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A note on coincidence points of multivalued weak G-contraction mappings

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Abstract

In this note, we discuss the definition of the multivalued weak contraction mappings defined in a metric space endowed with a graph as introduced by Hanjing and Suantai[A. Hanjing, S. Suantai, Fixed Point Theory Appl., **2015** (2015), 10 pages]. In particular, we show that this definition is not correct and give the correct definition of the multivalued weak contraction mappings defined in a metric space endowed with a graph. Then we prove the existence of coincidence points for such mappings.

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1. Introduction

Let G be a directed graph (digraph) with set of vertices $V(G)$ and set of edges $E(G)$ contains all the loops, that is, $(x, x) \in E(G)$ for any $x \in V(G)$. Such graphs are called reflexive. We also assume that G has no parallel edges. In the sequel, we assume that (X, d) is a metric space and G is a directed reflexive graph with set of vertices $V(G) = X$.

Generalizing the Banach contraction principle for multivalued mappings to metric spaces, Nadler [10] obtained the following result.

Theorem 1.1 ([10])**.** *Let (X, d) be a complete metric space. Denote by $\mathcal{CB}(\mathcal{X})$ the set of all nonempty closed bounded subsets of X. Let $F : X \to \mathcal{CB}(\mathcal{X})$ be a multivalued mapping. If there exists $k \in [0, 1)$ such that,*

$$H(F(x), F(y)) \le k\ d(x, y)$$

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for all $x, y \in X$, where H is the Hausdorff-Pompeiu distance on $CB(X)$, then F has a fixed point in X, that is, there exists $x \in X$ such that $x \in T(x)$.

A number of extensions and generalizations of Nadler's fixed point theorem were obtained by different authors; see for instance [6, 8] and references cited therein. Ran and Reurings [11] extended the Banach contraction principle to partially ordered metric spaces. Therefore, it was natural to find an extension of Nadler's fixed point theorem to partially ordered metric spaces. Beg and Butt [2] gave the first attempt. But their definition of multivalued monotone mappings was not correct which had the effect that the proof of their version of Nadler's fixed point theorem was wrong (see for example [1]).

Definition 1.2. Let (X, d) be a metric space. The Pompeiu-Hausdorff distance on $\mathcal{CB}(\mathcal{X})$ is defined by

$$H(A, B) := \max\{\sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B)\}$$

for $A, B \in \mathcal{CB}(\mathcal{X})$, where $d(a, B) := \inf_{b \in B} d(a, b)$.

The following technical result is useful to explain our definition later on.

Lemma 1.3 ([10])**.** *Let (X, d) be a metric space. For any $A, B \in \mathcal{CB}(\mathcal{X})$, $\varepsilon > 0$ and $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

By symmetry, for any $b \in B$, there exists $a \in A$ such that

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Note that from Lemma 1.3, whenever one uses multivalued mappings which involves the Pompeiu-Hausdorff distance, then one must assume that the multivalued mappings have bounded values. Otherwise, one has only to assume that the multivalued mappings have nonempty closed values.

In [10] Nadler introduced the concept of multivalued contraction mappings.

Definition 1.4. Let (X, d) be a metric space. The multivalued mapping $T : X \to \mathcal{CB}(\mathcal{X})$ is said to be a multivalued contraction mapping if there exists $k \in [0, 1)$ such that

$$H(T(x), T(y)) \leq k\ d(x, y)$$

for all $x, y \in X$.

Using Lemma 1.3, we see that T is a multivalued contraction if there exists $\alpha \in [0, 1)$ such that for any $x, y \in X$ and $a \in T(x)$, there exists $b \in T(y)$ such that

$$d(a, b) \leq \alpha\ d(x, y).$$

In their attempt to establish some coincidence point and fixed point theorems as an extension to the Mizoguchi-Takahashi fixed point theorem for Reich multivalued contraction mappings to metric spaces endowed with a graph, Hanjing and Suanti [7] introduced the following definition:

Definition 1.5. Let (X, d) be a metric space and G be a reflexive digraph with no-parallel edges. Let $T : X \to \mathcal{CB}(\mathcal{X})$ and $g : X \to X$. Then T is said to be a weak G-contraction with respect to g if for any $x, y \in X$ such that $x \neq y$ and $(x, y) \in E(G)$, we have

(i) $H(T(x), T(y)) \leq \alpha(d(x, y))\ d(x, y) + h(g(y))d(g(y), T(x))$,
(ii) if $u \in T(x)$ and $v \in T(y)$ are such that $d(u, v) \leq d(x, y)$, then $(u, v) \in E(G)$,

where $\alpha : (0, +\infty) \to [0, 1)$ satisfies $\limsup_{s \to t+} \alpha(s) < 1$, for any $t \in [0, +\infty)$ and $h : X \to [0, +\infty)$.

This definition is not appropriate because of the condition (ii). The following example explains our reasoning.

Example 1.6. Consider the space $\mathbb{R}^2$ endowed with the Euclidean distance d. Then consider the graph G obtained by the pointwise ordering of $\mathbb{R}^2$ defined by

$$((x_1, x_2), (y_1, y_2)) \in E(G) \text{ iff } x_1 \leq y_1 \ \& \ x_2 \leq y_2.$$

Let A be the unit ball of $\mathbb{R}^2$, that is, $A = \{(x_1, x_2) \in \mathbb{R}^2;\ d^2(x, 0) = x_1^2 + x_2^2 \leq 1\}$. Consider the multivalued map $T : \mathbb{R}^2 \to \mathcal{CB}(\mathbb{R}^2)$ defined by $T(x) = A$. Then we have $H(T(x), T(y)) = 0$ for any $x, y \in \mathbb{R}^2$. Since T is a constant multivalued mapping, then it is a contraction according to Nadler's definition. Therefore, T must be a weak G-contraction with respect to the identity map. The condition (i) is obviously satisfied but the condition (ii) fails. Indeed, set $x = (2, 0)$ and $y = (2, 2)$. Then $x \neq y$ and $(x, y) \in E(G)$. Since $d(x, y) = 2$, (ii) will hold if and only if for any $u, v \in A$ such that $d(u, v) \leq 2$, we must have $(u, v) \in E(G)$. This is not the case, if we take

$$u = (1, 0) \text{ and } v = (0, 1),$$

then $u, v \in A$, $d(u, v) = \sqrt{2}$, $(u, v) \notin E(G)$ and $(v, u) \notin E(G)$.

To give the correct definition of a multivalued weak G-contraction with respect to a function $g : X \to X$, we need the following remark.

Remark 1.7. Let (X, d) be a metric space. Let $T : X \to \mathcal{CB}(\mathcal{X})$ and $g : X \to X$. Assume there exists $\alpha : (0, +\infty) \to [0, 1)$ with $\limsup\limits_{s \to t+} \alpha(s) < 1$, for any $t \in [0, +\infty)$ and $h : X \to [0, +\infty)$ such that

$$H(T(x), T(y)) \leq \alpha(d(x, y))\ d(x, y) + h(g(y))d(g(y), T(x))$$

for any $x, y \in X$. Using Lemma 1.3, we can easily prove that for any $x, y \in X$ and $a \in T(x)$, there exists $b \in T(y)$ such that

$$d(a, b) \leq \beta(d(x, y))\ d(x, y) + h(g(y))d(g(y), T(x)),$$

where $\beta = \frac{1}{2}(1 + \alpha)$ which satisfies $\limsup\limits_{s \to t+} \beta(s) < 1$, for any $t \in [0, +\infty)$.

The following definition is more appropriate than Definition 1.5.

Definition 1.8. Let (X, d) be a metric space and G be a reflexive digraph with no-parallel edges. Denote by $\mathcal{C}(\mathcal{X})$ the set of all nonempty closed subsets of X. Let $T : X \to \mathcal{C}(\mathcal{X})$ and $g : X \to X$. Then T is said to be weak G-contraction with respect to g if for any $x, y \in X$ such that $x \neq y$ and $(x, y) \in E(G)$, we have

(i) if $a \in T(x)$, there exists $b \in T(y)$ such that $(a, b) \in E(G)$ and
(ii) $d(a, b) \leq \alpha(d(x, y))\ d(x, y) + h(g(y))\ d(g(y), T(x))$,

where $\alpha : (0, +\infty) \to [0, 1)$ satisfies $\limsup_{s \to t+} \alpha(s) < 1$, for any $t \in [0, +\infty)$ and $h : X \to [0, +\infty)$. A point $x \in X$ is a coincidence point of g and T if $g(x) \in T(x)$. If g is the identity map on X, then $x = g(x) \in T(x)$ is called a fixed point of T. The set of fixed points of T and the set of coincidence points of g and T are denoted by $Fix(T)$, $Coin(g, T)$, respectively.

The aim of this paper is to obtain sufficient conditions for the existence of coincidence points for the multivalued weak G-contraction mappings in metric spaces.

2. Main Results

Let (X, d) be a complete metric space and G be a reflexive digraph defined on X. We say that the triplet (X, d, G) has Property (P) if for any sequence $\{x_n\}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $(x_{n_k}, x) \in E(G)$ for some $k \in \mathbb{N}$.

Theorem 2.1. *Let (X, d) be a complete metric space and (X, d, G) has Property (P). Let $g : X \to X$ be a continuous self-mapping and $T : X \to \mathcal{C}(\mathcal{X})$ be a weak G-contraction mapping with respect to g. Suppose that $g(y) \in T(x)$ for all $(x, y) \in E(G)$ with $y \in T(x)$ and there is $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in T(x_0)$. Then $Coin(g, T) \cap Fix(T) \neq \emptyset$.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$ such that $(x_0, x_1) \in E(G)$. Thus by our assumption, we have $g(x_1) \in T(x_0)$. If $x_0 = x_1$ or $\alpha(d(x_0, x_1)) = 0$, then $x_0 \in Coin(g, T) \cap Fix(T)$. Suppose $x_0 \neq x_1$ and $\alpha(d(x_0, x_1)) \neq 0$. Using the weak contractive assumption of T, there exists $x_2 \in T(x_1)$ such that $(x_1, x_2) \in E(G)$ and

$$d(x_1, x_2) \leq \alpha(d(x_0, x_1))\ d(x_0, x_1) + h(g(x_1))\ d(g(x_1), T(x_0)) = \alpha(d(x_0, x_1))\ d(x_0, x_1).$$

By induction, we construct a sequence $\{x_n\}$ such that $x_{n+1} \in T(x_n)$, $(x_n, x_{n+1}) \in E(G)$ and

$$d(x_n, x_{n+1}) \leq \alpha(d(x_{n-1}, x_n))\ d(x_{n-1}, x_n)$$

for any $n \in \mathbb{N}$. Since $\alpha(t) < 1$, for any $t \in [0, +\infty)$, we conclude that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive numbers. Let

$$t_0 = \lim_{n \to +\infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}).$$

Since $\limsup_{s \to t_0+} \alpha(s) < 1$, there exist $\eta < 1$ and $n_0 \geq 1$ such that $\alpha(d(x_n, x_{n+1})) \leq \eta$, for any $n \geq n_0$. Then, we have

$$d(x_n, x_{n+1}) \leq \prod_{k=n_0}^{k=n-1} \alpha(d(x_k, x_{k+1}))\ d(x_{n_0}, x_{n_0+1}) \leq \eta^{n-n_0}\ d(x_{n_0}, x_{n_0+1})$$

for any $n \geq n_0$. This will imply that $\sum d(x_n, x_{n+1})$ is convergent. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then $\{x_n\}$ converges to some point $x \in X$. Using Property (P), there exists a subsequence $\{x_{\phi(n)}\}$ such that $(x_{\phi(n)}, x) \in E(G)$. Using the weak contractive assumption of T, there exists $y_n \in T(x)$ such that

$$d(x_{\phi(n)+1}, y_n) \leq \alpha(d(x_{\phi(n)}, x))\ d(x_{\phi(n)}, x) + h(g(x))\ d(g(x), T(x_{\phi(n)}))$$

for any $n \in \mathbb{N}$. Since $x_{\phi(n)+1} \in T(x_{\phi(n)})$, then $g(x_{\phi(n)+1}) \in T(x_{\phi(n)})$ which implies

$$d(x_{\phi(n)+1}, y_n) \leq \alpha(d(x_{\phi(n)}, x))\ d(x_{\phi(n)}, x) + h(g(x))\ d(g(x), g(x_{\phi(n)+1}))$$

for any $n \in \mathbb{N}$. Since g is continuous, we conclude that $\lim_{n \to +\infty} d(x_{\phi(n)+1}, y_n) = 0$. This will force $\{y_n\}$ to also converge to x. Since $T(x)$ is closed, we conclude that $x \in T(x)$, that is, x is a fixed point of T. Thus by our assumption, we have $g(x) \in T(x)$. Therefore, $x \in Coin(g, T) \cap Fix(T)$ as claimed. □

Remark 2.2. Notice that if $T(x)$ is g-invariant (that is, $g(T(x)) \subseteq T(x)$ for each $x \in X$), then our weaker condition $g(y) \in T(x)$ for all $(x, y) \in E(G)$ with $y \in T(x)$, is clearly satisfied.

Example 2.3. Let $X = \{\frac{1}{2^n}\ ;\ n \in \mathbb{N}\} \cup \{0\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Let $E(G) = \{(0,0), (\frac{1}{2^n}, \frac{1}{2^n}), (\frac{1}{2^n}, 0), (\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}})\ ;\ n \in \mathbb{N}\}$. Let $\alpha : (0, +\infty) \to [0, 1)$ be defined by $\alpha(t) = \frac{1}{2}$ for all $t \in (0, +\infty)$. Let $T : X \to \mathcal{C}(\mathcal{X})$ be defined by:

$$T(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \{0\} \cup \{\frac{1}{2^{2n+5}}, \frac{1}{2^{2n+6}}, \frac{1}{2^{2n+7}}, \frac{1}{2^{2n+9}}, \frac{1}{2^{2n+11}}, \cdots\} & \text{if } x = \frac{1}{2^{2n+1}},\ n \in \mathbb{N}, \\ \{1\} & \text{if } x = \frac{1}{2^{2n}}. \end{cases}$$

The multivalued mapping T is well defined on X. In fact, $T(x)$ is compact for any $x \in X$. Let $g : X \to X$ be defined by:

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2^{n+2}} & \text{if } x = \frac{1}{2^n}, \\ 1 & \text{if } x = 1. \end{cases}$$

Let $h : X \to [0, +\infty)$ be defined by:

$$h(x) = \begin{cases} 1 & \text{if } x = 0, \\ 2^{n+1} & \text{if } x = \frac{1}{2^n}, \\ 0 & \text{if } x = 1. \end{cases}$$

We claim that $T : X \to \mathcal{C}(\mathcal{X})$ is a weak G-contraction with respect to g. Indeed, let $x, y \in X$ such that $(x, y) \in E(G)$.

Case 1. If $x = y$, then there is nothing to prove.

Case 2. If $(x, y) = (\frac{1}{2^n}, 0)$. In this case, $T(0) = \{0\}$. If $n = 2k$ for some $k \in \mathbb{N}$, then we have $T(x) = \{1\}$ and

(a) $(1, 0) \in E(G)$,

(b) $\alpha(d(\frac{1}{2^n}, 0))\ d(\frac{1}{2^n}, 0) + h(g(0))\ d(g(0), \{1\}) = \frac{1}{2}\frac{1}{2^n} + 1 \geq d(1, 0)$.

If $n = 2k+1$ for some $k \in \mathbb{N}$, then we have $T(x) = \{0\} \cup \{\frac{1}{2^{2n+5}}, \frac{1}{2^{2n+6}}, \frac{1}{2^{2n+7}}, \frac{1}{2^{2n+9}}, \cdots\}$. Let $a \in T(x)$. If $a = 0$, then clearly the two conditions of Definition 1.8 are satisfied. If $a = \frac{1}{2^{2k+s}}$, for some $s \in \{5, 6, 7, 9, \cdots\}$. Then

(a) $(a, 0) \in E(G)$,

(b) $\alpha(d(\frac{1}{2^n}, 0))\ d(\frac{1}{2^n}, 0) + h(g(0))\ d(g(0), T(x)) = \frac{1}{2}\frac{1}{2^{2k+1}} \geq d(a, 0) = \frac{1}{2^{2k+s}}$.

Case 3. If $(x, y) = (\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}})$. In this case, $T(x) = \{1\}$ and $T(y) = \{0\} \cup \{\frac{1}{2^{2n+5}}, \frac{1}{2^{2n+6}}, \frac{1}{2^{2n+7}}, \frac{1}{2^{2n+9}}, \cdots\}$. For $a = 1$, pick $b = 0 \in T(y)$, then we have

(a) $(1, 0) \in E(G)$,

(b) $\alpha(d(\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}))\ d(\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}) + h(g(\frac{1}{2^{2n+1}}))\ d(g(\frac{1}{2^{2n+1}}), T(x)) = \frac{1}{2}\frac{1}{2^{2n+1}} + 2^{2n+4}(1 - \frac{1}{2^{2n+3}}) > d(1, 0) = 1$.

Therefore, $T : X \to \mathcal{C}(\mathcal{X})$ is a weak G-contraction with respect to g. It is easy to check that if $(x, y) \in E(G)$ with $y \in T(x)$, then $g(y) \in T(x)$. Notice that $g(T(\frac{1}{2^{2n+5}}))$ is not in $T(\frac{1}{2^{2n+5}})$, that is, $T(\frac{1}{2^{2n+5}})$ is not g-invariant. From Theorem 2.1, we know that $Coin(g, T) \cap Fix(T) \neq \emptyset$. In fact, we have $Coin(g, T) \cap Fix(T) = \{0, 1\}$.

Definition 2.4. Let (X, d) be a metric space and G be a reflexive digraph with no-parallel edges. Denote by $\mathcal{C}(\mathcal{X})$ the set of all nonempty closed subsets of X. The multivalued map $T : X \to \mathcal{C}(\mathcal{X})$ is said to be L-weak G-contraction if for any $x, y \in X$ such that $x \neq y$ and $(x, y) \in E(G)$, we have

(i) if $a \in T(x)$, there exists $b \in T(y)$ such that $(a, b) \in E(G)$ and

(ii) $d(a, b) \leq \alpha(d(x, y))\ d(x, y) + L\ d(g(y), T(x))$,

where $\alpha : (0, +\infty) \to [0, 1)$ satisfies $\limsup_{s \to t+} \alpha(s) < 1$, for any $t \in [0, +\infty)$ and $L \geq 0$.

The following corollary is a direct consequence of Theorem 2.1 by setting $h(x) = L$ for all $x \in X$ and some $L \geq 0$.

Corollary 2.5. *Let (X, d) be a complete metric space and (X, d, G) has Property (P). Let $g : X \to X$ be a continuous mapping and $T : X \to \mathcal{C}(\mathcal{X})$ be a L-weak G-contraction mapping with respect to g. Suppose that $T(x)$ is g-invariant and there is $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in T(x_0)$. Then $Coin(g, T) \cap Fix(T) \neq \emptyset$.*

The following fixed point result is a consequence of Theorem 2.1 by setting $g(x) = x$, the identity map.

Corollary 2.6. *Let (X, d) be a complete metric space and (X, d, G) has Property (P). Let $T : X \to \mathcal{C}(\mathcal{X})$ be a weak G-contraction mapping with respect to the identity map $g(x) = x$. Suppose that there is $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in T(x_0)$. Then $Fix(T) \neq \emptyset$ that is, $\exists\ x \in X$ such that $x \in T(x)$.*

Remark 2.7.

(i) Once Theorem 2.1 are established, it is easy to extend them to the case of uniformly locally contractive mappings in the sense of Edelstein [4] with or without a graph.

(ii) In Theorem 2.1, if we take G with $E(G) = X \times X$, we obtain Theorem 2.2 of Du [5].

(iii) In Theorem 2.1, if we take $h = 0$ and G with $E(G) = X \times X$, we obtain Mizoguchi-Takahasi Theorem [9].

(iv) In Theorem 2.1, if we take $g(x) = x$, the identity map, $h = L$ for some $L \geq 0$ and G with $E(G) = X \times X$, we obtain Theorem 4 of Berinde-Berinde [3].

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