General convolution identities for Apostol-Bernoulli, Euler and Genocchi polynomials

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Abstract

We perform a further investigation for the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. By making use of the generating function methods and summation transform techniques, we establish some general convolution identities for the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. These results are the corresponding extensions of some known formulas including the general convolution identities discovered by Dilcher and Vignat [K. Dilcher, C. Vignat, J. Math. Anal. Appl., 435 (2016), 1478–1498] on the classical Bernoulli and Euler polynomials.

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1. Introduction

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are usually defined by the following generating functions:

\[
\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi),
\]

\[
\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi),
\]

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and
\[
\frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < \pi).
\]

The rational numbers $B_n$, the integers $E_n$ and the rational numbers $G_n$ given by
\[
B_n = B_n(0), \quad E_n = 2^n E_n \left( \frac{1}{2} \right) \quad \text{and} \quad G_n = G_n(0)
\]
are called the classical Bernoulli numbers, the classical Euler numbers and the classical Genocchi numbers, respectively.

As is well-known, the classical Bernoulli, Euler and Genocchi polynomials and numbers play important roles in many different areas of mathematics such as number theory, combinatorics, special functions and mathematical analysis. Numerous interesting properties for them can be found in many books and papers; see, for example, [2, 3, 11, 13, 22, 23, 37]. The inspiration of the present paper stems from the general convolution identities for the classical Bernoulli and Euler polynomials due to Dilcher and Vignat [14] using identities for difference operators, techniques of symbolic computation and tools from the probability theory. We establish some general convolution identities for the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials by making use of the generating function methods and summation transform techniques. These results are the corresponding extensions of some known formulas including the general convolution identities on the classical Bernoulli and Euler polynomials due to Dilcher and Vignat [14] and the convolution identities for the classical Genocchi polynomials to Agoh [4].

We now turn to the Apostol-Bernoulli polynomials $B_n(x; \lambda)$, the Apostol-Euler polynomials $E_n(x; \lambda)$ and the Apostol-Genocchi polynomials $G_n(x; \lambda)$, which are usually defined by means of the following generating functions (see, e.g., [25, 27, 29]):
\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x; \lambda) \frac{t^n}{n!} \quad (|t + \log \lambda| < 2\pi), \quad (1.1)
\]
\[
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x; \lambda) \frac{t^n}{n!} \quad (|t + \log \lambda| < \pi), \quad (1.2)
\]
and
\[
\frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x; \lambda) \frac{t^n}{n!} \quad (|t + \log \lambda| < \pi). \quad (1.3)
\]

In particular, $B_n(\lambda)$, $E_n(\lambda)$ and $G_n(\lambda)$ given by
\[
B_n(\lambda) = B_n(0; \lambda), \quad E_n(\lambda) = 2^n E_n \left( \frac{1}{2}; \lambda \right) \quad \text{and} \quad G_n(\lambda) = G_n(0; \lambda)
\]
are called the Apostol-Bernoulli numbers, the Apostol-Euler numbers and the Apostol-Genocchi numbers, respectively. Clearly, $B_n(x; \lambda)$, $E_n(x; \lambda)$ and $G_n(x; \lambda)$ reduce to $B_n(x)$, $E_n(x)$ and $G_n(x)$ when $\lambda = 1$. It is worth mentioning that the Apostol-Bernoulli polynomials were firstly introduced by Apostol [6] (see also Srivastava [36] for a systematic study) in order to evaluate the value of the Hurwitz-Lerch zeta function. For some nice methods and results on these polynomials and numbers, one is referred to [7, 8, 26, 32, 33].

This paper is organized as follows. In the second section, we state some notation, recall the elementary and beautiful idea stemming from Euler to discover his famous pentagonal number theorem, and give some general convolution identities for the Apostol-Bernoulli polynomials. In the third section, we present some general convolution identities for the Apostol-Euler polynomials. The fourth section is contributed to the statement of some general convolution identities for the Apostol-Genocchi polynomials.

2. Convolution identities for Apostol-Bernoulli polynomials

For convenience, in the following we adopt the common notation described in the standard books [12, 38]. The rising factorial $(a)_k$ is defined for complex number $a$ and non-negative integer $k$ by
\[
(a)_k = \frac{e^{kt} - 1}{e^{t} - 1}.
\]
\[(a)_{0} = 1 \quad \text{and} \quad (a)_{k} = a(a+1) \cdots (a+k-1) \quad (k \geq 1).\]

The binomial coefficients \( \binom{a}{k} \) is defined for complex number \( a \) and non-negative integer \( k \) by
\[
\binom{a}{0} = 1 \quad \text{and} \quad \binom{a}{k} = \frac{a(a-1)(a-2) \cdots (a-k+1)}{k!} \quad (k \geq 1).
\]

The multinomial coefficient \( \binom{n}{r_{1}, \ldots, r_{k}} \) is defined for positive integer \( k \) and non-negative integers \( n, r_{1}, \ldots, r_{k} \) by
\[
\binom{n}{r_{1}, \ldots, r_{k}} = \frac{n!}{r_{1}! \cdots r_{k}!}.
\]

We also write, for a subset \( J \subseteq \{1, \ldots, k\} \) and complex numbers \( a_{1}, \ldots, a_{k} \), \( |J| \) as the cardinality of \( J \),
\[
a_{J} = \prod_{r \in J} a_{r} \quad \text{and} \quad J = \{1, \ldots, k\} \setminus J,
\]
and denote by \( [t_{1}^{j_{1}} \cdots t_{k}^{j_{k}}]f(t_{1}, \ldots, t_{k}) \) the coefficients of \( t_{1}^{j_{1}} \cdots t_{k}^{j_{k}} \) in \( f(t_{1}, \ldots, t_{k}) \) for positive integer \( k \) and non-negative integers \( j_{1}, \ldots, j_{k} \). Obviously,
\[
[t_{1}^{j_{1}} \cdots t_{k}^{j_{k}}]f(t_{1}, \ldots, t_{k}) = (j_{1}! \cdots j_{k}!)[t_{1}^{j_{1}} \cdots t_{k}^{j_{k}}]f(t_{1}, \ldots, t_{k}). \tag{2.1}
\]

We now recall Euler’s elementary and beautiful idea in the discovery of his famous pentagonal number theorem: for infinite number of complex numbers \( x_{1}, x_{2}, x_{3}, \ldots, \) (see, e.g., \[5, 9, 10\])
\[(1 + x_{1})(1 + x_{2})(1 + x_{3}) \cdots = (1 + x_{1}) + x_{2}(1 + x_{1}) + x_{3}(1 + x_{1})(1 + x_{2}) + \cdots. \tag{2.2}\]

We shall make use of the finite form of (2.2) to establish some general convolution identities for the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Clearly, the finite form of (2.2) can be written as
\[(1 + x_{1})(1 + x_{2})(1 + x_{3}) \cdots (1 + x_{n}) = (1 + x_{1}) + x_{2}(1 + x_{1}) + x_{3}(1 + x_{1})(1 + x_{2}) \]
\[\quad + \cdots + x_{n}(1 + x_{1})(1 + x_{2}) \cdots (1 + x_{n-1}), \quad (n \geq 1).\]  \tag{2.3}

We replace \( x_{r} \) by \( x_{r} - 1 \) for \( 1 \leq r \leq n \) in (2.3) to obtain
\[x_{1} \cdots x_{n} - 1 = \sum_{r=1}^{n} (x_{r} - 1)x_{1} \cdots x_{r-1}, \tag{2.4}\]
where the product \( x_{1} \cdots x_{r-1} \) is considered to be equal to 1 when \( r = 1 \). If we take \( x_{r} = \lambda_{r} e^{t_{r}} \) for \( 1 \leq r \leq n \) and substitute \( k \) for \( n \) in (2.4), then for positive integer \( k \),
\[\lambda_{1} \cdots \lambda_{k} e^{t_{1} + \cdots + t_{k}} - 1 = \sum_{r=1}^{k} (\lambda_{r} e^{t_{r}} - 1) \prod_{i=1}^{r-1} \lambda_{i} e^{t_{i}}. \tag{2.5}\]

It follows from (2.5) that
\[\prod_{i=1}^{k} \frac{t_{i} e^{x_{i}t_{i}}}{\lambda_{i} e^{t_{i}}} - 1 = \sum_{r=1}^{k} \frac{\lambda_{r} e^{t_{r}} - 1}{\lambda_{1} \cdots \lambda_{k} e^{t_{1} + \cdots + t_{k}} - 1} \prod_{i=1}^{r-1} \lambda_{i} e^{t_{i}} \prod_{i=r}^{k} \lambda_{i} e^{t_{i}} - 1}
\[= \sum_{r=1}^{k} \frac{t_{r} e^{t_{r}(x_{1} + \cdots + x_{r})}}{\lambda_{1} \cdots \lambda_{k} e^{t_{1} + \cdots + t_{k}} - 1} \prod_{i=1}^{r-1} \lambda_{i} e^{t_{i}} \prod_{i=r+1}^{k} \lambda_{i} e^{t_{i}} - 1}
\[= \sum_{r=1}^{k} \frac{t_{r} e^{(x_{r+1} - x_{r})t_{r}}}{\lambda_{1} \cdots \lambda_{k} e^{t_{1} + \cdots + t_{k}} - 1} \prod_{i=1}^{r-1} \lambda_{i} e^{t_{i}} \prod_{i=r+1}^{k} \lambda_{i} e^{t_{i}} - 1,\]
which means for non-negative integer \( n \) and complex numbers \( a_{1}, \ldots, a_{k} \),
\[
\sum_{j_1+\cdots+j_k=n+1 \atop j_1,\ldots,j_k \geq 0} \left( \begin{array}{c} -a_1 \\ j_1 \\ \vdots \\ -a_k \\ j_k \end{array} \right) \left( \sum_{r=1}^{k} t_r \prod_{i=1}^{k} \frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i} - 1} \right) \\
= \sum_{j_1+\cdots+j_k=n+1 \atop j_1,\ldots,j_k \geq 0} \left( \begin{array}{c} -a_1 \\ j_1 \\ \vdots \\ -a_k \\ j_k \end{array} \right) \left( \sum_{r=1}^{k} (t_1 + \cdots + t_k) e^{x_r (t_1 + \cdots + t_k)} / \lambda_1 \cdots \lambda_k e^{t_1 + \cdots + t_k} - 1 \right) \\
\times \prod_{i=1}^{r-1} \lambda_i t_i e^{x_i-x_i+1} t_i \prod_{i=r+1}^{k} \lambda_i e^{t_i} - 1)
\]

(2.6)

It follows from (1.1), (2.1), (2.6) and (2.10) that

\[
\sum_{j_1+\cdots+j_k=n+1 \atop j_1,\ldots,j_k \geq 0} \left( \begin{array}{c} -a_1 \\ j_1 \\ \vdots \\ -a_k \\ j_k \end{array} \right) \left( \sum_{r=1}^{k} (t_1 + \cdots + t_k) e^{x_r (t_1 + \cdots + t_k)} / \lambda_1 \cdots \lambda_k e^{t_1 + \cdots + t_k} - 1 \right) \\
= \sum_{j_1+\cdots+j_k=n+1 \atop j_1,\ldots,j_k \geq 0} \left( \begin{array}{c} -a_1 \\ j_1 \\ \vdots \\ -a_k \\ j_k \end{array} \right) \left( \sum_{r=1}^{k} \lambda_i t_i e^{x_i-t_i} \prod_{i=1}^{r-1} \lambda_i e^{t_i} - 1 \right) \\
\times \sum_{r=1}^{k} \sum_{l_r+1,\ldots,l_k \geq 0} \frac{B_{l_1+\cdots+l-r-1+(j_r-1)+l_r+1+\cdots+l_k}(x_r; \lambda_1 \cdots \lambda_k)}{l_1! \cdots l_{r-1}! \cdot (j_r - 1)! \cdot l_r + 1 \cdots l_k!}
\]

(2.7)

It is easily seen that for complex number \(a\) and non-negative integer \(k\),

\[
(k+1) \left( \begin{array}{c} -a \\ k + 1 \end{array} \right) = (-a-k) \left( \begin{array}{c} -a \\ k \end{array} \right) \quad \text{and} \quad (a)_k = (-1)^k k! \left( \begin{array}{c} -a \\ k \end{array} \right).
\]

(2.8)

(2.9)

On the other hand, since for positive integer \(k\) and non-negative integer \(N\) (see, e.g., [12, 38]),

\[
(t_1 + \cdots + t_k)^N = \sum_{l_1+\cdots+l_k=N \atop l_1,\ldots,l_k \geq 0} \left( \begin{array}{c} N \\ l_1, \ldots, l_k \end{array} \right) t_1^{l_1} \cdots t_k^{l_k},
\]

so by (1.1) and (2.9) we have

\[
\frac{(t_1 + \cdots + t_k) e^{x(t_1 + \cdots + t_k)}}{\lambda_1 \cdots \lambda_k e^{t_1 + \cdots + t_k} - 1} = \sum_{N=0}^{\infty} B_N(x; \lambda_1 \cdots \lambda_k) \sum_{l_1+\cdots+l_k=N \atop l_1,\ldots,l_k \geq 0} \frac{t_1^{l_1} \cdots t_k^{l_k}}{l_1! \cdots l_k!}.
\]

(2.10)

(2.11)
\[
\sum_{j_1 + \cdots + j_k = n + 1 \atop j_1, \ldots, j_k \geq 0} j_r \frac{(-a_r)}{j_1! \cdots j_k!} \left[ \frac{t_{j_1}^{l_1}}{j_1!} \cdots \frac{t_{j_k}^{l_k}}{j_k!} \right] \left( \sum_{r=1}^{k} t_r \prod_{i=1}^{k} \lambda_i e^{t_i} - 1 \right)
= \left( \frac{-1}{n!} \right)^{n+1} \sum_{r=1}^{k} a_r \sum_{l_1, \ldots, l_k = n \atop l_1, \ldots, l_k \geq 0} \left( \frac{n}{l_1, \ldots, l_k} \right) (a_1 + \cdots + a_k + n + 1 - l_r) \frac{\lambda_l B_{l_r} (x_r; \lambda_1 \cdots \lambda_k)}{\lambda_l B_{l_r} (x_r; \lambda_1 \cdots \lambda_k)}
\times \prod_{i=1}^{r-1} \lambda_i (a_i)_{l_i} B_{l_i} (x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^{k} (a_i)_{l_i} B_{l_i} (x_i - x_r; \lambda_i).
\]

Thus, by equations (2.8) and (2.14), we get the following result.

**Theorem 2.1.** Let \(a_1, \ldots, a_k\) be complex numbers with \(k\) being a positive integer. Then, for non-negative integer \(n\),

\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \frac{n}{j_1, \ldots, j_k} \right) \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_{n}} B_{j_1} (x_1; \lambda_1) \cdots B_{j_k} (x_k; \lambda_k)
= \sum_{r=1}^{k} \sum_{l_1, \ldots, l_k = n \atop l_1, \ldots, l_k \geq 0} \left( \frac{n}{l_1, \ldots, l_k} \right) \frac{a_r}{(a_1 + \cdots + a_k)_{n+1-l_r}} B_{l_r} (x_r; \lambda_1 \cdots \lambda_k)
\times \prod_{i=1}^{r-1} (a_i)_{l_i} \lambda_i B_{l_i} (x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^{k} (a_i)_{l_i} B_{l_i} (x_i - x_r; \lambda_i).
\]
It follows that we show some special cases of Theorem 2.1. Since the Apostol-Bernoulli polynomials satisfy the difference equation (see, e.g., [23]):

$$\lambda B_n(x + 1; \lambda) - B_n(x; \lambda) = nx^{n-1} \quad (n \geq 0),$$  \hfill (2.15)

so from (2.15) we have

$$\prod_{i=1}^{r-1} (a_i)_{l_i} B_{l_i}(x_i - x_r + 1; \lambda_i) = \sum_{J \subseteq \{1, \ldots, r-1\}} \prod_{i \in J} (a_i)_{l_i} B_{l_i}(x_i - x_r; \lambda_i) \prod_{i \in J^c} (a_i)_{l_i} B_{l_i}(x_i - x_r)^{l_i-1}. \hfill (2.16)$$

Thus, by applying (2.16) to Theorem 2.1 and then taking $x_1 = \cdots = x_k = x$, we get the general convolution identity for the Apostol-Bernoulli polynomials as follows.

**Corollary 2.2.** Let $a_1, \ldots, a_k$ be complex numbers with $k$ being a positive integer. Then, for non-negative integer $n$,

$$\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c} n \\ j_1, \ldots, j_k \end{array} \right) \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_n} B_{j_1}(x; \lambda_1) \cdots B_{j_k}(x; \lambda_k)$$

$$= \sum_{r=1}^{k} \sum_{|J|=r} \frac{n! \cdot a_j}{(n + 1 - r)!} \sum_{l_0, l_1, \ldots, l_{k-r} \geq 0} \left( \begin{array}{c} n + 1 - r \\ l_0, l_1, \ldots, l_{k-r} \end{array} \right) \frac{(a_{i_{r+1}})_{l_1} \cdots (a_{i_k})_{l_{k-r}}}{(a_1 + \cdots + a_k)_{n+1-l_0}}$$

$$\times B_{l_0}(x; \lambda_1 \cdots \lambda_k) B_{l_1}(\lambda_{i_{r+1}}) \cdots B_{l_{k-r}}(\lambda_{i_k}),$$

where $i_{r+1}, \ldots, i_k \in J$.

In particular, if we take $\lambda_1 = \cdots = \lambda_k = 1$ in Corollary 2.2 we get the following general convolution identity for the classical Bernoulli polynomials.

**Corollary 2.3.** Let $a_1, \ldots, a_k$ be complex numbers with $k$ being a positive integer. Then, for non-negative integer $n$,

$$\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c} n \\ j_1, \ldots, j_k \end{array} \right) \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_n} B_{j_1}(x) \cdots B_{j_k}(x)$$

$$= \sum_{r=1}^{k} \sum_{|J|=r} \frac{n! \cdot a_j}{(n + 1 - r)!} \sum_{l_0, l_1, \ldots, l_{k-r} \geq 0} \left( \begin{array}{c} n + 1 - r \\ l_0, l_1, \ldots, l_{k-r} \end{array} \right) \frac{(a_{i_{r+1}})_{l_1} \cdots (a_{i_k})_{l_{k-r}}}{(a_1 + \cdots + a_k)_{n+1-l_0}}$$

$$\times B_{l_0}(x) B_{l_1} \cdots B_{l_{k-r}},$$

where $i_{r+1}, \ldots, i_k \in J$.

The case $a_1, \ldots, a_k$ being positive real numbers in Corollary 2.3 is due to Dilcher and Vignat [14] Theorem 2], and leads to the corresponding higher-order convolution identity for the classical Bernoulli polynomials due to Agoh and Dilcher [4] Theorem 1] when $a_1 = \cdots = a_k = 1$. In the same way, if we take $a_1 = \cdots = a_k = 1$ in Corollary 2.2 we obtain the higher-order convolution identity for the Apostol-Bernoulli polynomials as follows.

**Corollary 2.4.** Let $k$ be a positive integer. Then, for non-negative integer $n$,

$$(n + k) \sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} B_{j_1}(x; \lambda_1) \cdots B_{j_k}(x; \lambda_k)$$

The case $a_1, \ldots, a_k$ being positive real numbers in Corollary 2.3 is due to Dilcher and Vignat [14] Theorem 2, and leads to the corresponding higher-order convolution identity for the classical Bernoulli polynomials due to Agoh and Dilcher [4] Theorem 1] when $a_1 = \cdots = a_k = 1$. In the same way, if we take $a_1 = \cdots = a_k = 1$ in Corollary 2.2 we obtain the higher-order convolution identity for the Apostol-Bernoulli polynomials as follows.
where \(i_1+\cdots+i_k \in J\).

**Remark 2.5.** Note that the corresponding higher-order convolution identity for the Apostol-Bernoulli polynomials stated in [18] Theorem 3.1] is only complete on condition that \(\lambda_1 = \cdots = \lambda_k\). The case \(k = 2\) in Corollary 2.3 can be easily used to give for positive integer \(n\) (see, e.g., [14]),

\[(n+2) \sum_{k=0}^{n} B_k(x)B_{n-k}(x) = 2 \sum_{k=0}^{n} \binom{n+2}{k+2} B_kB_{n-k}(x) + \binom{n+2}{3} B_{n-1}(x), \tag{2.17}\]

and for positive integer \(n \geq 2\),

\[\frac{n}{2} \sum_{k=1}^{n-1} \frac{B_k(x)B_{n-k}(x)}{k(n-k)} = \sum_{k=1}^{n} \binom{n}{k} \frac{B_kB_{n-k}(x)}{k} + \frac{n}{2} B_{n-1}(x) + H_{n-1}B_n(x), \tag{2.18}\]

where \(H_n\) is the \(n\)-th Harmonic numbers. And the case \(x = 0\) in (2.17) and (2.18) gives the famous identities of Matiyasevich [30] and Miki [31] for the classical Bernoulli numbers, respectively. For some equivalent versions and different proofs of (2.17) and (2.18), one is referred to [1 15 16 21 24 34]. For more applications of Corollary 2.3 see [14] for details.

### 3. Convolution identities for Apostol-Euler polynomials

We next apply (2.4) to establish some general convolution identities for the Apostol-Euler polynomials. By taking \(x_r = -\lambda_re^{r\xi}\) for \(1 \leq r \leq n\) and substituting \(k\) for \(n\) in (2.4), we get for positive integer \(k\),

\[(-1)^k \lambda_1 \cdots \lambda_k e^{t_1+\cdots+t_k} - 1 = \sum_{r=1}^{k} (-1)^r (\lambda_r e^{t_r} + 1) \prod_{i=1}^{r-1} \lambda_i e^{t_i}. \tag{3.1}\]

It follows from (3.1) that

\[\prod_{i=1}^{k} \frac{2e^{xt_i}}{\lambda_i e^{t_i}} + 1 = \sum_{r=1}^{k} (-1)^r (\lambda_1 \cdots \lambda_k e^{t_1+\cdots+t_k} - 1) \prod_{i=1}^{r-1} \lambda_i e^{t_i} \prod_{i=1}^{r} \frac{2e^{xt_i}}{\lambda_i e^{t_i}} + 1 \tag{3.2}\]

\[= \sum_{r=1}^{k} (-1)^r (\lambda_1 \cdots \lambda_k e^{t_1+\cdots+t_k} - 1) \prod_{i=1}^{r-1} \lambda_i e^{t_i} \prod_{i=r+1}^{k} \frac{2e^{(x_1-x_i)t_i}}{\lambda_i e^{t_i} + 1}. \]

We discuss (3.2) on two cases. We firstly consider the case \(k\) being an even integer. In this case, we get for non-negative integer \(n\) and complex numbers \(a_1, \ldots, a_k\),

\[
\sum_{j_1+\cdots+j_k=n+1} \binom{-a_1}{j_1} \cdots \binom{-a_k}{j_k} \left( \sum_{r=1}^{k} t_r \prod_{i=1}^{k} \frac{2e^{xt_i}}{\lambda_i e^{t_i} + 1} \right) \\
= 2 \sum_{j_1+\cdots+j_k=n+1} \binom{-a_1}{j_1} \cdots \binom{-a_k}{j_k} \left( \sum_{r=1}^{k} (-1)^r \frac{(t_1+\cdots+t_k)e^{(x_1+\cdots+t_k)}}{\lambda_1 \cdots \lambda_k e^{t_1+\cdots+t_k} - 1} \prod_{i=1}^{r-1} \lambda_i e^{t_i} + 1 \prod_{i=r+1}^{k} \frac{2e^{(x_1-x_i)t_i}}{\lambda_i e^{t_i} + 1} \right) \tag{3.3}
\]
If we make the operation \( \frac{t_{j_1}^{a_1}}{j_1!} \cdots \frac{t_{j_k}^{a_k}}{j_k!} \) in both sides of the above identity, in view of (1.2) and (2.7), the left hand side of (3.3) can be written as

\[
\sum_{j_1 + \cdots + j_k = n+1 \atop j_1, \ldots, j_k \geq 0} (-a_1) \cdots (-a_k) \left[ \frac{t_{j_1}^{a_1}}{j_1!} \cdots \frac{t_{j_k}^{a_k}}{j_k!} \right] \left( \sum_{r=1}^{k} t_r \prod_{i=1}^{k} \frac{2e^{x_t^i}}{\lambda_i e^{t_i} + 1} \right)
\]

\[
= \sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \sum_{r=1}^{k} (j_r + 1) \left[ \frac{(-a_r)}{j_r + 1} \prod_{i=1}^{k} \frac{(-a_i)}{j_i} \right] \left[ \frac{t_{j_1}^{a_1}}{j_1!} \cdots \frac{t_{j_k}^{a_k}}{j_k!} \right] \left( \prod_{i=1}^{k} \frac{2e^{x_t^i}}{\lambda_i e^{t_i} + 1} \right)
\]

\[
= \frac{(-1)^{n+1}(n + a_1 + \cdots + a_k)}{n!} \sum_{j_1, \ldots, j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c} n \\ j_1, \ldots, j_k \end{array} \right) (a_1)_{j_1} \cdots (a_k)_{j_k}
\times \mathcal{E}_{j_1}(x_1; \lambda_1) \cdots \mathcal{E}_{j_k}(x_k; \lambda_k),
\]

and in light of (1.2) and (2.10), the right hand side of (3.3) can be written as

\[
\sum_{j_1 + \cdots + j_k = n+1 \atop j_1, \ldots, j_k \geq 0} (-a_1) \cdots (-a_k) \left[ \frac{t_{j_1}^{a_1}}{j_1!} \cdots \frac{t_{j_k}^{a_k}}{j_k!} \right] \left( \sum_{r=1}^{k} t_r \prod_{i=1}^{k} \frac{2e^{x_t^i}}{\lambda_i e^{t_i} + 1} \right)
\]

\[
= 2 \sum_{j_1 + \cdots + j_k = n+1 \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c} -a_1 \\ j_1 \end{array} \right) \cdots \left( \begin{array}{c} -a_k \\ j_k \end{array} \right) \left( \sum_{r=1}^{k} \frac{(-1)^{r+1}}{l_1! \cdots l_r!} \left( \begin{array}{c} n+1 \\ l_1 + \cdots + l_k \end{array} \right) B_{l_r}(x_r; \lambda_1 \cdots \lambda_k) \times \sum_{l_1, \ldots, l_k \geq 0} \prod_{i=1}^{k} \left( \begin{array}{c} j_i \\ l_i \end{array} \right) \prod \mathcal{E}_{l_i}(x_i - x_r; \lambda_i) \right).
\]

It is easy to see from (2.12) and the famous Chu-Vandermonde convolution stated in [17] that for non-negative integers \( l_1, \ldots, l_k \) with \( l_1 + \cdots + l_k = n + 1 \),

\[
\sum_{j_1 + \cdots + j_k = n+1 \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c} -a_1 \\ j_1 \end{array} \right) \cdots \left( \begin{array}{c} -a_k \\ j_k \end{array} \right) \prod_{i=1}^{k} \left( \begin{array}{c} j_i \\ l_i \end{array} \right) = \prod_{i=1}^{k} \left( \begin{array}{c} l_i \\ j_i \end{array} \right) \frac{(-a_1 + \cdots + a_k) - (n + 1 - l_r)}{n - 1 - (n + 1 - l_r)}.
\]

Hence, applying (2.7) and (3.6) to (3.5) gives

\[
\sum_{j_1 + \cdots + j_k = n+1 \atop j_1, \ldots, j_k \geq 0} (-a_1) \cdots (-a_k) \left[ \frac{t_{j_1}^{a_1}}{j_1!} \cdots \frac{t_{j_k}^{a_k}}{j_k!} \right] \left( \sum_{r=1}^{k} t_r \prod_{i=1}^{k} \frac{2e^{x_t^i}}{\lambda_i e^{t_i} + 1} \right)
\]

\[
= 2 \frac{(-1)^{n+1}}{(n+1)!} \sum_{l_1, \ldots, l_k \geq 0} \left( \begin{array}{c} n+1 \\ l_1 + \cdots + l_k \end{array} \right) \prod_{i=1}^{k} \frac{\lambda_i (a_i)_{l_i}}{\mathcal{E}_{l_i}(x_i - x_r; \lambda_i) \prod_{i=1}^{k} (a_i)_{l_i}}
\times B_{l_r}(x_r; \lambda_1 \cdots \lambda_k)
\]

Thus, by equations (3.4) and (3.7), we obtain the following result.

**Theorem 3.1.** Let \( a_1, \ldots, a_k \) be complex numbers with \( k \) being an even positive integer. Then, for non-negative integer \( n \),

\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c} n \\ j_1, \ldots, j_k \end{array} \right) \prod_{i=1}^{k} \frac{(a_1)_{j_i} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_{n}} \mathcal{E}_{j_1}(x_1; \lambda_1) \cdots \mathcal{E}_{j_k}(x_k; \lambda_k)
\]
\[
\frac{2}{n+1} \sum_{r=1}^{k} (-1)^r \sum_{l_1 + \ldots + l_r = n+1 \atop l_1, \ldots, l_r \geq 0} \binom{n+1}{l_1, \ldots, l_k} \frac{B_{l_r}(x; \lambda_1 \ldots \lambda_k)}{(a_1 + \cdots + a_k)_{n+1-l_r}} \\
\times \prod_{i=1}^{r-1} (a_i)_{l_i} \lambda_i E_{l_i}(x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^{k} (a_i)_{l_i} E_{l_i}(x_i - x_r; \lambda_i).
\]

It follows that we give some special cases of Theorem 3.1. Since the Apostol-Euler polynomials satisfy the difference equation (see, e.g., [25])

\[
\lambda E_n(x + 1; \lambda) + E_n(x; \lambda) = 2x^n \quad (n \geq 0),
\]

so from (3.8) we have

\[
\prod_{i=1}^{r-1} \{-(a_i)_{l_i} \lambda_i E_{l_i}(x_i - x_r + 1; \lambda_i)\} = \sum_{J \subseteq \{1, \ldots, r-1\}} \prod_{j \in J} (a_i)_{l_i} E_{l_i}(x_i - x_r; \lambda_i) \prod_{i \notin J} \{-(2a_i)_{l_i} (x_i - x_r) \).
\]

Thus, applying (3.9) to Theorem 3.1 and then taking \(x_1 = \cdots = x_k = x\) gives the general convolution identity for the Apostol-Euler polynomials as follows.

**Corollary 3.2.** Let \(a_1, \ldots, a_k\) be complex numbers with \(k\) being an even positive integer. Then, for non-negative integer \(n\),

\[
\sum_{j_1 + \ldots + j_k = n \atop j_1, \ldots, j_k \geq 0} \binom{n}{j_1, \ldots, j_k} \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_n} E_{j_1}(x_1; \lambda_1) \cdots E_{j_k}(x; \lambda_k)
\]

\[
= \frac{1}{n+1} \sum_{r=1}^{k} \sum_{|J|=r} (-2)^r \sum_{l_0+l_1+\ldots+l_{k-r} = n+1 \atop l_0, l_1, \ldots, l_{k-r} \geq 0} \binom{n+1}{l_0, l_1, \ldots, l_{k-r}} \frac{(a_{i_r+1})_{l_i} \cdots (a_{i_k})_{l_{k-r}}}{(a_1 + \cdots + a_k)_{n+1-l_0}}
\]

\[
\times B_{l_0}(x; \lambda_1 \cdots \lambda_k) E_{l_1}(0; \lambda_{i_{r+1}}) \cdots E_{l_{k-r}}(0; \lambda_{i_k}),
\]

where \(i_{r+1}, \ldots, i_k \in J\).

If we take \(\lambda_1 = \cdots = \lambda_k = 1\) in Corollary 3.2, we get the following general convolution identity for the classical Euler polynomials.

**Corollary 3.3.** Let \(a_1, \ldots, a_k\) be complex numbers with \(k\) being an even positive integer. Then, for non-negative integer \(n\),

\[
\sum_{j_1 + \ldots + j_k = n \atop j_1, \ldots, j_k \geq 0} \binom{n}{j_1, \ldots, j_k} \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_n} E_{j_1}(x_1) \cdots E_{j_k}(x)
\]

\[
= \frac{1}{n+1} \sum_{r=1}^{k} \sum_{|J|=r} (-2)^r \sum_{l_0+l_1+\ldots+l_{k-r} = n+1 \atop l_0, l_1, \ldots, l_{k-r} \geq 0} \binom{n+1}{l_0, l_1, \ldots, l_{k-r}} \frac{(a_{i_r+1})_{l_i} \cdots (a_{i_k})_{l_{k-r}}}{(a_1 + \cdots + a_k)_{n+1-l_0}}
\]

\[
\times B_{l_0}(x) E_{l_1}(0) \cdots E_{l_{k-r}}(0),
\]

where \(i_{r+1}, \ldots, i_k \in J\).

If we take \(a_1 = \cdots = a_k = 1\) in Corollary 3.2, we get the following higher-order convolution identity for the Apostol-Euler polynomials.
Corollary 3.4. Let \( k \) be an even positive integer. Then, for non-negative integer \( n \),

\[
(n + k) \sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \mathcal{E}_{j_1}(x; \lambda_1) \cdots \mathcal{E}_{j_k}(x; \lambda_k)
\]

\[
= \sum_{r=1}^{k} \sum_{|J| = r} (-2)^r \sum_{l_0 + l_1 + \cdots + l_{k-r} = n+1 \atop l_0, l_1, \ldots, l_{k-r} \geq 0} \binom{n + k}{l_0} \mathcal{B}_{l_0}(x; \lambda_1 \cdots \lambda_k) \mathcal{E}_{l_1}(0; \lambda_{i_{r+1}}) \cdots \mathcal{E}_{l_{k-r}}(0; \lambda_{i_k}),
\]

where \( i_{r+1}, \ldots, i_k \in \mathcal{J} \).

We next consider the case \( k \) being an odd positive integer in (3.2). In this case, it is easily seen that for non-negative integer \( n \) and complex numbers \( a_1, \ldots, a_k \),

\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c}
-a_1 \\
1
\end{array} \right) \cdots \left( \begin{array}{c}
-a_k \\
1
\end{array} \right) \left( \prod_{i=1}^{k} \frac{2e^{a_i t_i}}{a_i e^{t_i} + 1} \right)
\]

\[
= \sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c}
-a_1 \\
1
\end{array} \right) \cdots \left( \begin{array}{c}
-a_k \\
1
\end{array} \right) \left( \sum_{r=1}^{k} (-1)^{r-1} \frac{2e^{x_i t_i} \lambda_{k+1}^{r-1}}{\lambda_1 \cdots \lambda_k e^{x_i t_i} + 1} + 1 \right)
\]

\[
\times \prod_{i=1}^{r-1} \lambda_i \frac{2e^{x_i t_i} \lambda_{k+1}^{r-1}}{\lambda_1 e^{x_i t_i} + 1} \prod_{i=r+1}^{k} \frac{2e^{x_i t_i} \lambda_{k+1}^{r-1}}{\lambda_1 e^{x_i t_i} + 1}.
\]

Notice that from (3.2) and (3.9) we have

\[
\frac{2e^{x_i t_i} \lambda_{k+1}^{r-1}}{\lambda_1 e^{x_i t_i} + 1} = \sum_{N=0}^{\infty} \mathcal{E}_N(x; \lambda_1 \cdots \lambda_k) \sum_{l_1 + \cdots + l_k = N \atop l_1, \ldots, l_k \geq 0} \frac{l_1^{j_1} \cdots l_k^{j_k}}{l_1! \cdots l_k!}.
\]

By making the operation \( \frac{j_1^{l_1}}{j_1} \cdots \frac{j_k^{l_k}}{j_k} \) in both sides of (3.10), in view of (3.2) and (3.11), we obtain

\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c}
-a_1 \\
1
\end{array} \right) \cdots \left( \begin{array}{c}
-a_k \\
1
\end{array} \right) \mathcal{E}_{j_1}(x_1; \lambda_1) \cdots \mathcal{E}_{j_k}(x_k; \lambda_k)
\]

\[
= \sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c}
-a_1 \\
1
\end{array} \right) \cdots \left( \begin{array}{c}
-a_k \\
1
\end{array} \right) \sum_{r=1}^{k} (-1)^{r-1} \sum_{l_1 + \cdots + l_k = n \atop l_1, \ldots, l_k \geq 0} \mathcal{E}_l(x_1; \lambda_1 \cdots \lambda_k)
\]

\[
\times \prod_{i=1}^{r-1} \frac{j_i}{l_i} \lambda_i \mathcal{E}_{l_i}(x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^{k} \frac{j_i}{l_i} \mathcal{E}_{l_i}(x_i - x_r; \lambda_i),
\]

which together with (3.7) and (3.6) yields the following result.

Theorem 3.5. Let \( a_1, \ldots, a_k \) be complex numbers with \( k \) being an odd positive integer. Then, for non-negative integer \( n \),

\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \frac{n}{(j_1 \cdots j_k)} \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_{n}} \mathcal{E}_{j_1}(x_1; \lambda_1) \cdots \mathcal{E}_{j_k}(x_k; \lambda_k)
\]
where we obtain the following higher-order convolution identity for the Apostol-Euler polynomials.

Corollary 3.6. Let \( a_1, \ldots, a_k \) be complex numbers with \( k \) being an odd positive integer. Then, for non-negative integer \( n \),

\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \binom{n}{j_1, \ldots, j_k} \frac{(a_1)_{j_1} \cdots (a_k)_{j_k} E_{j_1}(x; \lambda_1) \cdots E_{j_k}(x; \lambda_k)}{(a_1 + \cdots + a_k)_n}
\]

where \( i_r + 1, \ldots, i_k \in J \).

The case \( \lambda_1 = \cdots = \lambda_k = 1 \) in Corollary 3.6 gives the following general convolution identity for the classical Euler polynomials.

Corollary 3.7. Let \( a_1, \ldots, a_k \) be complex numbers with \( k \) being an odd positive integer. Then, for non-negative integer \( n \),

\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \binom{n}{j_1, \ldots, j_k} \frac{(a_1)_{j_1} \cdots (a_k)_{j_k} E_{j_1}(x) \cdots E_{j_k}(x)}{(a_1 + \cdots + a_k)_n}
\]

where \( i_r + 1, \ldots, i_k \in J \).

The case \( a_1, \ldots, a_k \) being positive real numbers in the above Corollaries 3.3 and 3.7 are due to Dilcher and Vignat [14, Theorem 4], which gives the corresponding higher-order convolution identity for the classical Euler polynomials due to Agoh and Dilcher [4, Theorems 2 and 3] when \( a_1 = \cdots = a_k = 1 \), respectively. In particular, the case \( k = 2 \) in Corollaries 3.3 and 3.7 will give some similar convolution identities for the classical Euler polynomials to (2.17) and (2.18) (see [14] for details). If we take \( a_1 = \cdots = a_k = 1 \) in Corollary 3.6, we obtain the following higher-order convolution identity for the Apostol-Euler polynomials.

Corollary 3.8. Let \( k \) be an odd positive integer. Then, for non-negative integer \( n \),

\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} E_{j_1}(x; \lambda_1) \cdots E_{j_k}(x; \lambda_k)
\]
so from (4.3) we have
\[
\sum_{r=1}^{k} \sum_{|j|=r} (-2)^{r-1} \sum_{l_0+l_1+\cdots+l_{k-r}=n} \left( \frac{n+k-1}{l_0} \right) e_{l_0}(x; \lambda_1 \cdots \lambda_k) e_{l_1}(0; \lambda_{i_1+1}) \cdots e_{l_{k-r}}(0; \lambda_{i_k}),
\]
where $i_{r+1}, \ldots, i_k \in \mathcal{J}$.

Remark 3.9. Note that the corresponding two higher-order convolution identities for the Apostol-Euler polynomials stated in [18, Theorem 3.2] are only complete on condition that $\lambda_1 = \cdots = \lambda_k$.

4. Convolution identities for Apostol-Genocchi polynomials

We finally apply (3.2) to establish some general convolution identities for the Apostol-Genocchi polynomials. By substituting $2t_i$ for $2$ in the left hand side of (3.2), we discover
\[
\prod_{i=1}^{k} \frac{2t_i e^{x t_i}}{\lambda_i e^{t_i} + 1} = \sum_{j_1+\cdots+j_k=n+1} \left( \frac{-a_1}{j_1} \right) \cdots \left( \frac{-a_k}{j_k} \right) \left( \sum_{r=1}^{k} t_r \prod_{i=1}^{k} \frac{2t_i e^{x t_i}}{\lambda_i e^{t_i} + 1} \right)
\]
\[
= 2 \sum_{j_1+\cdots+j_k=n+1} \left( \frac{-a_1}{j_1} \right) \cdots \left( \frac{-a_k}{j_k} \right) \left( \sum_{r=1}^{k} t_r \frac{(t_1+\cdots+t_k) e^{x t_1 + \cdots + t_k}}{\lambda_1 \cdots \lambda_k e^{t_1 + \cdots + t_k} - 1} \right)
\]
\[
\times \prod_{i=1}^{r-1} \lambda_i \frac{2t_i e^{x (x-r) t_i}}{\lambda_i e^{t_i} + 1} \prod_{i=r+1}^{k} \frac{2t_i e^{x (x-r) t_i}}{\lambda_i e^{t_i} + 1}.
\]

By making the operation $\left[ \frac{t_1^{j_1}}{j_1!} \cdots \frac{t_k^{j_k}}{j_k!} \right]$ in both sides of (4.2), in similar considerations to (2.8) and (2.14), we discover the following result.

Theorem 4.1. Let $a_1, \ldots, a_k$ be complex numbers with $k$ being an even positive integer. Then, for non-negative integer $n$,
\[
\sum_{j_1+\cdots+j_k=n} \left( \frac{n}{j_1, \ldots, j_k} \right) \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1+\cdots+a_k)_n} g_{j_1}(x_1; \lambda_1) \cdots g_{j_k}(x_k; \lambda_k)
\]
\[
= 2 \sum_{r=1}^{k} (-1)^r \sum_{l_1+\cdots+l_k=n} \left( \frac{n}{l_1, \ldots, l_k} \right) \frac{a_r}{(a_1+\cdots+a_k)_{n+1-l_r}} B_{l_r}(x; \lambda_1 \cdots \lambda_k)
\]
\[
\times \prod_{i=1}^{r-1} (a_i)_{l_i} g_{l_i}(x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^{k} (a_i)_{l_i} g_{l_i}(x_i - x_r; \lambda_i).
\]

It follows that we discuss some special cases of Theorem 4.1. Since the Apostol-Genocchi polynomials satisfy the difference equation (see, e.g., [25])
\[
\lambda g_n(x+1; \lambda) + g_n(x; \lambda) = 2nx^{n-1} \quad (n \geq 0),
\]
so from (4.3) we have
\[
\prod_{i=1}^{r-1} \{- (a_i) l_i \lambda_i \mathcal{G}_{l_i}(x_i - x_r + 1; \lambda_i)\} = \sum_{J \subseteq \{1, \ldots, r-1\}} \prod_{i \in J} (a_i) l_i \mathcal{G}_{l_i}(x_i - x_r; \lambda_i) \prod_{i \in J} \{- 2(a_i) l_i (x_i - x_r)^{l_i - 1}\}. \tag{4.4}
\]

Thus, applying Corollary 4.1 and then taking \(x_1 = \cdots = x_k = x\) gives the general convolution identity of the Apostol-Genocchi polynomials, as follows.

**Corollary 4.2.** Let \(a_1, \ldots, a_k\) be complex numbers with \(k\) being an even positive integer. Then, for non-negative integer \(n\),

\[
\sum_{j_1, \ldots, j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c} n \\ j_1, \ldots, j_k \end{array} \right) \frac{(a_1) j_1 \cdots (a_k) j_k}{(a_1 + \cdots + a_k)_n} \mathcal{G}_{j_1}(x; \lambda_1) \cdots \mathcal{G}_{j_k}(x; \lambda_k)
\]

\[
= \sum_{r=1}^{k} \sum_{|J| = r} (-2)^r \frac{n! \cdot a_J}{(n + 1 - r)!} \sum_{l_0, l_1, \ldots, l_{k-r} \geq 0} \left( \begin{array}{c} n + 1 - r \\ l_0, l_1, \ldots, l_{k-r} \end{array} \right) \frac{(a_{i_{r+1}}) l_1 \cdots (a_{i_k}) l_{k-r}}{(a_1 + \cdots + a_k)_{n+1-l_0}}
\]

\[
\times B_{l_0}(x; \lambda_1 \cdots \lambda_k) \mathcal{G}_{l_1}(\lambda_{i_{r+1}}) \cdots \mathcal{G}_{l_{k-r}}(\lambda_{i_k}),
\]

where \(i_{r+1}, \ldots, i_k \in J\).

It is obvious that the case \(\lambda_1 = \cdots = \lambda_k = 1\) in Corollary 4.2 gives the following general convolution identity for the classical Genocchi polynomials.

**Corollary 4.3.** Let \(a_1, \ldots, a_k\) be complex numbers with \(k\) being an even positive integer. Then, for non-negative integer \(n\),

\[
\sum_{j_1, \ldots, j_k = n \atop j_1, \ldots, j_k \geq 0} \left( \begin{array}{c} n \\ j_1, \ldots, j_k \end{array} \right) \frac{(a_1) j_1 \cdots (a_k) j_k}{(a_1 + \cdots + a_k)_n} \mathcal{G}_{j_1}(x) \cdots \mathcal{G}_{j_k}(x)
\]

\[
= \sum_{r=1}^{k} \sum_{|J| = r} (-2)^r \frac{n! \cdot a_J}{(n + 1 - r)!} \sum_{l_0, l_1, \ldots, l_{k-r} \geq 0} \left( \begin{array}{c} n + 1 - r \\ l_0, l_1, \ldots, l_{k-r} \end{array} \right) \frac{(a_{i_{r+1}}) l_1 \cdots (a_{i_k}) l_{k-r}}{(a_1 + \cdots + a_k)_{n+1-l_0}}
\]

\[
\times B_{l_0}(x) \mathcal{G}_{l_1} \cdots \mathcal{G}_{l_{k-r}},
\]

where \(i_{r+1}, \ldots, i_k \in J\).

If we take \(a_1 = \cdots = a_k = 1\) in Corollary 4.2 we get the following higher-order convolution identity for the Apostol-Genocchi polynomials.

**Corollary 4.4.** Let \(a_1, \ldots, a_k\) be complex numbers with \(k\) being an even positive integer. Then, for non-negative integer \(n\),

\[
(n + k) \sum_{j_1, \ldots, j_k = n \atop j_1, \ldots, j_k \geq 0} \mathcal{G}_{j_1}(x; \lambda_1) \cdots \mathcal{G}_{j_k}(x; \lambda_k)
\]

\[
= \sum_{r=1}^{k} \sum_{|J| = r} (-2)^r \sum_{l_0, l_1, \ldots, l_{k-r} = n+1-r \atop l_0, l_1, \ldots, l_{k-r} \geq 0} \left( \begin{array}{c} n + k \\ l_0 \end{array} \right) B_{l_0}(x; \lambda_1 \cdots \lambda_k) \mathcal{G}_{l_1}(\lambda_{i_{r+1}}) \cdots \mathcal{G}_{l_{k-r}}(\lambda_{i_k}),
\]

where \(i_{r+1}, \ldots, i_k \in J\).
In particular, the case $\lambda_1 = \cdots = \lambda_k = 1$ in Corollary 4.4 gives the higher-order convolution identity for the classical Genocchi polynomials as follows,

\[
(n + k) \sum_{j_1 + \cdots + j_k = n} G_{j_1}(x) \cdots G_{j_k}(x) = \sum_{r=1}^{k} \binom{k}{r} (-2)^r \sum_{l_0 + l_1 + \cdots + l_{k-r} = n+1-r}^{l_0, l_1, \cdots, l_{k-r} \geq 0} \binom{n+k}{l_0} B_{l_0}(x) G_{l_1} \cdots G_{l_{k-r}}, \tag{4.5}
\]

where $n$, $k$ are positive integers with $k$ being an even integer. If we take $k = 2$ in (4.5), in view of $G_0 = 0$ and $G_1 = 1$ (see, e.g., [27]), we get the convolution identity for the classical Genocchi polynomials due to Agoh [1, 20], namely

\[
\sum_{k=1}^{n-1} G_k(x) G_{n-k}(x) + \frac{4}{n+2} \sum_{k=0}^{n-2} \binom{n+2}{k} B_k(x) G_{n-k} = 0 \quad (n \geq 2).
\]

If we take $k = 2$ in Corollary 4.3 by applying the methods described in [14] to yield (2.18), we obtain another convolution identity for the classical Genocchi polynomials due to Agoh [1, 20], namely

\[
\sum_{k=1}^{n-1} G_k(x) G_{n-k}(x) + \frac{4}{n} \sum_{k=0}^{n-2} \binom{n}{k} \frac{B_k(x) G_{n-k}}{n-k} = 0 \quad (n \geq 2).
\]

We next consider the case $k$ being an odd positive integer in (4.1). Obviously, in this case, for non-negative integer $n$ and complex numbers $a_1, \ldots, a_k$,

\[
\sum_{j_1 + \cdots + j_k = n} \frac{(-a_1)}{j_1} \cdots \frac{(-a_k)}{j_k} \binom{k}{j_1} \frac{2t_je^{x_1 t_1}}{\lambda_{j_1} e^{t_1 + 1}} \cdots \frac{2t_je^{x_k t_k}}{\lambda_{j_k} e^{t_k + 1}}
\]

\[
= \sum_{j_1 + \cdots + j_k = n} \frac{(-a_1)}{j_1} \cdots \frac{(-a_k)}{j_k} \frac{(-1)^{r-1}}{\lambda_{j_1} \cdots \lambda_{j_k} e^{t_1 + \cdots + t_k} + 1} \frac{2t_je^{x_1 t_1}}{\lambda_{j_1} e^{t_1 + 1}} \cdots \frac{2t_je^{x_k t_k}}{\lambda_{j_k} e^{t_k + 1}}
\]

\[
\times \prod_{i=1}^{r-1} \lambda_{j_i} \frac{2t_je^{x_{i-r} t_i + 1}}{\lambda_{j_i} e^{t_i + 1}} \prod_{i=r}^{k} \frac{2t_je^{x_{i-r} t_i}}{\lambda_{j_i} e^{t_i + 1}}
\]

Since $G_0(x; \lambda) = 0$ (see, e.g., [27]), so by (2.9) we have

\[
\frac{2e^{x(t_1 + \cdots + t_k)}}{\lambda_{j_1} \cdots \lambda_{j_k} e^{t_1 + \cdots + t_k} + 1} = \frac{\sum_{N=0}^{\infty} G_{N+1}(x; \lambda_{1} \cdots \lambda_{k})}{N + 1} \sum_{l_1 + \cdots + l_k = N} \frac{l_1! \cdots l_k!}{l_1! \cdots l_k!}
\]

\[
= \sum_{l_1 + \cdots + l_k = N} \frac{G_{l_1}(x_r; \lambda_{1} \cdots \lambda_{k})}{l_r}
\]

By making the operation $\left[ \frac{t_1^{j_1}}{j_1!} \cdots \frac{t_k^{j_k}}{j_k!} \right]$ in both sides of (4.6), in view of (1.3) and (4.7), we get

\[
\sum_{j_1 + \cdots + j_k = n} \frac{(-a_1)}{j_1} \cdots \frac{(-a_k)}{j_k} G_{j_1}(x_1; \lambda_1) \cdots G_{j_k}(x_k; \lambda_k)
\]

\[
= \sum_{j_1 + \cdots + j_k = n} \frac{(-a_1)}{j_1} \cdots \frac{(-a_k)}{j_k} \sum_{r=1}^{k} (-1)^{r-1} j_r \sum_{l_1 + \cdots + l_k = N} \frac{G_{l_1}(x_r; \lambda_{1} \cdots \lambda_{k})}{l_r}
\]

\[
\times \prod_{i=1}^{r-1} \left( \frac{j_i}{l_i} \right) \lambda_i G_{l_i}(x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^{k} \left( \frac{j_i}{l_i} \right) G_{l_i}(x_i - x_r; \lambda_i).
\]
It is easily seen from (2.13) that for non-negative integers \( l_1, \ldots, l_k \) with \( l_1 + \cdots + l_k = n \),
\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} J_r \binom{k}{j_1} \cdots \binom{k}{j_k} \binom{n}{j_1} \cdots \binom{n}{j_k} = -a_r \prod_{i \neq r} \binom{-a_i}{l_i} \left( n - 1 \right) \binom{-a_1 + \cdots + a_k - 1 - (n - l_r)}{n - 1 - (n - l_r)}. \tag{4.9}
\]

Thus, by applying (4.9) to (4.8), in light of (2.7), we obtain the following result.

**Theorem 4.5.** Let \( a_1, \ldots, a_k \) be complex numbers with \( k \) being an odd positive integer. Then, for non-negative integer \( n \),
\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \binom{n}{j_1, \ldots, j_k} \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_n} G_{j_1}(x_1; \lambda_1) \cdots G_{j_k}(x_k; \lambda_k)
\]
\[
= \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{l_1, \ldots, l_k} \binom{n}{l_1, \ldots, l_k} \frac{a_r}{(a_1 + \cdots + a_k)_{n+1-l_r}} G_{l_r}(x_r; \lambda_1 \cdots \lambda_k)
\]
\[
\times \prod_{i=1}^{r-1} (a_i)_i \lambda_i \sum_{l_i, \ldots, l_k \geq 0} \binom{a_r}{l_i, \ldots, l_k} (\lambda_1 \cdots \lambda_k).
\]

If we apply (4.4) to Theorem 4.5 and then take \( x_1 = \cdots = x_k = x \), we get the general convolution identity for the Apostol-Genocchi polynomials, as follows.

**Corollary 4.6.** Let \( a_1, \ldots, a_k \) be complex numbers with \( k \) being an odd positive integer. Then, for non-negative integer \( n \),
\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \binom{n}{j_1, \ldots, j_k} \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_n} G_{j_1}(x_1; \lambda_1) \cdots G_{j_k}(x_k; \lambda_k)
\]
\[
= \sum_{r=1}^{k} \binom{k}{l_1, \ldots, l_k} \binom{n}{l_1, \ldots, l_k} (-2)^{r-1} n! \cdot a_r \frac{1}{(n+1-r)!} \sum_{l_0, l_1, \ldots, l_{k-r} = 0} \binom{n+1-r}{l_0, l_1, \ldots, l_{k-r}} \binom{a_{i+r+1}}{l_i} \cdots \binom{a_k}{l_{k-r}}
\]
\[
\times G_{l_0}(x; \lambda_1 \cdots \lambda_k) G_{l_1}(\lambda_{i+1}) \cdots G_{l_{k-r}}(\lambda_{i_k}),
\]
where \( i_{r+1}, \ldots, i_k \in J \).

It is clear that the case \( \lambda_1 = \cdots = \lambda_k = 1 \) in Corollary 4.6 gives the general convolution identity for the classical Genocchi polynomials, as follows.

**Corollary 4.7.** Let \( a_1, \ldots, a_k \) be complex numbers with \( k \) being an odd positive integer. Then, for non-negative integer \( n \),
\[
\sum_{j_1 + \cdots + j_k = n \atop j_1, \ldots, j_k \geq 0} \binom{n}{j_1, \ldots, j_k} \frac{(a_1)_{j_1} \cdots (a_k)_{j_k}}{(a_1 + \cdots + a_k)_n} G_{j_1}(x) \cdots G_{j_k}(x)
\]
\[
= \sum_{r=1}^{k} \binom{k}{l_1, \ldots, l_k} \binom{n}{l_1, \ldots, l_k} (-2)^{r-1} n! \cdot a_r \frac{1}{(n+1-r)!} \sum_{l_0, l_1, \ldots, l_{k-r} = 0} \binom{n+1-r}{l_0, l_1, \ldots, l_{k-r}} \binom{a_{i+r+1}}{l_i} \cdots \binom{a_k}{l_{k-r}}
\]
\[
\times G_{l_0}(x) G_{l_1}(x) \cdots G_{l_{k-r}},
\]
where \( i_{r+1}, \ldots, i_k \in J \).
If we take \( a_1 = \cdots = a_k = 1 \) in Corollary 4.6, we get the following higher-order convolution identity for the Apostol-Genocchi polynomials.

**Corollary 4.8.** Let \( \lambda \) Apostol-Genocchi polynomials described in [19, Theorem 2.3] are only complete on condition that \( \lambda_1 = \cdots = \lambda_k \). In particular, if we take \( \lambda_1 = \cdots = \lambda_k = 1 \) in Corollary 4.8, we get the following higher-order convolution identity for the classical Genocchi polynomials:

\[
(n + k) \sum_{j_1 + \cdots + j_k = n} G_{j_1}(x; \lambda_1) \cdots G_{j_k}(x; \lambda_k)
= \sum_{r=1}^{k} \binom{k}{r} (-2)^{r-1} \sum_{l_0, l_1, \ldots, l_{k-r} = n+1-r \atop l_0, l_1, \ldots, l_{k-r} \geq 0} \binom{n+k}{l_0} G_{l_0}(x; \lambda_1 \cdots \lambda_k) G_{l_1} \cdots G_{l_{k-r}}(\lambda_{l_1}),
\]

where \( i_{r+1}, \ldots, i_k \in J \).

The above Corollaries 4.4 and 4.8 imply the corresponding higher-order convolution identities for the Apostol-Genocchi polynomials described in [19, Theorem 2.3] are only complete on condition that \( \lambda_1 = \cdots = \lambda_k \). And the case \( k = 3 \) in (4.10) gives for positive integer \( n \geq 2 \),

\[
\sum_{j_1 + j_2 + j_3 = n} G_{j_1}(x) G_{j_2}(x) G_{j_3}(x) - \frac{3}{n+3} \sum_{j_1 + j_2 + j_3 = n} \binom{n+3}{j_3} G_{j_1} G_{j_2} G_{j_3}(x)
+ \frac{6}{n+3} \sum_{k=0}^{n-1} \binom{n+3}{k} G_{k}(x) G_{n-1-k} = \frac{4}{n+3} \binom{n+3}{5} G_{n-2}(x),
\]

which is very analogous to the convolution identity on the classical Bernoulli and Euler polynomials presented in [4 Corollaries 1 and 3]. In fact, by using the methods showed in [14], one can derive the similar convolution identity for the classical Genocchi polynomials to Corollary 9 stated in [14]. The details are left as an exercise for the interested readers. It is also informed that the results presented in [35, Corollary 2.3, (2.19)] are only complete on condition that \( \alpha_0 = \alpha_1 = \cdots = \alpha_{r-1} \), which implies the result stated in [35, (2.20)] is only complete when \( \alpha_0 = \alpha_1 \).

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