COUPLED FIXED POINT RESULTS FOR MIXED WEAKLY MONOTONE MAPPINGS IN PARTIALLY ORDERED G-METRIC SPACES

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Abstract. We introduce the concept of mixed weakly monotone mappings $f, g : X \times X \rightarrow X$ with respect to a map $H : X \times X \rightarrow X$ where $X$ is a partially ordered G-metric space. We establish $b$-coupled coincidence fixed point and $b$-common coupled fixed point theorems in such space. Some examples are given to illustrate our obtained results and as an application, we give an existence theorem of common solutions for a system of integral equations. The presented theorems generalize and extend several well known comparable results in the literature.

Keywords: $b$-coupled coincidence fixed point; $b$-common coupled fixed point; G-metric space; mixed weakly monotone maps.

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1. Introduction and Preliminaries

Fixed point theory has received much attention in metric spaces endowed with a partial ordering. Bhaskar and Lakshmikantham [4] introduced the concept of a coupled fixed point and the mixed monotone property. Furthermore, they proved some coupled fixed point theorems for mappings which satisfy the mixed monotone property and applied their theorems to problems of the existence of solution for a periodic boundary value problem. Subsequently, H. Aydi, B. Samet and C.Vetro introduced in [3] the concepts of $\tilde{w}$-compatible mappings, $b$-coupled coincidence point and $b$-common coupled fixed point and they established $b$-coupled coincidence and $b$-common coupled fixed point theorems.

Mustafa and Sims ([8], [9]) introduced a new concept of generalized metric spaces, called G-metric spaces. In such spaces every triplet of elements is assigned to a nonnegative real number. Based on the notion of G-metric spaces, Mustafa et al. [7] established fixed point theorems in G-metric spaces. Afterward, many fixed point results were proved in this space (see [1], [11], [2]).

The following definitions and results will be needed in the sequel.

Definition 1.1. ([8]). Let $X$ be a nonempty set. Suppose that a mapping $G : X \times X \times X \rightarrow \mathbb{R}_+$ satisfies:

$(G1) G(x, y, z) = 0$ if $x = y = z$;

$(G2) G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;

$(G3) G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

$(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables);

$(G5) G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then $G$ is called a $G$–metric on $X$ and $(X, G)$ is called a $G$–metric space.

Definition 1.2. ([8]). Let $X$ be a $G$–metric space and let $\{x_n\}$ be a sequence of points of $X$, a point $x \in X$ is said to be the limit of a sequence $\{x_n\}$ if $G(x, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow +\infty$ and sequence $\{x_n\}$ is said to be $G$–convergent to $x$.

From this definition, we obtain that if $x_n \rightarrow x$ in a $G$–metric space $X$, then for any $\varepsilon > 0$ there exists a positive integer $N$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

It has been shown in [8] that the $G$–metric induces a Hausdorff topology and the convergence described

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Lemma 1.2. Let $X$ be a $G$-metric space, a sequence $\{x_n\}$ is called $G$-Cauchy if for every $\varepsilon > 0$ there is a positive integer $N$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to +\infty$.

We next state the following lemmas.

Lemma 1.1. (8). If $X$ is a $G$-metric space, then the following are equivalent:

1. $\{x_n\}$ is $G$-convergent to $x$.
2. $G(x_n, x_n) \to 0$ as $n \to +\infty$.
3. $G(x_n, x_n) \to 0$ as $n \to +\infty$.
4. $G(x_m, x_n) \to 0$ as $n, m \to +\infty$.

Lemma 1.2. (8). If $X$ is a $G$-metric space, then the following are equivalent:

1. The sequence $\{x_n\}$ is $G$-Cauchy.
2. For every $\varepsilon > 0$, there exists a positive integer $N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m \geq N$.

Lemma 1.3. (8). If $X$ is a $G$-metric space then $G(x, y, z) \leq 2G(y, x, x)$ for all $x, y \in X$.

Definition 1.4. (8). Let $(X, G), (X', G')$ be two generalized metric spaces. A mapping $f : X \to X'$ is $G$-continuous at a point $x \in X$ if and only if it is $G$ sequentially continuous at $x$, that is, whenever $\{x_n\}$ is $G$-convergent to $x$, $\{f(x_n)\}$ is $G'$-convergent to $f(x)$.

Definition 1.5. (8). A $G$-metric space $X$ is said to be $G$-complete (or a complete $G$-metric space) if every $G$-Cauchy sequence in $X$ is convergent in $X$.

Definition 1.6. (8). Let $X$ be a $G$-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two $G$-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to $x$ and $y$, respectively, $\{F(x_n, y_n)\}$ is $G$-convergent to $F(x, y)$.

Definition 1.7. [4]. An element $(x, y) \in XX$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.8. [6]. An element $(x, y) \in X \times X$ is called

1. a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$, and $(gx, gy)$ is called coupled point of coincidence,
2. a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.9. [8] An element $(x, y) \in X \times X$ is called

1. a $b$-coupled coincidence point of mappings $F, G : X \times X \to X$ if $G(x, y) = F(x, y)$ and $G(y, x) = F(y, x)$, and $(G(x, y), G(y, x))$ is called $b$-coupled point of coincidence,
2. a $b$-common coupled fixed point of mappings $F, G : X \times X \to X$ if $x = G(x, y) = F(x, y)$ and $y = G(y, x) = F(y, x)$.

In [5], Madjid Eshaghi Gordji, Esmat Akbartabar, Yeol Je Cho, and Maryam Ramezani introduced the following definition.

Definition 1.10. [5] Let $(X, \leq)$ be a partially ordered set and $f ; g : X \times X \to X$ be mappings. We say that a pair $(f; g)$ has the mixed weakly monotone property on $X$ if, for any $x; y \in X$,

$$x \leq f(x; y); \ y \geq f(y; x)$$

and

$$x \leq g(x; y); \ y \geq g(y; x)$$

We have also

Definition 1.11. The mappings $F, G : X \times X \to X$ are called commutative if $F(G(x, y), G(y, x)) = G(F(x, y), F(y, x))$ for all $x, y \in X$.
2. Main results

Let $X$ be a non-empty set and $H : X \times X \to X$ be a given mapping. For every $x \in X$, we denote by $H^{-1}(x)$ the subset of $X \times X$ defined by:

$$H^{-1}(x) := \{(u, v) \in X \times X \text{ such that } H(u, v) = x\}.$$ 

We introduce the following concept

**Definition 2.1.** Let $(X, \leq)$ be a partially ordered set and $f, g, H : X \times X \to X$ be mappings such that

(*) for all $(x, y) \in X \times X$ there exists $(u, v) \in X \times X$ such that $H(u, v) = f(x, y)$ and $H(v, u) = f(y, x)$

and for all $(x', y') \in X \times X$ there exists $(u', v') \in X \times X$ such that $H(u', v') = f(x', y')$ and $H(v', u') = f(y', x')$.

We say that a pair $(f; g)$ has the mixed weakly monotone property on $X$ with respect to $H$ if, for any $x, y \in X$,

$$H(x, y) \leq f(x, y); \quad H(y, x) \geq f(y, x)$$

$$\implies \left\{ \begin{array}{ll}
    f(x, y) \leq g(u, v) \\
    f(y, x) \geq g(v, u)
  \end{array} \right.$$ 

for all $(u, v) \in X \times X$ such that $(u, v) \in H^{-1}(f(x, y))$ and $(v, u) \in H^{-1}(f(y, x))$

and

$$H(x, y) \leq g(x, y); \quad H(y, x) \geq g(y, x)$$

$$\implies \left\{ \begin{array}{ll}
    g(x, y) \leq f(u, v) \\
    g(y, x) \geq f(v, u)
  \end{array} \right.$$ 

for all $(u, v) \in X \times X$ such that $(u, v) \in H^{-1}(g(x, y))$ and $(v, u) \in H^{-1}(g(y, x))$

**Example 2.1.** Let $X = \mathbb{R}$ and $f, g, H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$f = \left\{ \begin{array}{ll}
    x - y & \text{if } x \geq y \\
    0 & \text{if } x < y
  \end{array} \right.; \quad g = \left\{ \begin{array}{ll}
    \frac{x - y}{4} & \text{if } x \geq y \\
    0 & \text{if } x < y
  \end{array} \right. \quad \text{and} \quad H = \left\{ \begin{array}{ll}
    \frac{x - y}{4} & \text{if } x \geq y \\
    0 & \text{if } x < y
  \end{array} \right.$$ 

We will show that the mappings $f$ and $g$ have the mixed weakly monotone property with respect to $H$. If $x = y$ then

$$f(x, y) = 0 = H(x, y) \quad \text{and} \quad f(y, x) = 0 = H(y, x).$$

If $x > y$ then

$$f(x, y) = x - y = H(4x, 4y) \quad \text{and} \quad f(y, x) = 0 = H(4y, 4x).$$

If $x < y$ then

$$f(x, y) = 0 = H(4x, 4y) \quad \text{and} \quad f(y, x) = y - x = H(4y, 4x).$$

Now, for $g$ and $H$ we have:

If $x = y$ then

$$g(x, y) = 0 = H(x, y) \quad \text{and} \quad g(y, x) = 0 = H(y, x).$$

If $x > y$ then

$$g(x, y) = \frac{x - y}{3} = H(\frac{4}{3}x, \frac{4}{3}y) \quad \text{and} \quad g(y, x) = 0 = H(\frac{4}{3}y, \frac{4}{3}x).$$

If $x < y$ then

$$g(x, y) = 0 = H(\frac{4}{3}x, \frac{4}{3}y) \quad \text{and} \quad g(y, x) = \frac{y - x}{3} = H(\frac{4}{3}y, \frac{4}{3}x).$$

So, it is clear that (*) is verified. We have:

**Part 1 :** $H(x, y) \leq f(x, y)$ and $H(y, x) \geq f(y, x)$ implies that $x - y \geq 0$. Let now $(u, v) \in \mathbb{R}^2$ such that $(u, v) \in H^{-1}(f(x, y))$ and $(v, u) \in H^{-1}(f(y, x))$, that is $H(u, v) = f(x, y)$ and $H(v, u) = f(y, x)$.

By the definition of $f$ we have $f(x, y) = x - y$ and then $H(u, v) = x - y$ and $H(v, u) = y - x$. Since $x - y \geq 0$, we distinguish the following two cases:

* case 1 :* If $x = y$ then by the definition of $H$ we have $u = v = x = y$ and then

$$f(x, y) = 0 \leq 0 = g(u, v)$$

and

$$f(y, x) = 0 \geq 0 = g(v, u).$$
• case 2: If \( x > y \) then by the definition of \( H \) we have \( u = 4x \) and \( v = 4y \) and then
\[
f(x, y) = x - y \leq \frac{4}{3}(x - y) = \frac{u - v}{3} = g(u, v)
\]
and
\[
f(y, x) = 0 \geq 0 = g(v, u)
\]

**Part II** \( H(x, y) \leq g(x, y) \) and \( H(y, x) \geq g(y, x) \) implies that \( x - y \geq 0 \). Let now \((u, v) \in \mathbb{R}^2 \) such that
\[
(u, v) \in H^{-1}(g(x, y)) \text{ and } (v, u) \in H^{-1}(g(y, x)),
\]
that is \( H(u, v) = g(x, y) \) and \( H(v, u) = g(y, x) \). We distinguish the following two cases:

• case 1: If \( x = y \) then by the definition of \( g \) we have \( g(x, y) = g(y, x) = 0 \) and then \( x = y = u = v \).

Clearly now
\[
g(x, y) = 0 \leq 0 = f(u, v)
\]
and
\[
g(y, x) = 0 \geq 0 = f(v, u)
\]

• case 2: If \( x > y \) then by the definition of \( g \) we have \( g(x, y) = \frac{2}{3}u \) and \( g(y, x) = 0 \) and then by the definition of \( H \) we have \( u = \frac{4}{3}x \) and \( v = \frac{4}{3}y \) and then
\[
g(x, y) = \frac{x - y}{3} \leq \frac{4}{3}(x - y) = u - v = f(u, v)
\]
and
\[
g(y, x) = 0 \geq 0 = f(v, u)
\]
Clearly \( f \) and \( g \) have the mixed weakly monotone property with respect to \( H \), but they don’t have the mixed weakly monotone property on \( \mathbb{R} \). Indeed, for any \((x, y)\) such that
\[
x < f(x, y) = x - y \quad \text{and} \quad y > f(y, x) = y - x
\]
we have
\[
f(x, y) = x - y > \frac{x - y}{3} = g(x - y, 0) = g(f(x, y), f(y, x))
\]

Like in [2], let \( \theta : [0, +\infty] \times [0, +\infty] \rightarrow [0, 1] \) which satisfies following condition: For any two sequences \( \{t_n\} \) and \( \{s_n\} \) of nonnegative real numbers,
\[
\theta(t_n, s_n) \rightarrow 1 \quad \Rightarrow \quad t_n, s_n \rightarrow 0.
\]
as \( n \rightarrow +\infty \). Now, we prove our first result.

**Theorem 2.1.** Let \((X, \leq)\) be a partially ordered set and \( G \) be a \( G \)-metric on \( X \) such that \((X, G)\) is complete. Let \( f, g, H : X \times X \rightarrow X \) be given mappings satisfying

(h1) for any \((x, y) \in X \times X \) and for any \((x', y') \in X \times X \), there exist \((u, v) \in X \times X \) and \((u', v') \in X \times X \) such that \( f(x, y) = H(u, v) \), \( f(x, y) = H(u, v) \), \( g(x, y) = H(u', v') \) and \( g(x', y') = H(v', u') \)

(h2) \( f \) and \( g \) have the mixed weakly monotone property on \( X \) with respect to \( H \)

(h3) \( f, g \) and \( H \) are continuous

(h4) \( f \) and \( H \) are commutative and \( g \) and \( H \) are commutative

(h5)
\[
G(f(x, y), f(u, v), g(w, z)) + G(f(y, x), f(v, u), g(z, w))
\]
\[
\leq \theta(G(H(x, y), H(u, v), H(w, z), G(H(y, x), H(v, u), H(z, w)))) \times (G(H(x, y), H(u, v), H(w, z)) + G(H(y, x), H(v, u), H(z, w)))
\]
and
\[
G(g(x, y), g(u, v), f(w, z)) + G(g(y, x), g(v, u), f(z, w))
\]
\[
\leq \theta(G(H(x, y), H(u, v), H(w, z), G(H(y, x), H(v, u), H(z, w)))) \times (G(H(x, y), H(u, v), H(w, z)) + G(H(y, x), H(v, u), H(z, w)))
\]

for all \( x, y, z, u, v, w \in X \) for which \( H(w, z) \leq H(u, v) \leq H(x, y) \) and \( H(y, x) \leq H(v, u) \leq H(z, w) \). If there exist \( x_0, y_0 \in X \) such that \( H(x_0, y_0) \leq f(x_0, y_0) \) and \( H(y_0, x_0) \geq f(y_0, x_0) \) or \( H(x_0, y_0) \leq g(x_0, y_0) \) and \( H(y_0, x_0) \geq g(y_0, x_0) \), then \( f, g \) and \( H \) have a \( b \)-coupled coincidence point \((x, y)\), that is, \( f(x, y) = g(x, y) = H(x, y) \) and \( f(y, x) = g(y, x) = H(y, x) \).
Proof 1. Let \( x_0, y_0 \in X \) such that \( H(x_0, y_0) \leq f(x_0, y_0) \) and \( H(y_0, x_0) \geq g(y_0, x_0) \). (The case \( H(x_0, y_0) \leq g(x_0, y_0) \) and \( H(y_0, x_0) \geq f(y_0, x_0) \) is similar). By (h1), there exists \((x_1, y_1) \in X \times X \) such that

\[
f(x_0, y_0) = H(x_1, y_1) \quad \text{and} \quad f(y_0, x_0) = H(y_1, x_1)
\]

Again by (h1), there exists \((x_2, y_2) \in X \times X \) such that

\[
g(x_1, y_1) = f(x_2, y_2) \quad \text{and} \quad g(y_1, x_1) = H(y_2, x_2).
\]

Continuing this process, we can construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) defined by

\[
\begin{align*}
H(x_{2n+1}, y_{2n+1}) &= f(x_{2n}, y_{2n}) \\
H(y_{2n+1}, x_{2n+1}) &= f(y_{2n}, x_{2n})
\end{align*}
\]

and

\[
\begin{align*}
H(x_{2n+2}, y_{2n+2}) &= g(x_{2n+1}, y_{2n+1}) \\
H(y_{2n+2}, x_{2n+2}) &= g(y_{2n+1}, x_{2n+1})
\end{align*}
\]

for all \( n \in \mathbb{N} \). By construction, we have \((x_1, y_1) \in H^{-1}(f(x_0, y_0)) \) and \((y_1, x_1) \in H^{-1}(f(y_0, x_0)) \). From (h2) we have

\[
H(x_0, y_0) \leq H(x_1, y_1) = f(x_0, y_0) \leq g(x_1, y_1) = H(x_2, y_2)
\]

and

\[
H(y_0, x_0) \geq H(y_1, x_1) = f(y_0, x_0) \geq g(y_1, x_1) = H(y_2, x_2)
\]

and by induction

\[
\begin{align*}
H(x_0, y_0) &\leq H(x_1, y_1) \leq \ldots \leq H(x_n, y_n) \\
H(y_0, x_0) &\geq H(y_1, x_1) \geq \ldots \geq H(y_n, x_n)
\end{align*}
\]

Let we denote, for all \( n \in \mathbb{N} \)

\[
\alpha_n = H(x_n, y_n) \quad \text{and} \quad \beta_n = H(y_n, x_n)
\]

If there exists \( n \in \mathbb{N}^* \) such that \( \alpha_n = \alpha_{n-1} \) and \( \beta_n = \beta_{n-1} \), then by construction of the sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) and by (1) with \( x = u = x_{2n} \), \( w = x_{2n-1} \), \( y = v = y_{2n} \) and \( z = y_{2n-1} \) and from the property G1, we have

\[
G(\alpha_{2n+1}, \alpha_{2n+1}, \alpha_{2n}) + G(\beta_{2n+1}, \beta_{2n+1}, \beta_{2n})
\]

\[
= G(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n}), g(x_{2n-1}, y_{2n-1}))
\]

\[
+ G(f(y_{2n}, x_{2n}), f(y_{2n}, x_{2n}), g(y_{2n-1}, x_{2n-1}))
\]

\[
\leq \theta(G(\alpha_{2n}, \alpha_{2n}, \alpha_{2n-1}), G(\beta_{2n}, \beta_{2n}, \beta_{2n-1}))
\]

\[
\times (G(\alpha_{2n}, \alpha_{2n}, \alpha_{2n-1}) + G(\beta_{2n}, \beta_{2n}, \beta_{2n-1})) = 0
\]

which implies that \( \alpha_{2n+1} = \alpha_{2n} \) and \( \beta_{2n+1} = \beta_{2n} \). Similarly with \( x = u = x_{2n+1} \), \( w = x_{2n} \), \( y = v = y_{2n+1} \) and \( z = y_{2n} \) and by (2), we obtain

\[
G(\alpha_{2n+2}, \alpha_{2n+2}, \alpha_{2n+1}) + G(\beta_{2n+2}, \beta_{2n+2}, \beta_{2n+1})
\]

\[
= G(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), f(x_{2n}, y_{2n}))
\]

\[
+ G(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), f(y_{2n}, x_{2n}))
\]

\[
\leq \theta(G(\alpha_{2n+1}, \alpha_{2n+1}, \alpha_{2n}), G(\beta_{2n+1}, \beta_{2n+1}, \beta_{2n}))
\]

\[
\times (G(\alpha_{2n+1}, \alpha_{2n+1}, \alpha_{2n}) + G(\beta_{2n+1}, \beta_{2n+1}, \beta_{2n})) = 0
\]

which implies that \( \alpha_{2n+2} = \alpha_{2n+1} \) and \( \beta_{2n+2} = \beta_{2n+1} \). This leads to \( \alpha_m = \alpha_{m-1} \) and \( \beta_m = \beta_{m-1} \) for every \( m \geq 2n \). This implies that \( \{\alpha_n\} \) and \( \{\beta_n\} \) are cauchy sequences. So from now on, we assume \( (\alpha_n, \beta_n) \neq (\alpha_n, \beta_n) \) for all \( n \in \mathbb{N} \), that is, we assume that either \( \alpha_{n+1} \neq \alpha_n \) or \( \beta_{n+1} \neq \beta_n \). For \( n \in \mathbb{N}^* \), let

\[
r_{2n-1} = G(\alpha_{2n}, \alpha_{2n}, \alpha_{2n-1}) + G(\beta_{2n}, \beta_{2n}, \beta_{2n-1})
\]

From property G2, we have \( r_{2n-1} > 0 \). Thanks to (3) and (4) and taking \( x = u = x_{2n} \), \( w = x_{2n-1} \), \( y = v = y_{2n} \) and \( z = y_{2n-1} \) in (1), we get

\[
r_{2n} = G(\alpha_{2n+1}, \alpha_{2n+1}, \alpha_{2n}) + G(\beta_{2n+1}, \beta_{2n+1}, \beta_{2n})
\]

\[
= G(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n}), g(x_{2n-1}, y_{2n-1}))
\]

\[
+ G(f(y_{2n}, x_{2n}), f(y_{2n}, x_{2n}), g(y_{2n}, x_{2n-1}))
\]

\[
\leq \theta(G(\alpha_{2n}, \alpha_{2n}, \alpha_{2n-1}), G(\beta_{2n}, \beta_{2n}, \beta_{2n-1}))r_{2n-1}
\]
Using $0 \leq \beta < 1$, we deduce then
\[ r_{2n} \leq r_{2n-1} \quad (6) \]
Similarly to this, one can find by (2) for $x = u = x_{2n+1}$, $w = x_{2n}$, $y = v = y_{2n+1}$ and $z = y_{2n}$ that
\[ r_{2n+1} \leq r_{2n} \quad (7) \]
Thus combining (6) together with (7) leads that for any $n \in \mathbb{N}$
\[ r_{n+1} \leq r_n \quad (8) \]
It follows that the sequence $\{r_n\}$ is monotonic decreasing. Hence, there exists $r \geq 0$ such that
\[ \lim_{n \to +\infty} r_n = r \quad (9) \]
Next, we claim that $r = 0$. Assume on contrary that $r > 0$, from (5), we have
\[ \frac{r_{2n}}{r_{2n-1}} \leq \theta(G(\alpha_{2n}, \alpha_{2n}, \alpha_{2n-1}), G(\beta_{2n}, \beta_{2n}, \beta_{2n-1})) < 1 \]
Letting $n \to +\infty$ in the above inequality, then thanks to (9), we obtain
\[ \lim_{n \to +\infty} \theta(G(\alpha_{2n}, \alpha_{2n}, \alpha_{2n-1}), G(\beta_{2n}, \beta_{2n}, \beta_{2n-1})) = 1 \]
and by definition of the function $\theta$, we obtain that
\[ \lim_{n \to +\infty} G(\alpha_{2n}, \alpha_{2n}, \alpha_{2n-1}) = 0 \quad \text{and} \quad \lim_{n \to +\infty} G(\beta_{2n}, \beta_{2n}, \beta_{2n-1}) = 0 \]
a contradiction. This implies that $r = 0$ and then
\[ \lim_{n \to +\infty} r_n = 0 \quad (10) \]
Now we prove that $\alpha_n$ and $\beta_n$ are Cauchy sequences in the $G$-metric space $(X, G)$. Following (3), it suffices to prove that $\alpha_{2n}$ and $\beta_{2n}$ are Cauchy sequences. On contrary, assume that at least one of $\alpha_{2n}$ or $\beta_{2n}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{l(k)\}$ such that for every natural number $k$
\[ m(k) > l(k) \geq k \]
and
\[ t_k = G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)}) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)}) \geq \varepsilon \quad (11) \]
Now, corresponding to $l(k)$ we choose $m(k)$ to be the smallest for which (11) holds. So
\[ G(\alpha_{2m(k)-2}, \alpha_{2m(k)-2}, \alpha_{2l(k)}) + G(\beta_{2m(k)-2}, \beta_{2m(k)-2}, \beta_{2l(k)}) < \varepsilon \quad (12) \]
Using (11), (12) and the rectangle inequality, we have
\[ \varepsilon \leq t_k \leq G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2m(k)-1}) + G(\alpha_{2m(k)-1}, \alpha_{2m(k)-1}, \alpha_{2m(k)-2}) + G(\alpha_{2m(k)-2}, \alpha_{2m(k)-2}, \alpha_{2m(k)-1}) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2m(k)-1}) + G(\beta_{2m(k)-1}, \beta_{2m(k)-1}, \beta_{2m(k)-2}) + G(\beta_{2m(k)-2}, \beta_{2m(k)-2}, \beta_{2l(k)}) \leq r_{2m(k)-1} + r_{2m(k)-2} + G(\alpha_{2m(k)-2}, \alpha_{2m(k)-2}, \alpha_{2l(k)}) + G(\beta_{2m(k)-2}, \beta_{2m(k)-2}, \beta_{2l(k)}) \leq r_{2m(k)-1} + r_{2m(k)-2} + \varepsilon \]
On taking limit as $k \to +\infty$ and using (10), we have
\[ \lim_{k \to +\infty} t_k = \varepsilon \quad (13) \]
Again, by the rectangle inequality and using that \( G(x, x, y) \leq 2G(x, y, y) \) for any \( x, y \in X \) and the property \( GA \), we get

\[
G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)}) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)}) \\
\leq G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)} - 1) + G(\alpha_{2l(k)}, \alpha_{2l(k)} - 1, \alpha_{2l(k)}) \\
+ G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)} - 1) + G(\beta_{2l(k)} - 1, \beta_{2l(k)} - 1, \beta_{2l(k)}) \\
\leq G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)} - 1) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)} - 1) \\
+ 2G(\alpha_{2l(k)}, \alpha_{2l(k)} - 1, \alpha_{2l(k)} - 1) + G(\beta_{2l(k)} - 1, \beta_{2l(k)} - 1, \beta_{2l(k)} - 1) \\
\leq G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)} - 1) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)} - 1) \\
+ 2r_{2l(k)} - 1.
\]

On taking limit as \( k \to +\infty \) and using (10) and (13), we have

\[
\lim_{k \to +\infty} G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)} - 1) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)} - 1) \geq \varepsilon
\]

(14)

On the other hand, the rectangle inequality gives also

\[
G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)} - 1) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)} - 1) \\
\leq G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)}) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)}) \\
+ G(\alpha_{2l(k)}, \alpha_{2l(k)} - 1, \alpha_{2l(k)} - 1) + G(\beta_{2l(k)}, \beta_{2l(k)} - 1, \beta_{2l(k)} - 1) \\
\leq G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)}) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)}) \\
+ 2G(\alpha_{2l(k)} - 1, \alpha_{2l(k)} - 1, \alpha_{2l(k)} - 1) + G(\beta_{2l(k)} - 1, \beta_{2l(k)} - 1, \beta_{2l(k)} - 1) \\
\leq G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)}) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)}) \\
+ 2r_{2m(k)} - 1.
\]

On taking limit as \( k \to +\infty \) and using (10) and (13), we have

\[
\lim_{k \to +\infty} G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)} - 1) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)} - 1) \leq \varepsilon
\]

(15)

Combining (14) to (15) gives

\[
\lim_{k \to +\infty} R_k \\
= \lim_{k \to +\infty} G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)} - 1) + G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)} - 1) \\
= \varepsilon
\]

(16)

We have also, by the rectangle inequality and the fact that \( G(x, x, y) \leq 2G(x, y, y) \) for any \( x, y \in X \)

\[
G(\alpha_{2m(k)} - 1, \alpha_{2m(k)} - 1, \alpha_{2l(k)} - 1) + G(\beta_{2m(k)} - 1, \beta_{2m(k)} - 1, \beta_{2l(k)} - 1) \\
\leq G(\alpha_{2m(k)} - 1, \alpha_{2m(k)} - 1, \alpha_{2l(k)}) + G(\alpha_{2m(k)} - 1, \alpha_{2m(k)} - 1, \alpha_{2l(k)} - 1) \\
+ G(\beta_{2m(k)} - 1, \beta_{2m(k)} - 1, \beta_{2l(k)}) + G(\beta_{2m(k)} - 1, \beta_{2m(k)} - 1, \beta_{2l(k)} - 1) \\
\leq 2G(\alpha_{2m(k)} - 1, \alpha_{2m(k)} - 1, \alpha_{2m(k)} - 1, \alpha_{2l(k)} - 1) + G(\alpha_{2m(k)} - 1, \alpha_{2m(k)} - 1, \alpha_{2l(k)} - 1) \\
+ 2G(\beta_{2m(k)} - 1, \beta_{2m(k)} - 1, \beta_{2m(k)} - 1, \beta_{2l(k)} - 1) + G(\beta_{2m(k)} - 1, \beta_{2m(k)} - 1, \beta_{2l(k)} - 1) \\
\leq G(\alpha_{2m(k)} - 1, \alpha_{2m(k)} - 1, \alpha_{2l(k)} - 1, \alpha_{2l(k)} - 1) + G(\beta_{2m(k)} - 1, \beta_{2m(k)} - 1, \beta_{2l(k)} - 1) \\
+ 2r_{2m(k)} - 1.
\]

On taking limit as \( k \to +\infty \) and using (10) and (13), we have

\[
\lim_{k \to +\infty} G(f(x_{2m(k)}), f(y_{2m(k)}), f(x_{2m(k)}), f(y_{2m(k)}), g(x_{2l(k)} - 1, y_{2l(k)} - 1)) \\
+ G(f(y_{2m(k)}), f(x_{2m(k)}), f(y_{2m(k)}), f(x_{2m(k)}), g(y_{2l(k)} - 1, x_{2l(k)} - 1)) \\
\geq \varepsilon
\]

(17)

We take now \( x = u = x_{2m(k)}, w = x_{2l(k)}, y = v = y_{2m(k)} \) and \( z = y_{2l(k)} - 1 \) in (1). Hence

\[
G(f(x_{2m(k)}), f(y_{2m(k)}), f(x_{2m(k)}), f(y_{2m(k)}), g(x_{2l(k)} - 1, y_{2l(k)} - 1)) \\
+ G(f(y_{2m(k)}), f(x_{2m(k)}), f(y_{2m(k)}), f(x_{2m(k)}), g(y_{2l(k)} - 1, x_{2l(k)} - 1)) \\
\leq \theta(G(\alpha_{2m(k)} - 1, \alpha_{2m(k)}, \alpha_{2l(k)} - 1), G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)} - 1)) \times R_k \\
< R_k
\]
Using (16), we obtain
\[
\lim_{k \to +\infty} G(f(x_{2m(k)}, y_{2m(k)}), f(x_{2m(k)}, y_{2m(k)}), g(x_{2l(k)-1}, y_{2l(k)-1})) \\
+ G(f(y_{2m(k)}, x_{2m(k)}), f(y_{2m(k)}, x_{2m(k)}), g(y_{2l(k)-1}, x_{2l(k)-1})) \\
\leq \varepsilon
\] (18)

Combining (17) to (18) yields
\[
\lim_{k \to +\infty} S_k \\
= \lim_{k \to +\infty} G(f(x_{2m(k)}, y_{2m(k)}), f(x_{2m(k)}, y_{2m(k)}), g(x_{2l(k)-1}, y_{2l(k)-1})) \\
+ G(f(y_{2m(k)}, x_{2m(k)}), f(y_{2m(k)}, x_{2m(k)}), g(y_{2l(k)-1}, x_{2l(k)-1})) \\
= \varepsilon
\] (19)

Therefore, since we are in the case \(\alpha_{2m(k)} \neq \alpha_{2l(k)-1}\) then writing
\[
\frac{S_k}{R_k} \leq \theta(G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)-1}), G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)-1})) < 1
\]
and using the fact that \(\varepsilon = \lim_{k \to +\infty} S_k = \lim_{k \to +\infty} R_k\) we get
\[
\lim_{k \to +\infty} \theta(G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)-1}), G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)-1})) = 1
\]

By the property of the function \(\theta\), we obtain
\[
\lim_{k \to +\infty} G(\alpha_{2m(k)}, \alpha_{2m(k)}, \alpha_{2l(k)-1}) = \lim_{k \to +\infty} G(\beta_{2m(k)}, \beta_{2m(k)}, \beta_{2l(k)-1}) = 0
\]
which is a contradiction with (16), and this since \(\varepsilon > 0\). We deduce then \(\{\alpha_n = H(x_n, y_n)\}\) and \(\{\beta_n = H(y_n, x_n)\}\) are Cauchy sequences in the complete \(G\)-metric space \((X, G)\). Then there exit \(x, y \in X\) such that
\[
\lim_{n \to +\infty} G(\alpha_n, \alpha_n, x) = \lim_{n \to +\infty} G(\alpha_n, x, x) = 0
\]
and
\[
\lim_{n \to +\infty} G(\beta_n, \beta_n, y) = \lim_{n \to +\infty} G(\beta_n, y, y) = 0
\]

Since \(H\) is continuous, then
\[
H(H(x_n, y_n), H(y_n, x_n)) \to H(x, y) \text{ as } n \to +\infty
\] (20)

and
\[
H(H(y_n, x_n), H(x_n, y_n)) \to H(y, x) \text{ as } n \to +\infty
\] (21)

Now, since \(f\) commute with \(H\), we have
\[
f(H(x_{2n}, y_{2n}), H(y_{2n}, x_{2n})) = H(f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n})) \\
= H(H(x_{2n+1}, y_{2n+1}), H(y_{2n+1}, x_{2n+1}))
\] (22)

and
\[
f(H(y_{2n}, x_{2n}), H(x_{2n}, y_{2n})) = H(f(y_{2n}, x_{2n}), f(x_{2n}, y_{2n})) \\
= H(H(y_{2n+1}, x_{2n+1}), H(x_{2n+1}, y_{2n+1}))
\] (23)

Now, by the continuity of \(f\), we obtain
\[
f(H(x_{2n}, y_{2n}), H(y_{2n}, x_{2n})) \to f(x, y) \text{ as } n \to +\infty
\] (24)

and
\[
f(H(y_{2n}, x_{2n}), H(x_{2n}, y_{2n})) \to f(y, x) \text{ as } n \to +\infty
\] (25)

Combining (20), (22) with (24), yields
\[
f(x, y) = H(x, y)
\]
and combining (21), (23) with (25), yields
\[
f(y, x) = H(y, x)
\]

Similarly, we prove that \(g(x, y) = H(x, y)\) and \(g(y, x) = H(y, x)\). Then we proved that \((x, y)\) is a \(b\)-coupled coincidence point of the mappings \(f, g\) and \(H\).
Now, we omit the continuity of \( f, g \) and \( H \) and the commutativity of the pairs \( (f, H) \) and \( (g, H) \), and we replace them by other conditions in order to find the same result. This will be the purpose of the next theorem:

We recall that, if \( (X, G) \) is a \( G \)-metric space, we endow the product set \( X \times X \) by the generalized metric \( G' \) defined by

\[
G'(x, y), (u, v), (w, z)) = G(x, u, w) + G(y, v, z), \quad \forall (x, y), (u, v), (w, z) \in X \times X.
\]

**Theorem 2.2.** Let \( (X, \preceq) \) be a partially ordered set and \( G \) be a \( G \)-metric on \( X \). Let \( f, g, H : X \times X \to X \) be given mappings satisfying

(h1) for any \( x, y \in X \times X \) and for any \( x', y' \in X \times X \), there exist \( u, v \in X \times X \) and \( u', v' \in X \times X \) such that \( f(x, y) = H(u, v), f(y, x) = H(v, u), g(x', y') = H(u', v') \) and \( g(y', x') = H(v', u') \)

(h2) \( f \) and \( g \) have the mixed weakly monotone property on \( X \) with respect to \( H \)

(h3) \( \{(H(x, y), H(y, x)), x, y \in X\} \) is a complete subspace of \( (X \times X, G') \)

(h4) \( X \) has the following property:

(i) if a non-decreasing sequence \( x_n \to x \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \),

(ii) if a non-increasing sequence \( y_n \to y \), then \( y_n \succeq y \) for all \( n \in \mathbb{N} \).

(h5) (1) and (2) holds for all \( x, y, z, u, v, w \in X \) for which \( H(w, z) \leq H(u, v) \leq H(x, y) \) and \( H(y, x) \leq H(v, u) \leq H(z, w) \). If there exist \( x_0, y_0 \in X \) such that \( H(x_0, y_0) \leq H(x, y_0) \) and \( H(y_0, x_0) \geq f(y_0, x_0) \) or \( H(x_0, y_0) \leq g(x_0, y_0) \) and \( H(y_0, x_0) \geq g(y_0, x_0) \), then \( f, g \) and \( H \) have a \( b \)-coupled coincidence point \((x, y)\), that is, \( f(x, y) = g(x, y) = H(x, y) \) and \( f(y, x) = g(y, x) = H(y, x) \).

**Proof.** We take the same sequences \( \{x_n\} \) and \( \{y_n\} \) as in the proof of Theorem 2.1. Since \( \{\alpha_n\} = \{H(x_n, y_n)\} \) and \( \{\beta_n\} = \{H(y_n, x_n)\} \) are cauchy sequences in the \( G \)-metric space \( (X, G) \). Then

\[
\lim_{n,m,l \to +\infty} G(\alpha_n, \alpha_m, \alpha_l) = 0
\]

and

\[
\lim_{n,m,l \to +\infty} G(\beta_n, \beta_m, \beta_l) = 0
\]

This implies that

\[
\lim_{n,m,l \to +\infty} G'(\alpha_n, \beta_n, (\alpha_m, \beta_m), (\alpha_l, \beta_l)) = 0
\]

Then \( \{(\alpha_n, \beta_n)\} \) is a cauchy sequence in the \( G' \)-metric space \( \{(H(x, y), H(y, x)), x, y \in X\} \) which is a complete subspace of \( (X \times X, G') \). Then there exit \( x, y \in X \) such that

\[
\lim_{n \to +\infty} G(\alpha_n, \alpha_n, H(x, y)) = \lim_{n \to +\infty} G(\alpha_n, H(x, y), H(x, y)) = 0
\]

and

\[
\lim_{n \to +\infty} G(\beta_n, \beta_n, H(y, x)) = \lim_{n \to +\infty} G(\beta_n, H(y, x), H(y, x)) = 0
\]

By the triangle inequality, (h4), (3), (4) and (1), we have

\[
G(f(x, y), f(x, y), H(x, y)) + G(f(y, x), f(y, x), H(y, x))
\]
\[
\leq G(f(x, y), f(x, y), H(x_2n, y_2n)) + G(H(x_2n, y_2n), H(x_2n, y_2n), H(x, y))
\]
\[
+ G(f(y, x), f(y, x), H(y_2n, x_2n)) + G(H(y_2n, x_2n), H(y_2n, x_2n), H(y, x))
\]
\[
\leq G(f(x, y), f(x, y), g(x_{2n-1}, y_{2n-1})) + G(H(x_2n, y_2n), H(x_2n, y_2n), H(x, y))
\]
\[
+ G(f(y, x), f(y, x), g(y_{2n-1}, x_{2n-1})) + G(H(y_2n, x_{2n}), H(y_2n, x_{2n}), H(y, x))
\]
\[
\leq \theta(G(\alpha_0, \alpha_0, \alpha_{2n-1}))(G(\beta, \beta, \beta_{2n-1})) + G(\beta_0, \beta_0, \alpha_{2n-1}) + G(\beta_0, \beta_0, \beta_{2n-1})
\]
\[
+ G(\alpha_0, \alpha_0, \alpha) + G(\beta_0, \beta_0, \beta)
\]

Letting \( n \to +\infty \) in the above inequality and using (26) and (27) we obtain that

\[
G(f(x, y), f(x, y), H(x, y)) + G(f(y, x), f(y, x), H(y, x)) = 0
\]
which implies that \( f(x, y) = H(x, y) \) and \( f(y, x) = H(y, x) \). Again, by the triangle inequality, (4), (3), (4) and (2), we have
\[
G(g(x, y), g(x, y), H(x, y)) + G(g(y, x), g(y, x), H(y, x)) \\
\leq G(g(x, y), g(x, y), H(x_{2n+1}, y_{2n+1})) + G(H(x_{2n+1}, y_{2n+1}), H(x, y)) \\
+ G(g(y, x), g(y, x), H(y_{2n+1}, x_{2n+1})) + G(H(y_{2n+1}, x_{2n+1}), H(y, x)) \\
\leq G(g(x, y), g(x, y), f(x_{2n}, y_{2n})) + G(H(x_{2n+1}, y_{2n+1}), H(x_{2n+1}, y_{2n+1}, H(x, y)) \\
+ G(g(y, x), g(y, x), f(y_{2n}, x_{2n})) + G(H(y_{2n+1}, x_{2n+1}), H(y_{2n+1}, x_{2n+1}, H(y, x)) \\
\leq \theta(G(\alpha_1, \alpha_2, \beta_1, \beta_2))(G(\alpha_1, \alpha_2) + G(\beta_1, \beta_2)) \\
+ G(\alpha_2, \alpha_3, \alpha_4, \beta) + G(\beta_1, \beta_2, \beta)
\]
Letting \( n \to +\infty \) in the above inequality and using (26) and (27) we obtain that
\[
G(g(x, y), g(x, y), H(x, y)) + G(g(y, x), g(y, x), H(y, x)) = 0
\]
which implies that \( g(x, y) = H(x, y) \) and \( g(y, x) = H(y, x) \). Then we proved that \((x, y)\) is a \( b\)-coupled coincidence point of the mappings \( f, g \) and \( H \).

As a consequence of Theorem 2.2, we give the following corollary.

**Corollary 2.1.** Let \((X, \leq)\) be a partially ordered set and \( G \) be a \( G \)-metric on \( X \). Let \( f, g : X \times X \to X \) and \( h : X \to X \) be mappings satisfying

(i) \( f(X \times X) \subset h(X) \) and \( g(X \times X) \subset h(X) \)

(ii) \( f \) and \( g \) have the mixed weakly monotone property on \( X \) with respect to \( h : (x, y) \in X \times X \mapsto h(x) \in X \)

(iii) \( h(X) \) is a complete subspace of \((X, G)\)

(iv) \( X \) has the following property:

(a) if a non-decreasing sequence \( x_n \to x \), then \( x_n \leq x \) for all \( n \in \mathbb{N} \),

(b) if a non-increasing sequence \( y_n \to y \), then \( y_n \geq y \) for all \( n \in \mathbb{N} \).

(v) \[
G(f(x, y), f(u, v), g(w, z)) + G(f(y, x), f(v, u), g(z, w)) \\
\leq G(h(x, h, h), G(h(y, h, h))(G(h(h, h, h) + G(h(h, h, h))))
\]
and
\[
G(g(x, y), g(u, v), f(w, z)) + G(g(y, x), g(v, u), f(z, w)) \\
\leq G(h(h, h, h), G(h(y, h, h))(G(h(h, h, h) + G(h(h, h, h))))
\]
for all \( x, y, z, u, v, w \in X \) for which \( h(w) \leq h(u) \leq h(x) \) and \( h(y) \leq h(v) \leq h(z) \). If there exist \( x_0, y_0 \in X \) such that \( h(x_0) \leq f(x_0, y_0) \) and \( h(y_0) \geq f(y_0, x_0) \) or \( h(x_0) \leq g(x_0, y_0) \) and \( h(y_0) \geq g(y_0, x_0) \), then \( f, g \) and \( h \) have a coupled coincidence point \((x, y)\), that is, \( f(x, y) = g(x, y) = h(x, y) \).

**Proof 3.** Consider the mapping \( H : X \times X \to X \) defined by
\[
H(x, y) = h(x, y), \quad \forall x, y \in X.
\]
We will check that all the hypotheses of Theorem 2.2 are satisfied.

Clearly (h2), (h4), (1) and (2) follow immediately. Let now \((x, y) \in X \times X\). Since \( f(X \times X) \subset g(X) \), there exists \( u \in X \) such that \( F(x, y) = hu = H(u, v) \) for any \( v \in X \) and similarly for \( g \). Then, (h1) is satisfied. For (h3). Let \( \{ x_n \} \) and \( \{ y_n \} \) be two sequences in \( X \) such that \( \{ (H(x_n, y_n), H(y_n, x_n)) \} \) is a Cauchy sequence in \( (X \times X, G') \). Then
\[
\lim_{n,m,l \to +\infty} G'(((H(x_n, y_n), H(y_n, x_n)), (H(x_m, y_m), H(y_m, x_m)), (H(x_l, y_l), H(y_l, x_l))) = 0
\]
that is,
\[
\lim_{n,m,l \to +\infty} G(H(x_n, y_n), H(x_m, y_m), H(x_l, y_l)) = 0
\]
and
\[
\lim_{n,m,l \to +\infty} G(H(y_n, x_n), H(y_m, x_m), H(y_l, x_l)) = 0
\]
This implies that \( \{ H(x_n, y_n) \} = \{ h_{x_n} \} \) and \( \{ H(y_n, x_n) \} = \{ h_{y_n} \} \) are Cauchy sequences in \( (h(X), G) \).
Since \( h(X) \) is complete, there exists \( x, y \in X \) such that \( h_{x_n} \to h(x) \) and \( h_{y_n} \to h(y) \), that is, \( H(x_n, y_n) \to H(x, y) \) and \( H(y_n, x_n) \to H(y, x) \). Therefore, \( (H(x_n, y_n), H(y_n, x_n)) \to (H(x, y), H(y, x)) \in (X \times \)
Then, \((H(x, y), H(y, x)) : x, y \in X\) is a complete subspace of \((X \times X, G')\). Then, \(f, g\) and \(H\) have a \(b\)-coupled coincidence point \((x, y) \in X \times X\), that is, \(f(x, y) = g(x, y) = H(x, y) = hx\) and \(f(y, x) = g(y, x) = H(y, x) = hy\). Thus, \((x, y)\) is a coupled coincidence point of \(f, g\) and \(h\).

**Example 2.2.** Let \(X = \mathbb{R}\) and let \(f, g : X \times X \to X\) and \(h : X \to X\) be defined as follows:

\[
f(x, y) = 3x - y - 1,\quad g(x, y) = 5x - 3y - 1\quad \text{and} \quad h(x) = 2x - 1
\]

We have that \(h(x) \leq f(x, y)\) and \(h(y) \geq f(y, x)\) implies that \(x \geq y\). Now for any \(x \geq y\), we can verify that

\[
f(x, y) = 3x - y - 1 \leq 9x - 7y - 1 = 5u - 3v - 1 = g(u, v)
\]

and

\[
f(y, x) = 3y - x - 1 \leq -7x + 9y - 1 = 5v - 3u - 1 = g(v, u)
\]

where \(u = \frac{3x}{2} - \frac{y}{2}\) and \(v = \frac{3y}{2} - \frac{x}{2}\).

Also, we have \(h(x) = g(x, y)\) and \(h(y) = g(y, x)\) implies that \(x \geq y\). Now for any \(x \geq y\), we can verify that

\[
g(x, y) = 5x - 3y - 1 \leq 9x - 7y - 1 = 3u - v - 1 = f(u, v)
\]

and

\[
g(y, x) = 5y - 3x - 1 \leq -7x + 9y - 1 = 3v - u - 1 = f(v, u)
\]

where \(u = \frac{5x}{2} - \frac{3y}{2}\) and \(v = \frac{5y}{2} - \frac{3x}{2}\). Then \(f\) and \(g\) have the mixed weakly monotone property with respect to \(H(x, y) = h(x)\).

**Remark 1.** If \(h : X \to X\) is the identity map, then \(f\) and \(g\) have the mixed weakly monotone property on \(X\) with respect to \(H : (x, y) \in X \times X \mapsto x \in X\) implies that \(f\) and \(g\) have the mixed weakly monotone property on \(X\) introduced in [5].

Now, we shall prove the existence and uniqueness theorem of a \(b\)-common fixed point: We consider the product space \(X \times X\) with following partial order: for all \((x, y), (u, v) \in X \times X\),

\[(x, y) \leq (u, v) \iff x \leq u; y \geq v\]

**Theorem 2.3.** In addition to the hypotheses of Theorem 2.1, suppose that for all \((x, y), (x^*, y^*) \in X \times X\), there exists \((u, v) \in X \times X\) such that \((f(u, v), f(v, u))\) is comparable with \((f(x, y), f(y, x))\) and \((f(x^*, y^*), f(y^*, x^*))\). Then \(f, g\) and \(H\) have a unique \(b\)-common coupled fixed point, that is there exists a unique \((x, y) \in X \times X\) such that \(x = H(x, y) = f(x, y) = g(x, y)\) and \(y = H(y, x) = f(y, x) = g(y, x)\).

**Proof 4.** From Theorem 2.1, the set of \(b\)-coupled coincidence points is not empty. We shall show that if \((x, y)\) and \((x^*, y^*) \in X \times X\) are \(b\)-coupled coincidence points, that is, if \(f(x, y) = g(x, y) = H(x, y), f(y, x) = g(y, x) = H(y, x)\), then

\[
H(x, y) = H(x^*, y^*)\quad \text{and} \quad H(y, x) = H(y^*, x^*)
\]

By assumption, there exists \((u, v) \in X \times X\) such that \((f(u, v), f(v, u))\) is comparable with \((f(x, y), f(y, x))\) and \((f(x^*, y^*), f(y^*, x^*))\). Without restriction to the generality, we can assume that

\[
(f(x, y), f(y, x)) \leq (f(u, v), f(v, u))
\]

and

\[
(f(x^*, y^*), f(y^*, x^*)) \leq (f(u, v), f(v, u))
\]

Put \((u_0, v_0) = (u, v)\) and choose \((u_1, v_1) \in X \times X\) such that \(H(u_1, v_1) = f(u_0, v_0)\) and \(H(v_1, u_1) = f(v_0, u_0)\). Then, similarly as in the proof of Theorem 2.1, we can inductively define sequences \((H(u_n, v_n))\) and \((H(v_n, u_n))\) in \(X\) by

\[
\begin{cases} H(u_{2n+1}, v_{2n+1}) = f(u_{2n}, v_{2n}) \\ H(v_{2n+1}, u_{2n+1}) = f(v_{2n}, u_{2n}) \end{cases} \quad \text{and} \quad \begin{cases} H(u_{2n+2}, v_{2n+2}) = f(u_{2n+1}, v_{2n+1}) \\ H(v_{2n+2}, u_{2n+2}) = f(v_{2n+1}, u_{2n+1}) \end{cases}
\]

for all \(n \in \mathbb{N}\). Further, set \((x, y) = (x_0, y_0)\) and \((x^*, y^*) = (x_0^*, y_0^*)\) and, in the same way, define the sequences \((H(x_n, y_n))\), \((H(y_n, x_n))\), \((H(x_0^*, y_0^*))\) and \((H(y_0^*, x_0^*))\). Since

\[
(f(x, y), f(y, x)) = (H(x_1, y_1), H(y_1, x_1)) = (H(x, y), H(y, x))
\]

\[
\leq (f(u, v), f(v, u)) = (H(u_1, v_1), H(v_1, u_1))
\]
then \( H(x, y) \leq H(u_1, v_1) \) and \( H(v_1, u_1) \leq H(y, x) \). Using that \( f \) and \( g \) have the mixed weakly monotone property on \( X \) with respect to \( H \), one can show easily that
\[
H(x, y) \leq H(u_n, v_n) \quad \text{and} \quad H(v_n, u_n) \leq H(y, x)
\]
for all \( n \in \mathbb{N} \). Thus, putting \( x = u = u_{2n}, v = v_{2n}, w = x \) and \( z = y \) in (1), and using that \( \theta(a, b) < 1 \) for all \( a, b \geq 0 \) and (31) we obtain
\[
G(H(u_{2n+1}, v_{2n+1}), H(u_{2n+1}, v_{2n+1}), H(x, y)) \\
+ G(H(v_{2n+1}, u_{2n+1}), H(v_{2n+1}, u_{2n+1}), H(y, x)) \\
= G(f(u_{2n}, v_{2n}), f(u_{2n}, v_{2n}), g(x, y)) \\
+ G(f(v_{2n}, u_{2n}), f(v_{2n}, u_{2n}), g(y, x)) \\
\leq G(H(u_{2n}, v_{2n}), H(u_{2n}, v_{2n}), H(x, y)) \\
+ G(H(v_{2n}, u_{2n}), H(v_{2n}, u_{2n}), H(y, x))
\]
(32)

Putting now \( x = x, y = y, u = w = u_{2n+1} \) and \( v = z = v_{2n+1} \) in (2), and using that \( \theta(a, b) < 1 \) for all \( a, b \geq 0 \) and (31) we obtain
\[
G(H(u_{2n+2}, v_{2n+2}), H(u_{2n+2}, v_{2n+2}), H(x, y)) \\
+ G(H(v_{2n+2}, u_{2n+2}), H(v_{2n+2}, u_{2n+2}), H(y, x)) \\
= G(g(u_{2n+1}, v_{2n+1}), g(u_{2n+1}, v_{2n+1}), f(x, y)) \\
+ G(g(v_{2n+1}, u_{2n+1}), g(v_{2n+1}, u_{2n+1}), f(y, x)) \\
\leq G(H(u_{2n+1}, v_{2n+1}), H(u_{2n+1}, v_{2n+1}), H(x, y)) \\
+ G(H(v_{2n+1}, u_{2n+1}), H(v_{2n+1}, u_{2n+1}), H(y, x))
\]
(33)

We combine (32) to (33) to remark that
\[
G(H(u_{n+1}, v_{n+1}), H(u_{n+1}, v_{n+1}), H(x, y)) \\
+ G(H(v_{n+1}, u_{n+1}), H(v_{n+1}, u_{n+1}), H(y, x)) \\
\leq G(H(u_n, v_n), H(u_n, v_n), H(x, y)) \\
+ G(H(v_n, u_n), H(v_n, u_n), H(y, x))
\]
for all \( n \in \mathbb{N} \). Let us denote
\[
\chi_n = G(H(u_n, v_n), H(u_n, v_n), H(x, y)) + G(H(v_n, u_n), H(v_n, u_n), H(y, x))
\]
The sequence \( \{\chi_n\} \) is non-increasing, so there exists \( r \geq 0 \) such that
\[
\chi_n \rightarrow r \quad \text{as} \quad n \rightarrow +\infty
\]
We know that
\[
\frac{\chi_{2n+1}}{\chi_{2n}} \leq \theta(G(H(u_{2n}, v_{2n}), H(u_{2n}, v_{2n}), H(x, y)), G(H(v_{2n}, u_{2n}), H(v_{2n}, u_{2n}), H(y, x))) < 1
\]
Letting \( n \rightarrow +\infty \) in the above inequality, then we obtain
\[
\lim_{n \rightarrow +\infty} \theta(G(H(u_{2n}, v_{2n}), H(u_{2n}, v_{2n}), H(x, y)), G(H(v_{2n}, u_{2n}), H(v_{2n}, u_{2n}), H(y, x))) = 1
\]
and this implies, by the property of the function \( \theta \), that
\[
\lim_{n \rightarrow +\infty} G(H(u_{2n}, v_{2n}), H(u_{2n}, v_{2n}), H(x, y)) = 0
\]
and
\[
\lim_{n \rightarrow +\infty} G(H(v_{2n}, u_{2n}), H(v_{2n}, u_{2n}), H(y, x)) = 0
\]
Hence
\[
\lim_{n \rightarrow +\infty} \chi_{2n} = 0
\]
We then write
\[
\lim_{n \rightarrow +\infty} \chi_n \\
\lim_{n \rightarrow +\infty} G(H(u_n, v_n), H(u_n, v_n), H(x, y)) + G(H(v_n, u_n), H(v_n, u_n), H(y, x)) \\
= 0
\]
(34)
The same idea yields
\[
\begin{align*}
\lim_{n \to +\infty} H(u_n, v_n) &= H(x^*, y^*) + G(H(u_n, v_n), H(x^*, y^*)), \\
\lim_{n \to +\infty} G(H(u_n, v_n), H(x^*, y^*)) + G(H(v_n, u_n), H(y^*, x^*)) &= 0
\end{align*}
\]

(34), (35) together with the fact that the limit is unique allows that
\[H(x, y) = H(x^*, y^*) \quad \text{and} \quad H(y, x) = H(y^*, x^*)\]

Let us denote
\[z = H(x, y) \quad \text{and} \quad w = H(y, x)\]

Now thanks to (30) and (34), we can write
\[
\begin{align*}
\lim_{n \to +\infty} f(u_{2n}, v_{2n}) &= H(x, y) = H(x^*, y^*) = z \\
\lim_{n \to +\infty} g(u_{2n+1}, v_{2n+1}) &= H(x, y) = H(x^*, y^*) = w
\end{align*}
\]

From the commutativity of \(f\) and \(H\), we have
\[f(H(u_{2n}, v_{2n})), H(u_{2n}, v_{2n})) = H(f(u_{2n}, v_{2n}), f(v_{2n}, u_{2n}))\]
and
\[f(H(v_{2n}, u_{2n})), H(v_{2n}, u_{2n}))) = H(f(v_{2n}, u_{2n}), f(u_{2n}, v_{2n}))\]

Letting \(n \to +\infty\) in the above equalities, using the continuity of \(f\) and \(g\) and (34), (35) and (36), we obtain
\[f(z, w) = H(z, w) \quad \text{and} \quad f(w, z) = H(w, z)\]

Similarly, we can prove also that
\[g(z, w) = H(z, w) \quad \text{and} \quad g(w, z) = H(w, z)\]

Then \((z, w)\) is a b-coupled coincidence point for \(f, g\) and \(H\). Since a b-coupled coincidence point for \(f, g\) and \(H\) is unique, then
\[H(x, y) = z = x \quad \text{and} \quad H(y, x) = w = y\]

This proves that \((x, y)\) is a b-common coupled fixed point of the mappings \(f, g\) and \(H\). Now our purpose is to check that such a point is unique. Suppose there is an other b-common coupled fixed point \((u, v)\), that is \(u = H(u, v) = f(u, v) = g(u, v)\) and \(v = H(v, u) = f(v, u) = g(v, u)\). Then \((u, v)\) is also a b-coupled coincidence point of \(f, g\) and \(H\) and then
\[H(u, v) = H(x, y) \quad \text{and} \quad H(v, u) = H(y, x)\]

Hence we get
\[u = H(u, v) = H(x, y) = x \quad \text{and} \quad v = H(v, u) = H(y, x) = y\]

which yields the uniqueness of the b-common coupled fixed point.

**Theorem 2.4.** Let \((X, \leq)\) be a partially ordered set and \(G\) be a \(G\)-metric on \(X\) such that \((X, G)\) is complete. Let \(f, g : X \times X \to X\) be given mappings satisfying
(i) \(f\) and \(g\) have the mixed weakly monotone property on \(X\)
(ii) \(X\) has the following property:
(a) if a non-decreasing sequence \(x_n \to x\), then \(x_n \leq x\) for all \(n \in \mathbb{N}\),
(b) if a non-increasing sequence \(y_n \to y\), then \(y_n \geq y\) for all \(n \in \mathbb{N}\).
(iii)
\[G(f(x, y), f(u, v), g(w, z)) + G(f(y, x), f(v, u), g(z, w)) \leq \theta(G(x, u, w), G(y, v, z))(G(x, u, w) + G(y, v, z))\]
and
\[G(g(x, y), g(u, v), f(w, z)) + G(g(y, x), g(v, u), f(z, w)) \leq \theta(G(x, u, w), G(y, v, z))(G(x, u, w) + G(y, v, z))\]
for all \(x, y, z, u, v, w \in X\) for which \(w \leq u \leq x \leq y \leq v \leq z\). If there exist \(x_0, y_0 \in X\) such that \(x_0 \leq f(x_0, y_0)\) and \(y_0 \geq f(y_0, x_0)\) or \(x_0 \leq g(x_0, y_0)\) and \(y_0 \geq g(y_0, x_0)\), then \(f\) and \(g\) have a coupled fixed point \((x, y)\). Furthermore, if \(y_0 \leq x_0\), then \(x = y\), that is, \(x = f(x, x) = g(x, x)\).
Proof 5. Following the proof of Theorem 2.2 and Corollary 2.1 with $H(x,y) = h(x) = x$ for all $x \in X$, we have only to show that $x = f(x,x)$. Since $y_0 \leq x_0$, we get

$$y \leq y_0 \leq \cdots \leq y_1 \leq y_0 \leq x_0 \leq x_1 \leq \cdots \leq x_n \leq x.$$ 

Thus, we have $y \leq x$. Suppose that $G(x,x,y) + G(y,y,x) > 0$. Using inequality (1), we have

$$G(x,x,y) + G(y,y,x)
= G(f(x,y),f(x,y),g(y,x)) + G(f(y,x),f(y,x),g(x,y))
\leq \theta(G(x,x,y),G(y,y,x))G(x,x,y) + G(y,y,x)$$

a contradiction, since $\theta(a,b) < 1$ for all $a,b \geq 0$. Then $G(x,x,y) + G(y,y,x) = 0$, which gives that $x = y = f(x,x) = g(x,x)$.

3. Example

Let $X = \mathbb{R}$ and $f, H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f = \begin{cases} \frac{x-y}{4} & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases} \quad \text{and} \quad H = \begin{cases} x-y & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$$

We will check that all the hypotheses of Theorem 2.1 are satisfied with $g = f$.

- Hypothesis (h1):
  - If $x = y$ then
    $$f(x,y) = 0 = H(x,y) \quad \text{and} \quad f(y,x) = 0 = H(y,x).$$
  - If $x > y$ then
    $$f(x,y) = \frac{x-y}{4} = H(4x,4y) \quad \text{and} \quad f(y,x) = 0 = H(4y,4x).$$
  - If $x < y$ then
    $$f(x,y) = 0 = H(4x,4y) \quad \text{and} \quad f(y,x) = \frac{y-x}{4} = H(4y,4x).$$

- Hypothesis (h2):
  - We have $H(x,y) \leq f(x,y)$ and $H(y,x) \geq f(y,x)$ implies that $x-y \leq 0$. Let now $(u,v) \in \mathbb{R}^2$ such that $(u,v) \in H^{-1}(f(x,y))$ and $(v,u) \in H^{-1}(f(y,x))$, that is $H(u,v) = f(x,y)$ and $H(v,u) = f(y,x)$.
  - By the definition of $f$ we obtain $H(u,v) = 0$ and $H(v,u) = y - x$. Since $x-y \leq 0$, we distinguish the following two cases:
    - case 1: If $x = y$ then by the definition of $H$ we have $u = v = x = y$ and then
      $$f(x,y) = 0 \leq 0 = f(u,v)$$
    and
    $$f(y,x) = 0 \geq 0 = f(v,u)$$
    - case 2: If $x < y$ then by the definition of $H$ we have $u = 4x$ and $v = 4y$ and then
      $$f(x,y) = 0 \leq 0 = f(4x,4y) = f(u,v)$$
    and
    $$f(y,x) = y - x \geq y - x = f(4y,4x) = f(v,u)$$

- Hypothesis (h3):
  - $f$ and $H$ are continuous.

- Hypothesis (h4):
  - If $x \leq y$ then
    $$f(H(x,y),H(y,x)) = f(0,0) = 0 = H(0,0) = H(f(x,y),f(y,x))$$
  - If $x > y$ then
    $$f(H(x,y),H(y,x)) = f(x-y,0) = \frac{x-y}{4} = H(\frac{x-y}{4},0) = H(f(x,y),f(y,x))$$
Then $f$ and $H$ are commutative.

- Hypothesis (h5):

Let $x, y, z, u, v, w \in \mathbb{R}$ be such that $w \leq u \leq x$ and $y \leq v \leq z$. We have

$$G(f(x, y), f(u, v), f(w, z))$$

$$= \frac{|x - y|}{4} - \frac{|u - v|}{4} + \frac{|x - y|}{4} - \frac{|w - z|}{4} + \frac{|u - v|}{4} - \frac{|w - z|}{4}$$

$$\leq \frac{1}{4}(|x - u| + |v - y| + |x - w| + |z - y| + |u - w| + |z - v|)$$

$$\leq \frac{1}{4}(G(x, u, w) + G(y, v, z))$$

Hence

$$G(f(x, y), f(u, v), f(w, z)) + G(f(y, x), f(v, u), f(z, w)) \leq \frac{1}{2}(G(x, u, w) + G(y, v, z))$$

Since now $H(0, 0) \leq f(0, 0)$ and $H(0, 0) \geq f(0, 0)$, all the required hypotheses of Theorem 2.1 and obviously hypotheses of Theorem 2.3 are satisfied. Consequently, $f$ and $H$ have a unique b-common coupled fixed point. In this example $f(0, 0) = H(0, 0) = 0$.

4. Application

In this section, we study the existence of common solutions of a system of integral equations using the results proved in Section 3.

Consider the following system of integral equations:

$$
\begin{align*}
    x(t) &= \int_0^T K_1(t, s, x(s), y(s))ds + a(t), \\
    y(t) &= \int_0^T K_2(t, s, x(s), y(s))ds + a(t), \\
    x(t) &= \int_0^T K_1(t, s, x(s), y(s))ds + a(t), \\
    y(t) &= \int_0^T K_2(t, s, x(s), y(s))ds + a(t),
\end{align*}
$$

(39)

where $t \in [0, T], T > 0$.

Let $X = C([0, T], \leq_X)$ be the set of continuous functions defined on $[0, T]$ endowed with the G-metric given by

$$G(x, y, z) = \sup_{t \in [0, T]}\{|x(t) - y(t)|, |x(t) - z(t)|, |y(t) - z(t)|\}$$

for all $x, y, z \in X$. $X$ can also be equipped with the partial order $\leq_X$ given by

$$\forall x, y \in X, x \leq_X y \iff x(t) = y(t), \forall t \in [0, T].$$

It is proved in [10] that $(X, \leq_X)$ verifies the property (h4) of Theorem 2.2.)

Now we will state the following theorem:

Theorem 4.1. Tex Suppose that the following hypotheses hold:

(i) $K_1, K_2 : [0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $a : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, (ii) for all $t, s \in [0, T]$ we have

$$K_1(t, s, x(t), y(t)) \leq K_2\left(t, s, \int_0^T K_1(s, \tau, x(\tau), y(\tau))d\tau + a(s), \int_0^T K_1(s, \tau, y(\tau), x(\tau))d\tau + a(s)\right)$$

and

$$K_2(t, s, x(t), y(t)) \leq K_1\left(t, s, \int_0^T K_2(s, \tau, x(\tau), y(\tau))d\tau + a(s), \int_0^T K_2(s, \tau, y(\tau), x(\tau))d\tau + a(s)\right)$$

(iii) there exists a continuous function $p : [0, T] \times [0, T] \rightarrow \mathbb{R}_+$ such that for all $1 \leq i, j \leq 2$

$$|K_i(t, s, x, y) - K_j(t, s, u, v)| \leq p(t, s)\ln\left(\frac{|x(s) - u(s)| + |y(s) - v(s)|}{2} + 1\right)$$

for all $t, s \in [0, T]$ and $x, y, u, v \in X$ such that $u \leq_X x$ and $y \leq_X y$. 

(iv) $\sup_{t \in [0, T]} p(t, s)ds \leq 1$.

(v) There exist $(\alpha, \beta) \in C([0, T]) \times C([0, T])$ such that

$$\alpha(t) \leq \int_0^T K_1(t, s, \alpha(s), \beta(s))ds + a(t)$$
and
\[ \beta(t) \geq \int_0^T K_1(t, s, \alpha(s), \beta(s)) ds + a(t) \]

Then the integral equations 39 have a solution \((x^*, y^*) \in X \times X\).

**Proof 6.** We need to define for \(t \in [0, T]\) the operator
\[
 f(x, y)(t) = \int_0^T K_1(t, s, x(s), y(s)) ds + a(t), \quad f(y, x)(t) = \int_0^T K_1(t, s, y(s), x(s)) ds + a(t)
\]

and
\[
 g(x, y)(t) = \int_0^T K_2(t, s, x(s), y(s)) ds + a(t), \quad g(y, x)(t) = \int_0^T K_2(t, s, y(s), x(s)) ds + a(t),
\]

By (ii), we have for all \(x, y \in X\)
\[
 f(x, y)(t) = \int_0^T K_1(t, s, x(s), y(s)) ds + a(t)
\]
\[
 \leq \int_0^T K_2 \left( t, s, \int_0^T K_1(t, s, x(s), y(s)) ds + a(t), \int_0^T K_1(t, s, y(s), x(s)) ds + a(t) \right) + a(t)
\]
\[
 \leq g \left( \int_0^T K_1(t, s, x(s), y(s)) ds + a(t), \int_0^T K_1(t, s, y(s), x(s)) ds + a(t) \right)(t)
\]
\[
 \leq g(f(x, y)(t), f(y, x)(t))(t)
\]

and similarly, we can prove that \(f(y, x)(t) \leq g(f(y, x)(t), f(x, y)(t))(t), g(x, y)(t) \leq f(g(x, y)(t), g(y, x)(t))(t)\)

and \(g(y, x)(t) \leq f(g(y, x)(t), g(x, y)(t))(t)\). So \(f\) and \(g\) have the mixed weakly monotone property. Moreover, we have
\[
 G(f(x, y), f(u, v), g(w, z)) = \sup_{t \in [0, T]} \{ |f(x, y)(t) - f(u, v)(t)|, |f(u, v)(t) - g(w, z)(t)|, |f(x, y)(t) - g(w, z)(t)| \}
\]

By (iii) and (iv) we obtain, for all \(x, y, u, v \in C([0, T])\) with \(u \leq_X x\) and \(y \leq_X y\)
\[
 |f(x, y)(t) - f(u, v)(t)| \leq \int_0^T |K_1(t, s, x(s), y(s)) - K_1(t, s, u(s), v(s))| ds
\]
\[
 \leq \int_0^T p(t, s) \ln \left( \frac{|x(s) - u(s)| + |y(s) - v(s)|}{2} + 1 \right) ds
\]
\[
 \leq \sup_{s \in [0, T]} \left( \ln \left( \frac{|x(s) - u(s)| + |y(s) - v(s)|}{2} + 1 \right) \right) \int_0^T p(t, s) ds
\]
\[
 \leq \sup_{s \in [0, T]} \ln \left( \frac{|x(s) - u(s)| + |y(s) - v(s)|}{2} + 1 \right)
\]

Then we have for all \(t \in [0, T]\),
\[
 |f(x, y)(t) - f(u, v)(t)| \leq \ln \left( \frac{G(x, u, w) + G(y, v, z)}{2} + 1 \right)
\]

Similarly, we may show that for all \(t \in [0, T]\), we have for all \(u, v, w, z \in C([0, T])\) with \(w \leq_X w\) and \(v \leq_X v\)
\[
 |f(u, v)(t) - g(w, z)(t)| \leq \ln \left( \frac{G(x, u, w) + G(y, v, z)}{2} + 1 \right)
\]

and for all \(x, y, w, z \in C([0, T])\) with \(w \leq_X x\) and \(y \leq_X y\)
\[
 |g(x, y)(t) - g(w, z)(t)| \leq \ln \left( \frac{G(x, u, w) + G(y, v, z)}{2} + 1 \right)
\]

So, for all \(x, y, u, v, w, z \in C([0, T])\) with \(w \leq_X w\) and \(y \leq_X v \leq_X z\)
\[
 G(f(x, y), f(u, v), g(w, z)) \leq \ln \left( \frac{G(x, u, w) + G(y, v, z)}{2} + 1 \right)
\]
We can prove also that
\[ G(f(y, x), f(v, u), g(z, w)) \leq \ln \left( \frac{G(x, u, w) + G(y, v, z)}{2} + 1 \right) \]
and then
\[ G(f(x, y), f(u, v), g(w, z)) + G(f(y, x), f(v, u), g(z, w)) \leq 2 \ln \left( \frac{G(x, u, w) + G(y, v, z)}{2} + 1 \right) \]
Hence, if we define
\[
\theta(t_1, t_2) = \frac{\ln \left( \frac{t_1 + t_2}{2} + 1 \right)}{t_1 + t_2}
\]
for \( t_1 > 0, t_2 > 0 \) and \( \theta(0, 0) = r \in [0, 1] \) then we obtain
\[
G(f(x, y), f(u, v), g(w, z)) + G(f(y, x), f(v, u), g(z, w)) \leq \theta(G(x, u, w), G(y, v, z))G(x, u, w) + G(y, v, z)
\]
for all \( x, y, u, v, w, z \in C([0, T]) \) with \( w \leq_X u \leq_X x \) and \( y \leq_X v \leq_X z \). Hence (28) is verified. We omit the proof of (29) since we follow the same steps as given for (28). Now, let \( \alpha, \beta \in C([0, 1]) \) be the functions given by (v). Then, we have
\[ \alpha \leq f(\alpha, \beta) \quad \text{and} \quad \beta \geq f(\beta, \alpha) \]
Now, all the required hypotheses of Corollary 2.1 are satisfied. Then \( f \) and \( g \) admits a coupled common fixed point \((u^*, v^*)\) and hence we obtain the existence of a common solution to system (39)

References