partial \eta_1} + \alpha_1 u^{i,k,m+1}_1 = \frac{\partial u^{i,k,m}_2}{\partial \eta_1} + \alpha_1 u^{i,k,m}_2, \quad \text{on } \Gamma_1
\end{aligned}
\]

and

(3.2)
\[
\begin{aligned}
\frac{u^{i,k,m+1}_3 - u^{i,k-1,m+1}_3}{\Delta t} - \Delta u^{i,k,m+1}_3 + a^{i,k}_0 u^{i,k,m+1}_3 \geq f^i \left( u^{i,m+1,k-1}_3 \right) \text{ in } \Omega_3, \\
u^{i,k,m+1}_3 = 0, \quad \text{on } \partial \Omega_3 \cap \partial \Omega, \\
\frac{\partial u^{i,k,m+1}_3}{\partial \eta_3} + \alpha_3 u^{i,k,m+1}_3 = \frac{\partial u^{i,k,m}_1}{\partial \eta_3} + \alpha_3 u^{i,k,m}_1, \quad \text{on } \Gamma_1.
\end{aligned}
\]

It can be taken \( e^{i,n+1,m}_1 = u^{i,n+1,m}_2 - u^{i,n+1,m}_3 \) on \( \Gamma_1 \), the difference between the overlapping and the nonoverlapping solutions \( u^{i,n+1,m}_2 \) and \( u^{i,n+1,m}_3 \) of the problems (2.14), (2.15) and (resp., (3.1) and (3.2)) in \( \Omega_3 \). Because both overlapping
and the nonoverlapping problems converge see [26] that is, \(u^{i,k,m}_{2}\) and \(u^{i,k,m}_{3}\) tend to \(u_{2}^{i}\) (resp. \(u_{3}^{i}\)), then \(\epsilon_{1}^{i,k,m}\) should tend to naught when \(m\) tends to infinity in \(V_{2}\).

By taking

\[
\Lambda_{3}^{i,k,m} = \frac{\partial u^{i,k,m}_{2}}{\partial \eta_{1}} + \alpha_{1} u^{i,k,m}_{2},
\]

\[
\Lambda_{1}^{i,k,m} = \frac{\partial u^{i,k,m}_{i}}{\partial \eta_{3}} + \alpha_{3} u^{i,k,m}_{i},
\]

(3.3)

\[
\Lambda_{3}^{i,k,m} = \frac{\partial u^{i,k,m}_{3}}{\partial \eta_{1}} + \alpha_{1} u^{i,k,m}_{3} + \frac{\partial \epsilon_{1}^{i,k,m}}{\partial \eta_{1}} + \alpha_{1} \epsilon_{1}^{i,k,m},
\]

\[
\Lambda_{1}^{i,k,m} = \frac{\partial u^{i,k,m}_{1}}{\partial \eta_{3}} + \alpha_{3} u^{i,k,m}_{1}.
\]

Under Green formula, (3.1) and (3.2) can be reformulated to the following system of elliptic variational equations

\[
b_{1}(u^{i,k,m+1}_{1}, v^{i}_{1} - u^{i,k,m+1}_{1}) + \left(\alpha_{1} u^{i,k,m+1}_{1}, v^{i}_{1} - u^{i,k,m+1}_{1}\right)_{\Gamma_{1}} \geq \left(F^{i}(u^{i,m+1,k-1}, v^{i}_{1} - u^{i,k,m+1}_{1}) + \left(\Lambda_{3}^{i,k,m}, v^{i}_{1} - u^{i,k,m+1}_{1}\right)_{\Gamma_{1}}, \forall v^{i}_{1} \in (V_{1})^{M}
\]

(3.4)

and

\[
b_{3}(u^{i,k,m+1}_{3}, v^{i}_{3} - u^{i,k,m+1}_{3}) + \left(\alpha_{3} u^{i,k,m+1}_{3}, v^{i}_{3} - u^{i,k,m+1}_{3}\right)_{\Gamma_{1}} \geq \left(F^{i}(u^{i,m+1,k-1}, v^{i}_{3} - u^{i,k,m+1}_{3}) + \left(\Lambda_{1}^{i,k,m}, v^{i}_{3} - u^{i,k,m+1}_{3}\right)_{\Gamma_{1}}, \forall v^{i}_{3} \in V_{3}.
\]

(3.5)

On the other hand by taking

\[
\theta_{1}^{i,k,m} = \frac{\partial \epsilon_{1}^{i,k,m}}{\partial \eta_{1}} + \alpha_{1} \epsilon_{1}^{i,k,m},
\]

(3.6)

we get

\[
\Lambda_{3}^{i,k,m} = \frac{\partial u^{i,k,m}_{3}}{\partial \eta_{1}} + \alpha_{1} u^{i,k,m}_{3} + \frac{\partial (u^{i,k,m}_{2} - u^{i,k,m}_{3})}{\partial \eta_{1}} + \alpha_{1} (u^{i,k,m}_{2} - u^{i,k,m}_{3})
\]

(3.7)

\[
= \frac{\partial u^{i,k,m}_{3}}{\partial \eta_{1}} + \alpha_{1} u^{i,k,m}_{3} + \frac{\partial \epsilon_{1}^{i,k,m}}{\partial \eta_{1}} + \alpha_{1} \epsilon_{1}^{i,k,m}
\]

\[= \frac{\partial u^{i,k,m}_{3}}{\partial \eta_{1}} + \alpha_{1} u^{i,k,m}_{3} + \theta_{1}^{i,k,m}.
\]
Using (3.6) we have
\[ \Lambda^{i,k,m+1}_3 = \frac{\partial u^{i,k,m+1}_3}{\partial \eta_1} + \alpha_1 u^{i,k,m+1}_3 + \theta^{i,k,m+1}_1 \]
\[ = -\frac{\partial u^{i,k,m+1}_3}{\partial \eta_3} + \alpha_1 u^{i,k,m+1}_3 + \theta^{i,k,m+1}_1 \]
\[ = \alpha_3 u^{i,k,m+1}_3 - \frac{\partial u^{i,k,m}_1}{\partial \eta_1} - \alpha_3 u^{i,k,m}_1 + \alpha_1 u^{i,k,m+1}_3 + \theta^{i,k,m+1}_1 \]
\[ = (\alpha_1 + \alpha_3) u^{i,k,m+1}_3 - \Lambda^{i,k,m}_1 + \theta^{i,k,m+1}_1 \]
and the last equation in (3.7), we have
\[ \Lambda^{i,k,m+1}_1 = -\frac{\partial u^{i,k,m+1}_1}{\partial \eta_1} + \alpha_3 u^{i,k,m+1}_1 \]
\[ = \alpha_1 u^{i,k,m+1}_1 - \frac{\partial u^{i,k,m}_2}{\partial \eta_1} - \alpha_1 u^{i,k,m}_2 + \alpha_3 u^{i,k,m+1}_1 + \alpha_3 u^{i,k,m+1}_1 \]
\[ = (\alpha_1 + \alpha_3) u^{i,k,m+1}_1 - \Lambda^{i,k,m}_3 + \theta^{i,k,m+1}_3 \].

From this result we can write the following algorithm which is equivalent to the auxiliary nonoverlapping problem (3.4), (3.5). We need this algorithm and two lemmas for obtaining an a posteriori error estimate for this problem.

3.1. Semi discrete algorithm. The sequences \((u^{i,k,m}_1, u^{i,k,m}_3)_{m \in \mathbb{N}}\) solutions of (3.4), (3.5) satisfy the following domain decomposition algorithm:

**Step 1:** \(k = 0\).

**Step 2:** Let \(\Lambda^{i,k,0}_1 \in W^*_1\) be an initial value, \(s = 1, 3\) \((W^*_1 is the dual of \(W_1\)).

**Step 3:** Given \(\Lambda^{i,k,m}_1 \in W^*\) solve for \(s, t = 1, 3, s \neq t\) : Find \(v^{i,k,m+1}_s \in V_s\) solution of
\[ b^i_s(u^{i,k,m+1}_s, v^i_s - u^{i,k,m+1}_s) + (\alpha_s u^{i,k,m+1}_s, v^i_s)_{\Gamma_s} \geq (F^i(u^{i,k-1,m+1}_s), v^i_s)_{\Omega_s} + \]
\[ + (\Lambda^{i,k,m+1}_t, v^i_s)_{\Gamma_s}, \forall u_s \in V_s. \]

**Step 4:** Compute
\[ \theta^{i,k,m+1}_1 = \frac{\partial \epsilon^{i,k,m+1}_1}{\partial \eta_1} + \alpha_1 \epsilon^{i,k,m+1}_1. \]

**Step 5:** Compute new data \(\Lambda^{s,k+1,m}_t \in W^*\) solve for \(s, t = 1, 3, 3\), from
\[ (\Lambda^{s,k+1,m}_t, \varphi)_{\Gamma_t} = (\alpha_s u^{i,k,m+1}_s, v^i_s)_{\Gamma_s} - \]
\[ (\Lambda^{s,k+1,m}_t, \varphi)_{\Gamma_s} + (\theta^{k,m+1}_t, \varphi)_{\Gamma_t}, \forall \varphi \in W_s, s \neq t. \]
Step 6: Set $m = m + 1$ go to Step 3.

Step 7: Set $k = k + 1$ go to Step 2.

Lemma 1. Let $u^{i,k}_s = u^{i,k}_{1s}$, $e^{i,k,m+1}_s = u^{i,k,m+1}_s - u^{i,k}_s$ and $\eta^{i,k,m+1}_s = \Lambda^{i,k,m+1}_s - \Lambda^{i,k}_s$. Then for $s, t = 1, 3, s \neq t$, we have

\begin{equation}
(\alpha_s e^{i,k,m+1}_s, v^i_s - e^{i,k,m+1}_s)_{\Gamma_s} = \left(\eta^{i,k,m}_s, v^i_s - e^{i,k,m+1}_s\right)_{\Gamma_s}, \forall v^i_s \in V^i_s
\end{equation}

and

\begin{equation}
(\eta^{i,k,m+1}_s, \varphi)_{\Gamma_s} = \left((\alpha_s + \alpha_1)e^{i,k,m+1}_s, v^i_s\right)_{\Gamma_s} - \left(\eta^{i,k,m}_s, \varphi\right)_{\Gamma_s} + \left(\theta^{i,k,m+1}_t, \varphi\right)_{\Gamma_s}, \forall \varphi \in W_1.
\end{equation}

Proof. 1. We have

\begin{equation}
\begin{aligned}
&b^i_s(u^{i,k,m+1}_s, v^i_s - u^{i,k,m+1}_s) + \left(\alpha_s u^{i,k,m+1}_s, v^i_s - u^{i,k,m+1}_s\right)_{\Gamma_s}, \\
&\quad \geq (F^i(u^{i,k-1,m+1}_s), v^i_s - u^{i,k}_s)_{\Omega_s}, \\
&\quad + \left(\Lambda^{i,k}_s, v^i_s - u^{i,k}_s\right)_{\Gamma_s}, \forall v^i_s \in V^i_s
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
&b^i_s(u^{i,k}_s, v^i_s - u^{i,k}_s) + \left(\alpha_s u^{i,k}_s, v^i_s - u^{i,k}_s\right)_{\Gamma_s}, \\
&\quad \geq (F^i(u^{i,k-1}_s), v^i_s - u^{i,k}_s)_{\Omega_s}, \\
&\quad + \left(\Lambda^{i,k}_s, v^i_s - u^{i,k}_s\right)_{\Gamma_s}, \forall v^i_s \in V^i_s.
\end{aligned}
\end{equation}

Since $b^i(\cdot, \cdot)$ is a coercive bilinear form, it can be deduced

\begin{equation}
\begin{aligned}
&b^i_s(u^{i,k,m+1}_s - u^{i,k+1}_s, v^i_s) + \left(\alpha_s u^{i,k,m+1}_s - u^{i,k+1}_s, v^i_s\right)_{\Gamma_s} \geq \left(\Lambda^{i,k,m}_s - \Lambda^{i,k}_s, v^i_s\right)_{\Gamma_s}, \forall v^i_s \in V^i_s
\end{aligned}
\end{equation}

and so

\begin{equation}
\begin{aligned}
&b^i_s(e^{i,k,m+1}_s, v^i_s - e^{i,k,m+1}_s) + \left(\alpha_s e^{i,k,m+1}_s, v^i_s - e^{i,k,m+1}_s\right)_{\Gamma_s} \geq \left(\eta^{i,k,m}_s, v^i_s - e^{i,k,m+1}_s\right)_{\Gamma_s}, \forall v^i_s \in V^i_s.
\end{aligned}
\end{equation}

2. We have \( \lim_{m \to +\infty} \epsilon^{i,k+1,1}_1 = \lim_{m \to +\infty} \theta^{i,k+1,1}_1 = 0 \). Then

\[ \Lambda^{i,k}_s = (\alpha_1 + \alpha_3)u^{i,k}_s - \Lambda^{i,k}_t. \]

Therefore

\[ \eta^{i,k,m+1}_s = \Lambda^{i,k,m+1}_s - \Lambda^{i,k}_s \]

\[ = (\alpha_1 + \alpha_3)u^{i,k,m+1}_s - \Lambda^{i,k,m}_s + \theta^{i,k,m+1}_t - (\alpha_1 + \alpha_3)u^{i,k}_s + \Lambda^{i,k}_j \]

\[ = (\alpha_1 + \alpha_3)(u^{i,k,m+1}_1 - u^{i,k}_s) - (\Lambda^{i,k,m}_t - \Lambda^{i,k}_t) + \theta^{i,k,m+1}_t. \]
Lemma 2. By letting $C$ be a generic constant which has different values at different places, we get for $s,t=1,3, s\neq t$

\begin{equation}
\begin{aligned}
(\eta^{i,k,m-1}_s - \alpha_se^{i,k,m}_s,w^i)_{\Gamma_s} &\leq C\|e^{i,k,m}_s\|_{1,\Omega_s}\|w^i\|_{W}\ 
\text{and}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
(\alpha_sw^i_s + \theta^{i,k,m+1}_s, e^{i,k,m+1}_s)_{\Gamma_s} &\leq C\|e^{i,k,m+1}_s\|_{1,\Omega_s}\|w^i\|_{W}.
\end{aligned}
\end{equation}

Proof. Using Lemma 1 and the fact of the inverse of the trace mapping $Tr^{-1} : W_1 \rightarrow V_s$ is continuous we have for $s,t=1,3, s\neq t$

\begin{equation}
\begin{aligned}
(\eta^{i,k,m-1}_s - \alpha_se^{i,k,m}_s,w^i)_{\Gamma_s} = (e^{i,k,m}_s, Tr^{-1}w^i)_{\Omega_i} + \\
+ (\alpha_se^{i,k,m}_s, Tr^{-1}w^i)_{\Omega_i} + \lambda (e^{i,k,m}_s, Tr^{-1}w^i)_{\Omega_i} \\
\leq |e^{i,k,m}_s|_{1,\Omega_s} |Tr^{-1}w^i|_{1,\Omega_s} + \|\alpha\|_{\infty} \|e^{i,k,m}_s\|_{0,\Omega_s} \|Tr^{-1}w^i\|_{0,\Omega_s} \\
+ |\lambda| \|e^{i,k,m}_s\|_{0,\Omega_s} \|Tr^{-1}w^i\|_{0,\Omega_s} \\
\leq C\|e^{i,k,m}_s\|_{1,\Omega_s}\|w^i\|_{W}.
\end{aligned}
\end{equation}

For the second estimate, we have

\begin{equation}
\begin{aligned}
(\alpha_sw^i_s + \theta^{i,k,m+1}_s, e^{i,k,m+1}_s)_{\Gamma_s} = (\alpha_sw^i_s + \theta^{i,k,m+1}_s, e^{i,k,m+1}_s)_{\Gamma_s} \\
\leq \|\alpha_sw^i_s + \theta^{i,k,m+1}_s\|_{0,\Gamma_1} \|e^{i,k,m+1}_s\|_{0,\Gamma_1} \\
\leq (\|\alpha_s\|_{\infty} \|w^i\|_{0,\Gamma_1} + \|\theta^{i,k,m+1}_s\|_{0,\Gamma_1}) \|e^{i,k,m+1}_s\|_{0,\Gamma_1} \\
\leq \max(|\alpha_s|, \|\theta^{i,k,m+1}_s\|_{0,\Gamma_1}) \|w^i\|_{0,\Gamma_1} \|e^{i,k,m+1}_s\|_{0,\Gamma_1} \\
\leq C\|e^{i,k,m+1}_s\|_{0,\Gamma_1} \|w^i\|_{0,\Gamma_1} \leq C\|e^{i,k,m+1}_s\|_{0,\Gamma_1} \|w^i\|_{W}.
\end{aligned}
\end{equation}

Thus, it can be deduced

\begin{equation}
\begin{aligned}
|\alpha_s| \|w^i\|_{0,\Gamma_1} + \|\theta^{i,k,m+1}_s\|_{0,\Gamma_1} \leq \max(|\alpha_s|, \|\theta^{i,k,m+1}_s\|_{0,\Gamma_1}) \|w^i\|_{0,\Gamma_1}.
\end{aligned}
\end{equation}

\hfill \Box

Proposition 1. For the sequences $(u^{i,k,m}_1, u^{i,k,m}_3)_{m\in\mathbb{N}}$ solutions of (3.4), (3.5) we have the following a posteriori error estimation

\begin{equation}
\begin{aligned}
\|u^{i,k,m+1}_1 - u^{i,k}_1\|_{1,\Omega_1} + \|u^{i,k,m}_3 - u^{i,k}_3\|_{3,\Omega_3} \leq C\|u^{i,k,m+1}_1 - u^{i,k,m}_3\|_{W}.
\end{aligned}
\end{equation}
Proof. From (3.8), (3.10) and we take \( v_1^i = v_1^i - u^{i,k,m+1} \) in (3.4), then we have

\[
b_1^i (e_1^{i,k,m+1}, v_1^i) + b_3^i (e_3^{i,k,m}, v_3^i) = \]
\[
\left( \eta_3^{i,k,m} - \alpha_1 e_1^{i,k,m+1}, v_1^i \right)_{\Gamma_1} + \left( \eta_1^{i,k,m-1} - \alpha_3 e_3^{i,k,m}, v_3^i \right)_{\Gamma_1} \]
\[
= \left( \eta_3^{i,k,m} - \alpha_1 e_1^{i,k,m+1}, v_1^i \right)_{\Gamma_1} + \left( \eta_1^{i,k+1,m-1} - \alpha_3 e_3^{i,k+1,m}, v_3^i \right)_{\Gamma_1} \]
\[
+ \left( \eta_1^{i,k+1,m-1} - \alpha_3 e_3^{i,k+1,m}, v_3^i - v_1^i \right)_{\Gamma_1} .
\]

Thus, we have

\[
(3.14)
\]
\[
b_1^i (e_1^{i,k,m+1}, v_1^i) + b_3^i (e_3^{i,k,m}, v_3^i) =
\]
\[
\left( \eta_3^{i,n+1,m} - \alpha_1 e_1^{i,k,m+1} + \eta_1^{i,k+1,m-1} - \alpha_3 e_3^{i,k+1,m}, v_1^i \right)_{\Gamma_1} \]
\[
+ \left( \eta_1^{i,k+1,m-1} - \alpha_3 e_3^{i,k+1,m}, v_3^i - v_1^i \right)_{\Gamma_1}.
\]

Taking \( v_1^i = e_1^{i,k+1,m+1} \) and \( v_3^i = e_3^{i,k+1,m} \) in (3.4), then using \( \frac{1}{2} (a + b) \leq a^2 + b^2 \)
and the lemma 2, we get

\[
\frac{1}{2} \left( \left\| u^{i,k,m+1} - u^{i,k+1}_1 \right\|_{1,\Omega_1} + \left\| u^{i,k,m} - u^{i,k+1}_3 \right\|_{3,\Omega_3} \right)^2 \leq \left\| u^{i,k,m+1} - u^{i,k}_1 \right\|_{1,\Omega_1}^2 + \left\| u^{i,k,m} - u^{i,k}_3 \right\|_{3,\Omega_3}^2
\]
\[
\leq \left\| e_1^{i,k,m+1} \right\|_{1,\Omega_1}^2 + \left\| e_3^{i,k,m} \right\|_{3,\Omega_3}^2 \leq \left( \nabla e_1^{i,k,m+1}, \nabla e_1^{i,k,m+1} \right)_{\Omega_1} + \left( a_0 e_1^{i,k,m+1} + e_1^{i,k,m+1} \right)_{\Omega_1}
\]
\[
+ \left( \nabla e_3^{i,k,m}, \nabla e_3^{i+1,k,m} \right)_{\Omega_1} + \left( a_0 e_3^{i,k,m}, e_3^{i,k,m} \right)_{\Omega_1} \leq \left( \nabla e_1^{i,k,m+1}, \nabla e_1^{i,k,m+1} \right)_{\Omega_1} + \left\| a_0 \right\|_{\infty} \left( e_1^{i,k,m+1}, e_1^{i,k,m+1} \right)_{\Omega_1}
\]
\[
+ \left( \nabla e_3^{i,k,m}, \nabla e_3^{i,k,m} \right)_{\Omega_1} + \left\| a_0 \right\|_{\infty} \left( e_3^{i,k,m}, e_3^{i,k,m} \right)_{\Omega_3}.
\]
Then
\[ \frac{1}{2} \left( \left\| u^{i,k,m+1}_1 - u^{i,k+1}_1 \right\|_{1,\Omega_1} + \left\| u^{i,k,m}_3 - u^{i,k+1}_3 \right\|_{3,\Omega_3} \right)^2 \]
\[ \leq \max(1, \| q_0^\|_\infty) \left( b_1 \left( e^{i,k,m+1}_1, e^{i,k,m+1}_1 \right) + b_3 \left( e^{i,k,m}_3, e^{i,k,m}_3 \right) \right) \]
\[ = \max(1, \| q_0^\|_\infty) \left( \alpha_1 (e^{i,k,m}_3 - e^{i,k,m+1}_1) + \theta_1 (e^{i,k,m}_1, e^{i,k,m+1}_1) \right) + \left( \eta_1 (e^{i,k,m-1} - \alpha_3 e^{i,k,m}_3, e^{i,k,m}_3 - e^{i,k,m+1}_1) \right) \Gamma_1 \]
\[ \leq C_1 \left[ \left\| e^{i,k,m+1}_1 \right\|_{1,\Omega_1} + \left\| e^{i,k,m}_3 \right\|_{3,\Omega_3} \right] + \left\| e^{i,k,m}_3 - e^{i,k,m+1}_1 \right\|_{W_1} \]
\[ \leq C_1 \left[ \left\| e^{i,k,m+1}_1 \right\|_{1,\Omega_1} + \left\| e^{i,k,m}_3 \right\|_{3,\Omega_3} \right] + \left\| e^{i,k,m}_3 - e^{i,k,m+1}_1 \right\|_{W_1} \]
thus
\[ \left\| e^{i,k+1,m+1}_1 \right\|_{1,\Omega_1} + \left\| e^{i,k+1,m}_3 \right\|_{3,\Omega_3} \leq \left\| e^{i,k+1,m+1}_1 - e^{i,k+1,m}_3 \right\|_{W_1} . \]

Therefore
\[ \left\| u^{i,k+1,m+1}_1 - u^{i,k+1}_1 \right\|_{1,\Omega_1} + \left\| u^{i,k+1,m}_3 - u^{i,k+1}_3 \right\|_{3,\Omega_3} \leq 2C_1 \left\| u^{n+1,m+1}_1 - u^{n+1,m}_3 \right\|_{W_1} . \]

In the similar way, we define another nonoverlapping auxiliary problems over \((\Omega_2, \Omega_4)\), we get the same result.

**Proposition 2.** For the sequences \((u^{i,k,m}_2, u^{i,k,m}_4)_{m \in \mathbb{N}}\). We get the the similar following a posteriori error estimation

\[ \left\| u^{i,k,m+1}_2 - u^{i,k}_2 \right\|_{2,\Omega_2} + \left\| u^{i,k,m}_4 - u^{i,k}_4 \right\|_{4,\Omega_4} \leq C \left\| u^{i,k,m+1}_2 - u^{i,k}_4 \right\|_{W_2} . \]

**Proof.** The proof is very similar to proof of Proposition 1. \(\square\)

**Theorem 1.** Let \(u^{i,k}_s = u^{i,k}_s\). For the sequences \((u^{i,k,m}_1, u^{i,k,m}_2)_{m \in \mathbb{N}}\) solutions of problems (3.1),(3.2), one have the following result

\[ \left\| u^{i,k+1,m}_1 - u^{i,k}_1 \right\|_{1,\Omega_1} + \left\| u^{i,k+1,m}_2 - u^{i,k}_2 \right\|_{2,\Omega_2} \leq \]
\[ C \left( \left\| u^{i,k+1,m}_1 - u^{i,k}_1 \right\|_{W_1} + \left\| u^{i,k+1,m}_2 - u^{i,k}_2 \right\|_{W_2} + \right) \]
\[ + \left\| e^{i,k}_1 \right\|_{W_1} + \left\| e^{i,k}_2 \right\|_{W_2} . \]
Proof. We use two nonoverlapping auxiliary problems over \((\Omega_1, \Omega_3)\) and over \((\Omega_2, \Omega_4)\) resp. From the previous two propositions, we have

\[ \begin{align*}
&\|u^{i, k, m+1}_1 - u^{i, k}_1\|_{1, \Omega_1} + \|u^{i, k, m}_2 - u^{i, k}_2\|_{2, \Omega_2} \\
&\leq \|u^{i, k, m+1}_1 - u^{i, k}_1\|_{1, \Omega_1} + \|u^{i, k, m}_3 - u^{i, k}_3\|_{3, \Omega_3} \\
&+ \|u^{i, k, m}_2 - u^{i, k+1}_2\|_{2, \Omega_2} + \|u^{i, k, m-1}_4 - u^{i, k+1}_4\|_{4, \Omega_4} \\
&\leq C \left( \left\|u^{i, k, m+1}_1 - u^{i, k+1}_3\right\|_{W_1} + C \left\|u^{i, k, m}_2 - u^{i, k}_2\right\|_{W_2} \right),
\end{align*} \]

then

\[ \begin{align*}
&\|u^{i, k, m+1}_1 - u^{i, k}_1\|_{1, \Omega_1} + \|u^{i, k, m}_2 - u^{i, k}_2\|_{2, \Omega_2} \\
&
\leq C \left( \left\|u^{i, k, m+1}_1 - u^{i, k}_2 + \epsilon_1^{i, k, m+1}\right\|_{W_1} + \left\|u^{i, k, m}_2 - u^{i, k}_1 + \epsilon_2^{i, k, m-1}\right\|_{W_2} \right),
\end{align*} \]

\[ \square \]

4. A Posteriori Error Estimate in the Discrete Case

4.1. The space discretization. Let \(\Omega\) be decomposed into triangles and \(\tau_h\) denote the set of all those elements \(h > 0\) is the mesh size. We assume that the family \(\tau_h\) is regular and quasi-uniform. We consider the usual basis of affine functions \(i = \{1, \ldots, m(h)\}\) defined by \(\varphi_i(M_j) = \delta_{ij}\), where \(M_j\) is a vertex of the considered triangulation.

We discretize in space, i.e., that we approach the space \(H_0^1\) by a space discretization of finite dimensional \(V^h \subset H_0^1\). In a second step, we discretize the problem with respect to time using the semi-implicit scheme. Therefore, we search a sequence of elements \(u^h_n \in V^h\) which approaches \(u_h(t_n, \cdot), t_n = n\Delta t, k = 1, \ldots, n\), with initial data \(u^h_0 = u_{0h}\).

Let \(u^{i, k, m+1}_n \in V^h\) be the solution of the discrete problem associated with (3.1), \(u^{m, h}_n = u^{m+1, h}_n\).

We construct the sequences \((u^{i, k, m+1}_n)_{m \in \mathbb{N}}, u^{i, k, m+1}_s \in K_{i, h}, (s = 1, 2)\) solutions of discrete problems associated with (3.1), (3.2).

where \(K_h\) is a suitable set given by

\[ K_h = \left\{ \left( u^i_h, \ldots, u^M_h \right) \in \left( L^2 \left( 0, T, H^1_0 (\Omega) \right) \right)^M : \left( u^i_h \right) \leq r_h \left( l + u^{i+1}_h \right), \right. \]

\[ u^i_h = 0 \text{ in } \Gamma, u^i_h(t_0) = u^i_{h, 0} \text{ in } \Omega, \]

\[ (4.1) \]
where \( r_h \) is the usual interpolation operator defined by

\[
(4.2) \quad r_h v = \sum_{i=1}^{m(h)} v(M_j) \varphi_i(x).
\]

In similar manner to that of the previous section, we introduce two auxiliary problems, we define for \((\Omega_1, \Omega_3)\) the following full-discrete problems: find \( u_{1,1}^{i,k,m+1} \in K_h \) solution of

\[
(4.3) \quad \begin{cases}
    b_i^1(u_{1,1}^{i,k,m+1}, \tilde{v}_{1,1} - u_{1,1}^{i,k,m+1}) + \left( \alpha_{1,1}^i u_{1,1}^{i,k,m+1}, \tilde{v}_{1,1} - u_{1,1}^{i,k,m+1} \right)_{\Gamma_i} \\
    \geq \left( F(u_{1,1}^{i,k,m+1}), \tilde{v}_{1,1} - u_{1,1}^{i,k,m+1} \right)_{\Omega_1}, \quad \tilde{v}_{1,1} \in K_h, \\
    u_{1,1}^{i,k,m+1} = 0, \quad \text{on } \partial \Omega_1 \cap \partial \Omega, \\
    \frac{\partial u_{1,1}^{i,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,1}^{i,k,m+1} = \frac{\partial u_{2,2}^{i,k,m}}{\partial \eta_1} + \alpha_2 u_{2,2}^{i,k,m}, \quad \text{on } \Gamma_1.
\end{cases}
\]

by taking the trial function \( \tilde{v}_{1,1} = v_{1,1} - u_{1,1}^{i,k,m+1} \) in \((4.2)\), we get

\[
(4.4) \quad \begin{cases}
    b_i^1(u_{1,1}^{i,k,m+1}, v_{1,1}^i) + \left( \alpha_{1,1}^i u_{1,1}^{i,k,m+1}, v_{1,1}^i \right)_{\Gamma_i} \\
    \leq \left( F(u_{1,1}^{i,k,m+1}), v_{1,1}^i \right)_{\Omega_1}, \quad v_{1,1}^i \in K_h, \\
    u_{1,1}^{i,k,m+1} = 0, \quad \text{on } \partial \Omega_1 \cap \partial \Omega, \\
    \frac{\partial u_{1,1}^{i,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,1}^{i,k,m+1} = \frac{\partial u_{2,2}^{i,k,m}}{\partial \eta_1} + \alpha_2 u_{2,2}^{i,k,m}, \quad \text{on } \Gamma_1.
\end{cases}
\]

Similarly, we get

\[
(4.5) \quad \begin{cases}
    b_i^1(u_{3,3}^{i,k,m+1}, v_{3,3}^i) + \left( \alpha_{3,3}^i u_{3,3}^{i,k,m+1}, v_{3,3}^i \right)_{\Gamma_1} \leq \left( F^i(u_{3,3}^{i,k,m+1}), v_{3,3}^i \right)_{\Omega_3}, \\
    u_{3,3}^{i,k,m+1} = 0, \quad \text{on } \partial \Omega_3 \cap \partial \Omega, \\
    \frac{\partial u_{3,3}^{i,k,m+1}}{\partial \eta_3} + \alpha_3 u_{3,3}^{i,k,m+1} = \frac{\partial u_{1,1}^{i,k,m}}{\partial \eta_3} + \alpha_3 u_{1,1}^{i,k,m}, \quad \text{on } \Gamma_1.
\end{cases}
\]
For \((\Omega_2, \Omega_1)\), we have
\[
\begin{cases}
\beta^i(u_{2,h}^{i,k,m+1}, v_{2,h}^i) + (\alpha_{2,h} u_{2,h}^{i,k,m+1}, v_{2,h}^i) \leq \left( F^i(u_{2,h}^{i,k-1,m+1}, v_{2,h}^i) \right)_{\Omega_2}, \\
u_{2,h}^{i,k,m+1} = 0, \text{ on } \partial \Omega_2 \cap \partial \Omega,
\end{cases}
\]
\[
\frac{\partial u_{2,h}^{i,k,m+1}}{\partial \eta_2} + \alpha_2 u_{2,h}^{i,k,m+1} = \frac{\partial u_{1,h}^{i,k,m}}{\partial \eta_2} + \alpha_2 u_{1,h}^{i,k,m}, \text{ on } \Gamma_2
\]
and
\[
\begin{cases}
\beta^i(u_{4,h}^{i,k,m+1}, v_{4,h}^i) + (\alpha_{4,h} u_{4,h}^{i,k,m+1}, v_{4,h}^i) \leq \left( F^i(u_{4,h}^{i,k-1,m+1}, v_{4,h}^i) \right)_{\Omega_4}, \\
u_{4,h}^{n+1,m+1} = 0, \text{ on } \partial \Omega_1 \cap \partial \Omega,
\end{cases}
\]
\[
\frac{\partial u_{4,h}^{n+1,m+1}}{\partial \eta_4} + \alpha_4 u_{4,h}^{n+1,m+1} = \frac{\partial u_{2,h}^{n+1,m}}{\partial \eta_4} + \alpha_4 u_{2,h}^{n+1,m}, \text{ on } \Gamma_2.
\]

**Theorem 2.** [29] The solution of the system of QVI (4.3), (4.4), and (4.5) is the maximum element the set of discrete subsolutions.

We can obtain the discrete counterparts of propositions 1 and 2 by doing almost the same analysis as in section above (i.e., passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore,

\[
\left\| u_{1}^{i,k,m+1} - u_{1}^{i,k} \right\|_{1,\Omega_1} + \left\| u_{3}^{i,k,m} - u_{3}^{i,k} \right\|_{1,\Omega_3} \leq C \left\| u_{1}^{i,k,m+1} - u_{3}^{i,k,m} \right\|_{W_1}
\]

and

\[
\left\| u_{2}^{i,k,m+1} - u_{2}^{i,k} \right\|_{1,\Omega_2} + \left\| u_{4}^{i,k,m} - u_{4}^{i,k} \right\|_{1,\Omega_4} \leq C \left\| u_{2}^{i,k,m+1} - u_{4}^{i,k,m} \right\|_{W_2}.
\]

Similar to that in the proof of Theorem 1 we get the following discrete estimates

\[
C \left( \left\| u_{1}^{i,k,m+1} - u_{1}^{i,k} \right\|_{1,\Omega_1} + \left\| u_{2}^{i,k,m} - u_{2}^{i,k} \right\|_{1,\Omega_2} \right) \leq
\]

\[
\left( \left\| u_{1}^{i,k,m+1} - u_{1}^{i,k} \right\|_{W_1} + \left\| u_{2}^{i,k,m} - u_{2}^{i,k,m-1} \right\|_{W_2} + \left\| u_{1}^{i,k,m+1} \right\|_{W_1} + \left\| u_{2}^{i,k,m-1} \right\|_{W_2} \right).
\]

Next we will obtain an error estimate between the approximated solution \(u_{s,h}^{i,k,m+1}\) and the semi discrete solution in time \(u^{i,k}\). We introduce some necessary notations. We denote by

\[
\varepsilon_h = \{ E \in T : T \in \tau_h \text{ and } E \notin \partial \Omega \}
\]
and for every \(T \in \tau_h \text{ and } E \in \varepsilon_h\), we define

\[
\omega_T = \{ T' \in \tau_h : T' \cap T \neq \emptyset \}, \quad \omega_E = \{ T' \in \tau_h : T' \cap E \neq \emptyset \}.
\]
The right hand side \( f \) is not necessarily continuous function across two neighboring elements of \( \tau_h \) having \( E \) as a common side, \([f]\) denotes the jump of \( f \) across \( E \) and \( \eta_E \) the normal vector of \( E \).

We have the following theorem which gives an a posteriori error estimate for the discrete GODDM.

**Theorem 3.** Let \( u^k_s = u^k |_{\Omega_s} \) where \( u \) is the solution of problem (1.1), the sequences \( (u_{1,h}^{i,k,m+1}, u_{2,h}^{i,k,m}) \) are solutions of problems (3.4) and (2.16). Then there exists a constant \( C \) independent of \( h \) such that

\[
\left\| u_{1,h}^{i,k,m+1} - u_1 \right\|_{1,\Omega_1} + \left\| u_{2,h}^{i,k,m} - u_2 \right\|_{1,\Omega_2} \leq C \left\{ \sum_{i=1}^{2} \sum_{T \in \tau_h} (\eta_T^T) + \eta_s \right\},
\]

where

\[
\eta_s = \left\| u_h^{i,k,*} - u_{h,t}^{i,k,*-1} \right\|_{W_{h,s}} + \left\| c_{i,h}^{i,k,*} \right\|_{W_{h,s}}
\]

and

\[
\eta_s^T = h_T \left\| F \left( u_{h,s}^{i,k,*} \right) + u_{h,s}^{i,k-1} + \Delta u_{h,s}^{i,k,*} - \left( 1 + \lambda \alpha_{h,0}^{i,k} \right) u_{h,s}^{i,k} \right\|_{0,E} + \sum_{E \in \tau_h} h_E^2 \left\| \frac{\partial u_{h,s}^{i,k,*}}{\partial n_E} \right\|_{0,E}^T,
\]

where \( c \) is Lipschitz constant of the right hand side and the symbol * is corresponds to \( m+1 \) when \( s = 1 \) and to \( m \) when \( s = 2 \).

**Proof.** The proof is based on the technique of the residual a posteriori estimation see [26] and Theorem 2. We give the main steps by the triangle inequality we have

\[
(4.9) \sum_{s=1}^{2} \left\| u_s^{i,k} - u_s^{i,k,*} \right\|_{1,\Omega_s} \leq \sum_{s=1}^{2} \left\| u_s^{i,k} - u_{h,s}^{i,k} \right\|_{1,\Omega_s} + \sum_{s=1}^{2} \left\| u_{h,s}^{i,k} - u_{h,s}^{i,k,*} \right\|_{1,\Omega_s}.
\]

The second term on the right hand side of (4.8) is bounded of (3.15) as in (3.10) by

\[
\sum_{s=1}^{2} \sum_{i=1}^{2} \left\| u_{h,s}^{i,k} - u_{h,s}^{i,k,*} \right\|_{1,\Omega_s} \leq C \sum_{s=1}^{2} \eta_s.
\]

To bound the first term on the right hand side of (4.8) we use the residual equation and apply the technique of the residual a posteriori error estimation [22], to get for \( v_h \in K_h \) and \( v_{h,s} \),

\[
\begin{align*}
& b^i (u_s^{i,k} - u_{h,s}^{i,k}, v_h) = b^i (u_s^{i,k} - u_{h,s}^{i,k}, v_h - v_{h,s}) \\
& \leq \sum_{T \in \Omega_s} \int_T \left( F^i (u_{h,s}^{i,k}) + u_{h,s}^{i,k-1} + \Delta u_{h,s}^{i,k,*} - \left( 1 + \lambda \alpha_{h,0}^{i,k} \right) u_{h,s}^{i,k} \right) (v_h - v_{h,s}) \, ds \\
& - \sum_{E \in \Omega_s} \int_E \left( \frac{\partial u_{h,s}^{i,k,*}}{\partial n_E} \right) (v_h - v_{h,s}) \, ds - \sum_{E \in \Gamma_s} \int_E \left( \frac{\partial u_{h,s}^{i,k,*}}{\partial n_E} \right) (v_h - v_{h,s}) \, ds' \\
& + \sum_{E \in \Omega_s} \int_T \left( F^i (u_s^{i,k}) - F^i (u_{h,s}^{i,k}) \right) (v_h - v_{h,s}) \, ds + \left( \frac{\partial u_{h,s}^{i,k,*}}{\partial n_s} \right) (v_h - v_{h,s}) \, ds',
\end{align*}
\]
where \( F^i (u_{i,h,s}^{i,k}) \) is any approximation of \( F^i (u_{i,s}^i) \). Therefore

\[
\sum_{s=1}^{2} b^i (u_s^i - u_{i,h,s}^{i,k}, v_s^i) \leq \sum_{s=1}^{2} \left\| F^i (u_{i,h,s}^{i,k}) + u_{i,k}^{i,k-1} + \lambda \Delta u_{i,h,s}^{i,k} - \left( 1 + \lambda a_{i,0}^{i,k} \right) u_{i,s}^i \right\|_{0,T} \left\| v_s^i - v_{i,h,s}^i \right\|_{0,T} \\
+ \sum_{s=1}^{2} \sum_{E \subset \Omega_s} \left\| \frac{\partial u_{i,k}^{i,k}}{\partial \eta_E} \right\|_{0,E} \left\| v_s^i - v_{i,h,s}^i \right\|_{0,E} + \sum_{s=1}^{2} \sum_{E \subset \Gamma_s} \left\| \frac{\partial u_{i,k}^{i,k}}{\partial \eta_s} \right\|_{0,E} \left\| v_s^i - v_{i,h,s}^i \right\|_{0,E} \\
+ \sum_{s=1}^{2} \sum_{1 \leq \tau \leq \Omega_s} c \left\| u_{i,k}^{i,k} - u_{i,h,s}^{i,k} \right\|_{0,T} \left\| v_s^i - v_{i,h,s}^i \right\|_{0,T} + \sum_{s=1}^{2} \sum_{1 \leq \tau \leq \Omega_s} \left\| \frac{\partial u_{i,k}^{i,k}}{\partial \eta_s} \right\|_{0,T} \left\| v_s^i - v_{i,h,s}^i \right\|_{0,T}.
\]

Using the following fact

\[
\left\| v_s^i - u_{i,h,s}^{i,k} \right\|_{1,\Omega_s} \leq \sup_{v_s^i \in K} \frac{b^i (u_s^i - u_{i,h,s}^{i,k}, v_s^i + ch_T^i)}{||v_s^i + ch_T^i||_{1,\Omega_s}},
\]

we get

\[
\sum_{s=1}^{2} b^i (u_s^i - u_{i,h,s}^{i,k}, v_s^i + ch_T^i) \leq \sum_{s=1}^{2} \left( \sum_{1 \leq \tau \leq \Omega_s} \eta_s^i \right) \sum_{s=1}^{2} \left\| v_s^i \right\|_{1,\Omega_s}.
\]

Finally, by combining (4.8), (4.9) and (4.10) the required result follows. \( \square \)

5. An asymptotic behavior for the problem

5.1. A fixed point mapping associated with discrete problem. We define for \( i = 1, 2, 3, 4 \) the following mapping

\[
T_h : K_h \rightarrow H_0^1 (\Omega_s)
\]

(5.1)

\[
W_s \rightarrow TW_s = \xi_{h,s}^{i,m+1} = \partial_h (F(w_s)),
\]

where \( \xi_{h,s}^i \) is the solution of the following problem

\[
b^i (\xi_{s,h}^{i,k,m+1}, v_s^i - \xi_{s,h}^{i,k,m+1}) \left( \alpha_{s,h} \xi_{s,h}^{i,k,m+1} + \xi_{s,h}^{i,k,m+1} - \xi_{s,h}^{i,k,m+1} \right)_{\Gamma_s} \\
\geq \left( F(w_s), v_s,h - \xi_{s,h}^{i,k,m+1} \right)_{\Omega_s},
\]

(5.2)

\[
\xi_{s,h}^{i,k,m+1} = 0, \text{ on } \partial \Omega_s \cap \partial \Omega,
\]

\[
\frac{\partial \xi_{s,h}^{i,k,m+1}}{\partial \eta_s} + \alpha_s \xi_{s,h}^{i,k,m+1} = \frac{\partial \xi_{s,h}^{i,k,m}}{\partial \eta_s} + \alpha_s \xi_{s,h}^{i,k,m}, \text{ on } \Gamma_s, \ s = 1, ..., 4, \ t = 1, 2.
\]
5.2. **An iterative discrete algorithm.** Choosing \( u_{h,s}^{i,0} = u_{h,0,s}^i \in \left( H_0^1(\Omega_s) \cap C(\Omega_s) \right)^M, i = 1, \ldots, M \), the solution of the following discrete equation

\[
\Delta u_{h,s}^{i,0} + a_0^i u_{h,s}^{i,0} = g_i^0,
\]

where \( g_i^0 \) is a regular function.

Now we give the following discrete algorithm

\[
u_{s,h}^{i,k,m+1} = T_h u_{s,h}^{i,k-1,m+1}, k = 1, \ldots, n, i = 1, \ldots, M, s = 1, \ldots, 4,
\]

where \( u_{h,s}^{i,k} \) is the solution of the problem (5.2).

**Proposition 3.** Let \( \xi_{h,s}^{i,k} \) be a solution of the problem (5.2) with the right hand side \( F^i(w_s^i) \) and the boundary condition \( \frac{\partial \xi_{h,s}^{i,k,m+1}}{\partial \eta_i} + a_s \xi_{h,s}^{i,k,m+1} = T_h \xi_{h,s}^{i,k,m} \), \( \xi_{h,s}^{i,k} \) the solution for \( F^i(\tilde{w}_s^i) \) and \( \frac{\partial \tilde{\xi}_{h,s}^{i,k,m+1}}{\partial \eta_i} + a_s \tilde{\xi}_{h,s}^{i,k,m+1} = T_h \tilde{\xi}_{h,s}^{i,k,m} \). The mapping \( T_h \) is a contraction in \( K_h \) with the rate of contraction \( \frac{\lambda + c}{\beta + \lambda} \). Therefore, \( T_h \) admits a unique fixed point which coincides with the solution of the problem (5.2).

**Proof.** We note that

\[
\phi = \frac{1}{\beta + \lambda} \left\| F^i(w_s^i) - F^i(\tilde{w}_s^i) \right\|_1.
\]

Setting

\[
\xi_{h,s}^{i,k,m+1} = \phi, v_{h,s}^i, \phi = \left( F^i(w_s^i) + a_0^i \phi, v_{h,s}^i + \phi \right),
\]

\[
\xi_{h,s}^{i,k,m+1} = 0, \text{ on } \partial \Omega_s \cap \partial \Omega,
\]

\[
\frac{\partial \xi_{h,s}^{i,k,m+1}}{\partial \eta_i} + a_s \xi_{h,s}^{i,k,m+1} = \frac{\partial \tilde{\xi}_{h,s}^{i,k,m}}{\partial \eta_i} + a_s \tilde{\xi}_{h,s}^{i,k,m} \text{, on } \Gamma_s, s = 1, \ldots, 4, t = 1, 2.
\]

From assumption (1.3), we have

\[
F^i(w_s^i) \leq F^i(\tilde{w}_s^i) + \left\| F^i(w_s^i) - F^i(\tilde{w}_s^i) \right\|_1
\]

\[
\leq F^i(\tilde{w}_s^i) + \frac{a_0^i}{\beta + \lambda} \left\| F^i(w_s^i) - F^i(\tilde{w}_s^i) \right\|_1
\]

\[
\leq F^i(\tilde{w}_s^i) + a_0^i \phi.
\]

We know by [29], if \( F^i(w_s^i) \geq F^i(\tilde{w}_s^i) \) then \( \xi_{h,s}^{i,k,m+1} \geq \tilde{\xi}_{h,s}^{i,k,m+1} \). Thus

\[
\xi_{h,s}^{i,k,m+1} \leq T_h \xi_{h,s}^{i,k,m} + \phi.
\]

But the role of \( w_s^i \) and \( \tilde{w}_s^i \) are symmetrical, thus we have the similar prof

\[
\tilde{\xi}_{h,s}^{i,k,m+1} \leq T_h \tilde{\xi}_{h,s}^{i,k,m} + \phi.
\]
Proposition 4. Under the previous hypotheses and notations, we have the following estimate of convergent
\[
\|T(w_i^k) - T(\tilde{w}_i)\|_{\infty} \leq \frac{1}{\beta + \lambda} \|F(w_i^k) - F(\tilde{w}_i)\|_1
\]
\[
= \frac{1}{\beta + \lambda} \|f^i(w_i^k) + \lambda w_i - f^i(\tilde{w}_i) - \lambda \tilde{w}_i\|_1
\]
\[
\leq \frac{\lambda + c}{\beta + \lambda} \|w_i - \tilde{w}_i\|_1.
\]

\]

Proof. The proof is similar to that in [6] which has been treated the evolutionary HJB equation with non linear source terms.

Theorem 4. Under the previous hypotheses, notations, we have
\[
\sum_{s=1}^{2} \left\| u^{i,n,m+1}_{k,s} - u^{i,\infty,m+1}_{k,s} \right\|_1 \leq C \left( \sum_{s=1}^{2} \sum_{T \in \mathcal{T}_h} (\eta_s^T + \eta_{\Gamma_s}) + \left( \frac{1 + c(\Delta t)}{1 + \beta(\Delta t)} \right)^n \right).
\]

Proof. Using Theorem 3 and Proposition 4, we get (5.5).

6. Numerical example

In this section we give a simple numerical example. Consider the following evolutionary HJB equation
\[
\begin{cases}
\max_{1 \leq i \leq 2} (\frac{\partial u^i}{\partial t} + A^i u^i - f^i) = 0, \text{ in } \Omega \times [0, T], \\
u(0, t) \text{ in } \Omega = 0,
\end{cases}
\]

where \(\Omega = [0,1] \times [0,T] = [0,1] \times [0,1]\) and
\[
A^1 u = \frac{\partial^2 u}{\partial x^2}, \quad A^2 u = \frac{\partial^2 u}{\partial x^2} + u, \quad f^1(u) = f^2(u) = \max(A^1 u, A^2 u).
\]

The exact solution of the problem is
\[
u(x, t) = (x^4 - x^5) \sin(10x) \cos(20\pi t).
\]

For the finite element approximation, we take uniform partition and linear conforming element. For the domain decomposition, we use the following decompositions \(\Omega_1 = [0,0.55], \Omega_2 = [0.45,1]\).

We compute the bilinear semi-implicit scheme combined with Galerkin solution in \(\Omega\) and we apply the generalized overlapping domain decomposition method to compute the bilinear sequences \(u^{i,k,m+1}_{k,s}, (s = 1, 2)\) to be able to look at the
behavior of the constant $C$, where the space steps $h = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$ and the time steps of discretization $\Delta t = \frac{1}{10}, \frac{1}{50}, \frac{1}{100}$.

We denote by $E_s = \left\| u_i^{k} - u_{i,h,s}^{k} \right\|_{1,\Omega_s}$, $T_1 = \left\| u_{i,h,1}^{k,m+1} - u_{i,h,2}^{k,m} \right\|_{W_h^1}$ and $T_2 = \left\| u_{i,h,2}^{k,m} - u_{i,h,1}^{k,m} \right\|_{W^2_h}$.

The generalized overlapping domain decomposition method, with $\alpha_1 = \alpha_2 = 0.55$, converges. The iterations have been stopped when the relative error between two subsequent iterates is less than $10^{-6}$, we get the following results:

$$\Delta t = \frac{1}{10}, \quad h = 1/10, 1/100, 1/1000$$

<table>
<thead>
<tr>
<th></th>
<th>$E_s$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5081043</td>
<td>0.264825</td>
<td>0.475905</td>
<td>8</td>
</tr>
<tr>
<td>$E_s$</td>
<td>0.6265874</td>
<td>0.3852017</td>
<td>0.3837247</td>
<td>14</td>
</tr>
<tr>
<td>$T_1$</td>
<td>0.9650827</td>
<td>0.573981</td>
<td>0.1286211</td>
<td>20</td>
</tr>
<tr>
<td>$T_2$</td>
<td>0.892843</td>
<td>0.6418371</td>
<td>0.9430526</td>
<td></td>
</tr>
</tbody>
</table>

Finally, we can deduce the result of the asymptotic behavior $A_s = \sum_{s=1}^{2} \left\| u_{i,n,m+1}^{h,s} - u^{i,\infty} \right\|_1$ for $\Delta t = 1/1000$, i.e., $n = 1000$ as the following result:

<table>
<thead>
<tr>
<th></th>
<th>$E_s$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.4759595</td>
<td>0.8496273</td>
<td>0.9482601</td>
<td>8</td>
</tr>
<tr>
<td>$E_s$</td>
<td>0.5083649</td>
<td>0.7892758</td>
<td>0.8542894</td>
<td>14</td>
</tr>
<tr>
<td>$T_1$</td>
<td>0.7592478</td>
<td>0.927307</td>
<td>0.9785809</td>
<td>20</td>
</tr>
<tr>
<td>$T_2$</td>
<td>0.8584208</td>
<td>0.855012</td>
<td>0.9438526</td>
<td></td>
</tr>
</tbody>
</table>

In the tables above also we see that the iteration number is roughly related to $h$ and $\Delta t$ and the order of convergence is a good agreement with our estimates (5.5). Under adequate assumption, we can prove that $u^{i,\infty} \leq u_{i,h,s}^{1000,m+1} + \sum_{s=1}^{2} \sum_{t \in T_h} (\eta_t^T + \eta_{T_s}) + 1$, where $c = \beta = 1$ without the assumption of the discret maximum principle (see [8]).
Conclusion

In this paper, a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the interfaces for the discrete solutions on subdomains of evolutionary HJB equation with nonlinear source terms are derived using the semi-implicit time scheme combined with a finite element spatial approximation. Also the techniques of the residual a posteriori error analysis are used. Then a result of asymptotic behavior in uniform norm is deduced using Bensoussan-Lions’ Algorithm. Furthermore the results of some numerical experiments are presented to support the theory. In the future work, the a posteriori error analysis for similar results will be obtained in the general case of more than two subdomains.

References


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