On a solvable for some systems of rational difference equations

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Abstract

In this paper, we study the existence of solutions for a class of rational systems of difference equations of order four in four-dimensional case

\[ x_{n+1} = \frac{x_{n-3} \pm t_n \pm y_{n-1} \pm z_{n-2} \pm x_{n-3}}{1 \pm x_n \pm y_n \pm z_n \pm x_{n-1} \pm y_{n-1} \pm z_{n-1} \pm x_{n-2} \pm y_{n-2} \pm z_{n-2}} \]

\[ y_{n+1} = \frac{y_{n-3} \pm t_n \pm x_{n-1} \pm z_{n-2} \pm y_{n-3}}{1 \pm x_n \pm y_n \pm z_n \pm x_{n-1} \pm y_{n-1} \pm z_{n-1} \pm x_{n-2} \pm y_{n-2} \pm z_{n-2}} \]

\[ z_{n+1} = \frac{z_{n-3} \pm t_n \pm y_{n-1} \pm x_{n-2} \pm z_{n-3}}{1 \pm x_n \pm y_n \pm z_n \pm x_{n-1} \pm y_{n-1} \pm z_{n-1} \pm x_{n-2} \pm y_{n-2} \pm z_{n-2}} \]

\[ t_{n+1} = \frac{t_{n-3} \pm t_n \pm y_{n-1} \pm x_{n-2} \pm t_{n-3}}{1 \pm x_n \pm y_n \pm z_n \pm x_{n-1} \pm y_{n-1} \pm z_{n-1} \pm x_{n-2} \pm y_{n-2} \pm z_{n-2}} \]

with the initial conditions are real numbers. Also, we study some behavior such as the periodicity and boundedness of solutions for such systems. Finally, some numerical examples are given to confirm our theoretical results and graphed by Matlab. ©2016 All rights reserved.

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1. Introduction

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have experienced enormous growth in many areas. Many applications of discrete dynamical systems and difference equations have appeared recently in the areas of biology, economics, physics, resource management and others. The theory of difference equations occupies a central position in applicable analysis. There is no suspicion that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance.

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in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, psychology, engineering, physics, probability theory, economics, genetics, physiology and resource management. It is very interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the local asymptotic stability of their equilibrium points. There are many papers deal with the difference equations system \( [1, 33] \). For example, the dynamical behavior of positive solution for the system

\[
x_{n+1} = \frac{x_{n-m+1}}{A + y_{n-1}x_{n-m+1}}, \quad y_{n+1} = \frac{y_{n-m+1}}{A + x_{n-1}x_{n-m+1}},
\]

has been studied by Sroysang in \([24]\).

In \([12]\), El-Metwally presented solutions of the following sixteen systems of difference equations

\[
x_n = \frac{x_{n-2}y_{n-1}}{\pm x_{n-2} \pm y_{n-3}}, \quad y_n = \frac{y_{n-2}x_{n-1}}{\pm y_{n-2} \pm x_{n-3}}.
\]

In \([26]\), Stević et al. studied the solutions of rational difference equations

\[
x_n = \frac{x_{n-k}y_{n-k}}{b_nx_{n-k} + a_ny_{n-k}}, \quad y_n = \frac{y_{n-k}x_{n-k}}{d_nx_{n-k} + c_ny_{n-k}}.
\]

In \([4]\), Din has investigated the dynamics of a system of fourth-order rational difference equations

\[
x_{n+1} = \frac{\alpha_1 x_{n-3}}{\beta_1 + \gamma_1 y_{n-1}x_{n-2}x_{n-3}}, \quad y_{n+1} = \frac{\alpha_2 y_{n-3}}{\beta_2 + \gamma_2 x_{n-1}y_{n-2}y_{n-3}}.
\]

Yalçinkaya and Çinar \([29]\) got the periodicity of the positive solutions of the nonlinear difference equations system

\[
x_{n+1} = \frac{1}{z_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \quad z_{n+1} = \frac{1}{x_{n-1}}.
\]

El-Dessoky et al. \([11]\) obtained the solutions of the difference equation systems

\[
x_{n+1} = \frac{x_{n-1}}{1 + y_nx_{n-1}}, \quad y_{n+1} = \frac{y_{n-1}}{1 + x_ny_{n-1}}, \quad z_{n+1} = \frac{z_{n-1}}{x_{n-1}y_{n}}.
\]

Özkan et al. \([23]\) investigated the periodical solutions of the third order rational difference equations

\[
x_{n+1} = \frac{y_{n-2}}{-1 \pm y_{n-2}x_{n-1}y_{n}}, \quad y_{n+1} = \frac{x_{n-2}}{-1 \pm x_{n-2}y_{n-1}x_{n}}, \quad z_{n+1} = \frac{x_{n-3} + y_{n-2}}{-1 \pm x_{n-3}y_{n-1}x_{n}}.
\]

Yazlik et al. \([31]\) studied the behaviour of solutions of the systems of difference equations

\[
x_{n+1} = \frac{y_{n-2}x_{n-3}y_{n-4}}{y_{n}y_{n-1}x_{n}(\pm 1 \pm y_{n-3}x_{n}y_{n-4})}, \quad y_{n+1} = \frac{x_{n-3}y_{n-3}y_{n-4}}{x_{n}y_{n-1}(\pm 1 \pm x_{n-3}y_{n-4})}.
\]

El-Dessoky et al. \([9]\) investigated the form of the solution of the systems of difference equations

\[
x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_{n-2}z_{n-1}y_{n}}, \quad y_{n+1} = \frac{y_{n-2}}{\pm 1 + y_{n-2}x_{n-1}z_{n}}, \quad z_{n+1} = \frac{z_{n-2}}{\pm 1 + z_{n-2}y_{n-1}x_{n}}.
\]

Also, in \([19]\), Kurbanli studied a three-dimensional system of rational difference equations

\[
x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_{n-1}}, \quad y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_{n-1}}, \quad z_{n+1} = \frac{x_{n}}{z_{n-1}}.
\]

To be motivated by the above studies, our aim in this paper is to obtain the existence of solutions for the rational systems of difference equations of order four in four-dimensional case

\[
x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm t_nz_{n-1}y_{n-2}z_{n-3}}, \quad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm x_{n-1}y_{n-2}z_{n-3}}, \quad z_{n+1} = \frac{z_{n-3}}{\pm 1 \pm y_{n-1}x_{n-2}z_{n-3}}, \quad t_{n+1} = \frac{t_{n-3}}{\pm 1 \pm z_{n-1}y_{n-2}t_{n-3}},
\]

where \( n \in \mathbb{N}_0 \) and the initial conditions \( x_i, y_i, z_i, t_i \) for \( i = -3, -2, -1, 0 \) are arbitrary real numbers. We
study the dynamics of these solutions such as the periodicity and boundedness and give some numerical examples for the systems.

2. Systems and the expressions of their solutions

Here we interest to investigate the following system of some rational difference equations

\[
x_{n+1} = \frac{x_{n-3}}{-1 - t_n x_{n-1} y_n - 2 x_{n-3}},
\]

\[
y_{n+1} = \frac{y_{n-3}}{-1 - x_n t_{n-1} y_{n-2} y_{n-3}},
\]

\[
t_{n+1} = \frac{t_{n-3}}{1 + y_n x_{n-1} t_{n-2} z_{n-3}},
\]

where \(n \in \mathbb{N}_0\) and the initial conditions \(x_i, y_i, z_i, t_i\) for \(i = -3, -2, -1, 0\) are arbitrary real numbers.

**Theorem 2.1.** Assume that \(\{x_n, y_n, z_n, t_n\}\) are solutions of system (2.1), then for \(n = 0, 1, 2, \ldots\), we obtain

\[
x_{4n-3} = (-1)^n x_{-3} \prod_{i=0}^{n-1} \left(1 + (2i)x_{-3}y_{-2}z_{-1}t_0\right),
\]

\[
x_{4n-1} = (-1)^n x_{-1} \prod_{i=0}^{n-1} \left(1 + (2i+1)x_{-3}y_{-2}z_{-1}t_0\right),
\]

\[
y_{4n-3} = (-1)^n y_{-3} \prod_{i=0}^{n-1} \left(1 + (2i)y_{-3}z_{-2}t_{-1}x_0\right),
\]

\[
y_{4n-1} = (-1)^n y_{-1} \prod_{i=0}^{n-1} \left(1 + (2i+2)y_{-3}z_{-2}t_{-1}x_0\right),
\]

\[
z_{4n-3} = z_{-3} \prod_{i=0}^{n-1} \left(1 + (2i)z_{-3}t_{-2}x_{-1}y_0\right),
\]

\[
z_{4n-1} = z_{-1} \prod_{i=0}^{n-1} \left(1 + (2i+1)z_{-3}t_{-2}x_{-1}y_0\right),
\]

and

\[
t_{4n-3} = t_{-3} \prod_{i=0}^{n-1} \left(1 + (2i+1)t_{-3}z_{-2}t_{-1}x_0\right),
\]

\[
t_{4n-1} = t_{-1} \prod_{i=0}^{n-1} \left(1 + (2i+1)t_{-3}z_{-2}t_{-1}x_0\right),
\]

where \(\prod_{i=0}^{n-1} A_i = 1\).

**Proof.** For \(n = 0\) the result holds. Now let \(n > 1\) and that our assumption holds for \(n - 1\), that is,

\[
x_{4n-7} = (-1)^{n-1} x_{-3} \prod_{i=0}^{n-2} \left(1 + (2i)x_{-3}y_{-2}z_{-1}t_0\right),
\]

\[
x_{4n-5} = (-1)^{n-1} x_{-1} \prod_{i=0}^{n-2} \left(1 + (2i+1)x_{-3}y_{-2}z_{-1}t_0\right),
\]

\[
y_{4n-7} = (-1)^{n-1} y_{-3} \prod_{i=0}^{n-2} \left(-1 + (2i)y_{-3}z_{-2}t_{-1}x_0\right),
\]

\[
y_{4n-5} = (-1)^{n-1} y_{-1} \prod_{i=0}^{n-2} \left(-1 + (2i+2)y_{-3}z_{-2}t_{-1}x_0\right),
\]

\[
z_{4n-7} = z_{-3} \prod_{i=0}^{n-2} \left(1 + (2i)z_{-3}t_{-2}x_{-1}y_0\right),
\]

\[
z_{4n-5} = z_{-1} \prod_{i=0}^{n-2} \left(1 + (2i+1)z_{-3}t_{-2}x_{-1}y_0\right),
\]

\[
x_{4n-6} = (-1)^{n-1} x_{-2} \prod_{i=0}^{n-2} \left(1 + (2i+1)t_{-3}x_{-2}y_{-1}t_0\right),
\]

\[
x_{4n-4} = (-1)^{n-1} x_0 \prod_{i=0}^{n-2} \left(-1 + (2i+1)t_{-3}y_{-2}z_{-1}t_0\right),
\]

\[
y_{4n-6} = (-1)^{n-1} y_{-2} \prod_{i=0}^{n-2} \left(1 + (2i+1)z_{-3}y_{-2}z_{-1}t_0\right),
\]

\[
y_{4n-4} = (-1)^{n-1} y_0 \prod_{i=0}^{n-2} \left(-1 + (2i+1)z_{-3}t_{-2}z_{-1}x_0\right),
\]

\[
z_{4n-6} = z_{-2} \prod_{i=0}^{n-2} \left(-1 + (2i)y_{-3}z_{-2}t_{-1}x_0\right),
\]

\[
z_{4n-4} = z_{-2} \prod_{i=0}^{n-2} \left(-1 + (2i)z_{-3}t_{-2}x_{-1}y_0\right),
\]

and the result holds for \(n > 1\). Therefore, our algorithm is verified.
\[ z_{4n-4} = z_0 \prod_{i=0}^{n-2} \frac{(1 + (2i+1)x_{3}y_{2}z_{1}t_{0})}{(1 + (2i+2)x_{3}y_{2}z_{1}t_{0})}, \]
\[ t_{4n-6} = t_{-2} \prod_{i=0}^{n-2} \frac{(1 + (2i+1)z_{3}t_{-2}x_{1}y_{0})}{(1 + (2i+2)z_{3}t_{-2}x_{1}y_{0})}, \]
\[ t_{4n-6} = t_{0} \prod_{i=0}^{n-2} \frac{(1 + (2i+1)x_{3}y_{2}z_{1}t_{0})}{(1 + (2i+2)x_{3}y_{2}z_{1}t_{0})}. \]

From system (2.1), we deduce that

\[ x_{4n-3} = \frac{\pi_{4n-7}}{-1 - \pi_{4n-5}y_{4n-6}x_{4n-7}}, \]

where

\[ x_{4n-7} = (1) \prod_{i=0}^{n-2} \frac{(1 + (2i)x_{3}y_{2}z_{1}t_{0})}{(1 + (2i+1)x_{3}y_{2}z_{1}t_{0})}, \]

\[ y_{4n-3} = \frac{y_{4n-7}}{-1 - x_{4n-4}y_{4n-5}z_{4n-6}y_{4n-7}}, \]

and

\[ y_{4n-7} = (1) \prod_{i=0}^{n-2} \frac{(1 + (2i+1)y_{3}z_{2}t_{-1}x_{0})}{(1 + (2i+2)y_{3}z_{2}t_{-1}x_{0})}. \]
Also, from system (2.1), we see that

\[ z_{4n-3} = \frac{z_{4n-7}}{1 + y_{4n-4} x_{4n-5}^2 x_{4n-6} z_{4n-7}} \]

\[ = \frac{z_3^{-2} \prod_{i=0}^{n-2} \left( \frac{1 + (2i)z_{3-2} x_{2y-1} z_0}{1 + (2i+1) z_{3-2} x_{2y-1} z_0} \right)}{1 + \left[ \prod_{i=0}^{n-2} \left( \frac{1 + (2i+1) x_{3y-2} x_{2y-1} z_0}{1 + (2i+2) x_{3y-2} x_{2y-1} z_0} \right) \right] \left[ \prod_{i=0}^{n-2} \left( \frac{1 + (2i) x_{3y-2} x_{2y-1} z_0}{1 + (2i+1) x_{3y-2} x_{2y-1} z_0} \right) \right]} \]

Finally, from system (2.1), we see that

\[ t_{4n-3} = \frac{t_{4n-7}}{1 + z_{4n-4} y_{4n-5} x_{4n-6}^2 x_{4n-7}} \]

\[ = \frac{t_3^{-2} \prod_{i=0}^{n-2} \left( \frac{1 + (2i) t_{3x-2y} y_{2z-1} t_0}{1 + (2i+1) t_{3x-2y} y_{2z-1} t_0} \right)}{1 + \left[ \prod_{i=0}^{n-2} \left( \frac{1 + (2i+1) t_{3x-2y} y_{2z-1} t_0}{1 + (2i+2) t_{3x-2y} y_{2z-1} t_0} \right) \right] \left[ \prod_{i=0}^{n-2} \left( \frac{1 + (2i) t_{3x-2y} y_{2z-1} t_0}{1 + (2i+1) t_{3x-2y} y_{2z-1} t_0} \right) \right]} \]

Similarly, we can prove the other relations. This completes the proof.

Lemma 2.2. If \( x_i, y_i, z_i, t_i \) for \( i = -3, -2, -1, 0 \) be arbitrary real numbers and let \( \{ x_n, y_n, z_n, t_n \} \) are solutions of system (2.1), then the following conditions hold:

(i) If \( x_{-3} = 0 \), then we have \( x_{n-3} = 0 \) and \( y_{n-2} = (-1)^n y_{-2} \), \( z_{n-1} = z_{-1} \), \( t_{n} = t_{0} \).

(ii) If \( x_{-2} = 0 \), then we have \( x_{n-2} = 0 \) and \( y_{n-1} = (-1)^n y_{-1} \), \( z_{n} = z_{0} \), \( t_{n-3} = t_{3} \).

(iii) If \( x_{-1} = 0 \), then we have \( x_{n-1} = 0 \) and \( y_{n} = (-1)^n y_{0} \), \( z_{n-3} = z_{3} \), \( t_{n-2} = t_{2} \).

(iv) If \( x_{0} = 0 \), then we have \( x_{n} = 0 \) and \( y_{n-3} = (-1)^n y_{3} \), \( z_{n-2} = z_{2} \), \( t_{n-1} = t_{1} \).

(v) If \( y_{-3} = 0 \), then we have \( y_{n-3} = 0 \) and \( x_{n} = (-1)^n x_{0} \), \( z_{n-2} = z_{2} \), \( t_{n-4} = t_{4} \).

(vi) If \( y_{-2} = 0 \), then we have \( y_{n-2} = 0 \) and \( x_{n-3} = (-1)^n x_{3} \), \( z_{n-1} = z_{1} \), \( t_{n} = t_{0} \).

(vii) If \( y_{-1} = 0 \), then we have \( y_{n-1} = 0 \) and \( x_{n-2} = (-1)^n x_{2} \), \( z_{n} = z_{0} \), \( t_{n-3} = t_{3} \).

(viii) If \( y_{0} = 0 \), then we have \( y_{n} = 0 \) and \( x_{n-1} = (-1)^n x_{1} \), \( z_{n-3} = z_{3} \), \( t_{n-2} = t_{2} \).
Proof. The proof follows directly from the expressions of the solutions of system \((2.1)\).

**Theorem 2.3.** Assume that \(\{x_n, y_n, z_n, t_n\}\) are solutions of the system

\[
\begin{align*}
x_{n+1} &= \frac{x_{n-3}}{1 + t_n y_n - 2 x_{n-3}}, \\
y_{n+1} &= \frac{y_{n-3}}{1 - y_n x_n - 2 z_{n-3}}, \\
t_{n+1} &= \frac{t_{n-3}}{1 + z_n y_n - 2 t_{n-3}},
\end{align*}
\]

with \(x_{-3} y_{-2} z_{-1} t_0 \neq \pm 1, t_{-3} x_{-2} y_{-1} z_0 \neq -1, t_{-3} x_{-2} y_{-1} z_0 \neq -\frac{1}{2}, z_{-3} t_{-3} x_{-1} y_0 \neq \pm 1, y_{-3} z_{-2} t_{-1} x_0 \neq 1, y_{-3} z_{-2} t_{-1} x_0 \neq \frac{1}{2}\), they take the form

\[
\begin{align*}
x_{4n+3} &= \frac{x_{-3}}{(1 + x_{-3} y_{-2} z_{-1} t_0)^n}, \\
x_{4n+1} &= \frac{x_{-1}}{(1 + z_{-3} t_{-2} x_{-1} y_0)^n}, \\
y_{4n+3} &= \frac{y_{-3}}{(1 - y_{-3} z_{-2} t_{-1} x_0)^n}, \\
y_{4n+1} &= \frac{y_{-1} (1 + 2 t_{-3} x_{-2} y_{-1} z_0)^n}{(1 + t_{-3} x_{-2} y_{-1} z_0)^n}, \\
z_{4n+3} &= \frac{z_{-3}}{(1 + z_{-3} t_{-2} x_{-1} y_0)^n}, \\
z_{4n+1} &= \frac{z_{-1}}{(1 - x_{-3} y_{-2} z_{-1} t_0)^n}, \\
t_{4n+3} &= \frac{t_{-3}}{(1 + t_{-3} x_{-2} y_{-1} z_0)^n}, \\
t_{4n+1} &= \frac{t_{-1} (1 + 2 y_{-3} z_{-2} t_{-1} x_0)^n}{(1 - y_{-3} z_{-2} t_{-1} x_0)^n}.
\end{align*}
\]

Proof. For \(n = 0\) the result holds. Now suppose that \(n > 0\) and that our assumption holds for \(n-1\), that is,

\[
\begin{align*}
x_{4n+7} &= \frac{x_{-3}}{(1 + x_{-3} y_{-2} z_{-1} t_0)^n-1}, \\
x_{4n+5} &= \frac{x_{-1}}{(1 + z_{-3} t_{-2} x_{-1} y_0)^n-1}, \\
y_{4n+7} &= \frac{y_{-3}}{(1 - y_{-3} z_{-2} t_{-1} x_0)^n-1}, \\
y_{4n+5} &= \frac{y_{-1} (1 + 2 t_{-3} x_{-2} y_{-1} z_0)^n-1}{(1 + t_{-3} x_{-2} y_{-1} z_0)^n-1}, \\
z_{4n+7} &= \frac{z_{-3}}{(1 - z_{-3} t_{-2} x_{-1} y_0)^n-1}, \\
z_{4n+5} &= \frac{z_{-1}}{(1 - x_{-3} y_{-2} z_{-1} t_0)^n-1}.
\end{align*}
\]

\[
\begin{align*}
x_{4n+6} &= \frac{x_{-2} (1 + 2 t_{-3} x_{-2} y_{-1} z_0)^{n-1}}{(1 + 2 t_{-3} x_{-2} y_{-1} z_0)^{n-1}}, \\
x_{4n+4} &= \frac{x_{0} (1 - y_{-3} z_{-2} t_{-1} x_0)^{n-1}}, \\
y_{4n+6} &= \frac{y_{-2} (1 + x_{-3} y_{-2} z_{-1} t_0)^{n-1}}, \\
y_{4n+4} &= \frac{y_{0} (1 + z_{-3} t_{-2} x_{-1} y_0)^{n-1}}, \\
z_{4n+6} &= \frac{z_{-2} (1 + y_{-3} z_{-2} t_{-1} x_0)^{n-1}}{(1 + 2 y_{-3} z_{-2} t_{-1} x_0)^{n-1}}, \\
z_{4n+4} &= \frac{z_{0} (1 + t_{-3} x_{-2} y_{-1} z_0)^{n-1}}{1 + t_{-3} x_{-2} y_{-1} z_0)^{n-1}}.
\end{align*}
\]
It follows that, from system (2.2), we have

\[ z_{4n-5} = \frac{z-1}{(1-x_{3y-2z-1}t_0)^{n-1}}, \quad z_{4n-4} = z_0 \left(1 + t_{-3x-2y-1}t_0\right)^{n-1}, \]

\[ t_{4n-7} = \frac{t_{-3}}{(1-t_{-3x-2y-1}t_0)^{n-1}}, \quad t_{4n-6} = t_{-2} \left(1 - z_{-3t-2x-1}t_0\right)^{n}, \]

\[ t_{4n-5} = \frac{t_{-1} (1 + 2y_{-3y-2}t_{-1}x_0)^{n-1}}{(1 + y_{-3y-2}t_{-1}x_0)^{n-1}}, \quad t_{4n-4} = t_0 \left(1 - x_{3y-2z-1}t_0\right)^{n-1}. \]

Also, we can prove the other relations similarly. The proof is complete. □

**Theorem 2.4.** Let \( \{x_n, y_n, z_n, t_n\} \) are solutions of the system

\[ x_{n+1} = \frac{x_n - 3}{1 + t_n z_{n-1}y_n - 2x_n - 3}, \quad y_{n+1} = \frac{y_{n-3}}{-1 - x_n t_{n-1}z_{n-2}y_n}, \]

\[ z_{n+1} = \frac{z_n - 3}{1 + y_n x_{n-1} t_n - 2z_n - 3}, \quad t_{n+1} = \frac{t_n - 3}{1 + z_n y_{n-1} x_{n-2}t_n}. \]  

(2.3)
with the initial values are arbitrary real numbers satisfy $x_{-3}y_{-2}z_{-1}t_0 \neq 1$, $x_{-3}y_{-2}z_{-1}t_0 \neq \frac{1}{2}$, $z_{-3}t_{-2}x_{-1}y_0 \neq -1$, $z_{-3}t_{-2}x_{-1}y_0 \neq -\frac{1}{2}$, $t_{-3}x_{-2}y_{-1}z_0 \neq \pm 1$ and $y_{-3}z_{-2}t_{-1}x_0 \neq \pm 1$. Then the solution is given by the following formula for $n = 0, 1, 2, \ldots$,

\[
\begin{align*}
x_{4n-3} &= \frac{x_{-3}}{(-1 + x_{-3}y_{-2}z_{-1}t_0)^n}, \\
x_{4n-1} &= \frac{(1 + 2z_{-3}t_{-2}x_{-1}y_0)^n}{(1 + z_{-3}t_{-2}x_{-1}y_0)^n}, \\
y_{4n-3} &= \frac{y_{-3} \pm \sqrt{y_{-3}^2 - 4(1 + y_{-3}z_{-2}t_{-1}x_0)^n}}{2y_{-3}}, \\
y_{4n-1} &= \frac{(1 + t_{-3}x_{-2}y_{-1}z_0)^n}{y_{-1}}, \\
z_{4n-3} &= \frac{z_{-3}}{(1 + z_{-3}t_{-2}x_{-1}y_0)^n}, \\
z_{4n-1} &= \frac{(1 + 2x_{-3}y_{-2}z_{-1}t_0)^n}{(1 + x_{-3}y_{-2}z_{-1}t_0)^n}, \\
t_{4n-3} &= \frac{t_{-3}}{(1 + t_{-3}x_{-2}y_{-1}z_0)^n}, \\
t_{4n-1} &= \frac{(1 + y_{-3}z_{-2}t_{-1}x_0)^n}{(-1 + y_{-3}z_{-2}t_{-1}x_0)^n}.
\end{align*}
\]

**Proof.** As the proof of Theorem 2.3.

Here for confirming the results of this section, we consider an interesting numerical examples of the systems (2.1) – (2.2).

**Example 2.5.** We consider the system (2.1) with the initial conditions $x_{-3} = 0.6$, $x_{-2} = 3$, $x_{-1} = 0.9$, $x_0 = 1.3$, $y_{-3} = 2$, $y_{-2} = 1.3$, $y_{-1} = -0.5$, $y_0 = 0.1$, $z_{-3} = 1.1$, $z_{-2} = 0.6$, $z_{-1} = -0.7$, $z_0 = 1.5$, $t_{-3} = -2$, $t_{-2} = 0.9$, $t_{-1} = -3$ and $t_0 = 0.8$, see Figure 1.

![Figure 1](image)

**Figure 1:** Sketch the behavior of the solution of system (2.1)

**Example 2.6.** See Figure 2 for an example for the system (2.2) with the initial values $x_{-3} = 0.46$, $x_{-2} = 0.23$, $x_{-1} = 0.29$, $x_0 = 1.16$, $y_{-3} = 0.2$, $y_{-2} = 1.3$, $y_{-1} = -0.5$, $y_0 = 0.61$, $z_{-3} = 0.21$, $z_{-2} = 0.26$, $z_{-1} = 0.27$, $z_0 = 1.85$, $t_{-3} = 0.2$, $t_{-2} = 0.09$, $t_{-1} = 0.28$ and $t_0 = 0.58$. 
3. Systems have a periodic solutions:

In this section, we study the solutions and periodic nature of the solutions of the following system of four nonlinear difference equations

\[
\begin{align*}
x_{n+1} &= \frac{x_{n-3}}{-1 - t_n z_{n-1} y_n - 2 x_{n-3}}, \\
y_{n+1} &= \frac{y_{n-3}}{-1 - t_n z_{n-1} y_n - 2 y_{n-3}}, \\
z_{n+1} &= \frac{z_{n-3}}{-1 - y_n x_n - 1 t_n z_{n-3}}, \\
t_{n+1} &= \frac{t_{n-3}}{-1 - z_n y_{n-1} x_n - 2 t_{n-3}},
\end{align*}
\]

where \( n \in \mathbb{N}_0 \) and the initial conditions are arbitrary real numbers.

**Theorem 3.1.** Assume that \( x_{-3}y_{-2}z_{-1}t_0 \neq -1, t_{-3}x_{-2}y_{-1}z_0 \neq -1, z_{-3}t_{-2}x_{-1}y_0 \neq -1 \) and \( x_{-3}y_{-2}z_{-1}t_0 \neq -2, t_{-3}x_{-2}y_{-1}z_0 \neq -2, z_{-3}t_{-2}x_{-1}y_0 \neq -2 \) and \( y_{-3}z_{-2}t_{-1}x_0 \neq -2 \), then all solutions of the system (3.1) are unbounded and given by the expressions

\[
\begin{align*}
x_{4n-3} &= \frac{x_{-3}}{(-1 - x_{-3}y_{-2}z_{-1}t_0)^n}, \\
x_{4n-1} &= \frac{x_{-1}}{(-1 - z_{-3}t_{-2}x_{-1}y_0)^n}, \\
y_{4n-3} &= \frac{y_{-3}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)^n}, \\
y_{4n-1} &= \frac{y_{-1}}{(-1 - t_{-3}x_{-2}y_{-1}z_0)^n}, \\
z_{4n-3} &= \frac{z_{-3}}{(-1 - z_{-3}t_{-2}x_{-1}y_0)^n}, \\
z_{4n-1} &= \frac{z_{-1}}{(-1 - x_{-3}y_{-2}z_{-1}t_0)^n}, \\
t_{4n-3} &= \frac{t_{-3}}{(-1 - t_{-3}x_{-2}y_{-1}z_0)^n}, \\
t_{4n-1} &= \frac{t_{-1}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)^n},
\end{align*}
\]

**Proof.** For \( n = 0 \) the result holds. Now suppose that \( n > 0 \) and that our assumption holds for \( n - 1 \), that is,

\[
\begin{align*}
x_{4n-7} &= \frac{x_{-3}}{(-1 - x_{-3}y_{-2}z_{-1}t_0)^{n-1}}, \\
x_{4n-5} &= \frac{x_{-1}}{(-1 - z_{-3}t_{-2}x_{-1}y_0)^{n-1}}, \\
x_{4n-6} &= \frac{x_{-2}(-1 - t_{-3}x_{-2}y_{-1}z_0)^n}{(-1 - t_{-3}x_{-2}y_{-1}z_0)^{n-1}}, \\
x_{4n-4} &= \frac{x_{-2}(-1 - y_{-3}z_{-2}t_{-1}x_0)^n}{(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1}},
\end{align*}
\]
It follows that, from system (3.1), we have

\[
y_{n-7} = \frac{y-3}{(-1 - y_3 z_2 t_1 x_0)^{n-1}}, \\
y_{n-5} = \frac{y-1}{(-1 - t_3 x_2 y_2 - 1 z_0)^{n-1}}, \\
z_{n-7} = \frac{z-1}{(1 - z_3 t_2 z_1 y_0)^{n-1}}, \\
z_{n-5} = \frac{z_1}{(1 - x_3 y_2 z_1 t_0)^{n-1}}, \\
t_{n-7} = \frac{t_1}{(-1 - t_3 x_2 y_2 - 1 z_0)^{n-1}}, \\
t_{n-5} = \frac{t_{1-1}}{(-1 - y_3 z_2 t_2 - 1 t_0)^{n-1}}.
\]

Also, we can prove the other relations similarly. The proof is complete. 

**Theorem 3.2.** If the sequences \( \{x_n, y_n, z_n, t_n\} \) are solutions of difference equation system (3.1) such that \( x_{-3} y_{-2} z_{-1} t_0 = t_3 x_2 y_2 - 1 z_0 = -2 \), then all solutions of the system are
periodic with period four and takes the form

\[ x_{4n-3} = x_{-3}, \quad x_{4n-2} = x_{-2}, \quad x_{4n-1} = x_{-1}, \quad x_{4n} = x_0, \]
\[ y_{4n-3} = y_{-3}, \quad y_{4n-2} = y_{-2}, \quad y_{4n-1} = y_{-1}, \quad y_{4n} = y_0, \]
\[ z_{4n-3} = z_{-3}, \quad z_{4n-2} = z_{-2}, \quad z_{4n-1} = z_{-1}, \quad z_{4n} = z_0, \]
\[ t_{4n-3} = t_{-3}, \quad t_{4n-2} = t_{-2}, \quad t_{4n-1} = t_{-1}, \quad t_{4n} = t_0 \]

or

\[ \{x_n\} = \{x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, x_{-2}, \ldots\}, \]
\[ \{y_n\} = \{y_{-3}, y_{-2}, y_{-1}, y_0, y_{-3}, y_{-2}, \ldots\}, \]
\[ \{z_n\} = \{z_{-3}, z_{-2}, z_{-1}, z_0, z_{-3}, z_{-2}, \ldots\}, \]
\[ \{t_n\} = \{t_{-3}, t_{-2}, t_{-1}, t_0, t_{-3}, t_{-2}, \ldots\}. \]

**Proof.** The proof follows from the previous theorem and will be omitted.

**Example 3.3.** We put the initial conditions as follows \( x_{-3} = 0.6, \ x_{-2} = 3, \ x_{-1} = 0.9, \ x_0 = 1.3, \ y_{-3} = 0.22, \ y_{-2} = 1.3, \ y_{-1} = -0.5, \ y_0 = 0.1, \ z_{-3} = 1.1, \ z_{-2} = 0.6, \ z_{-1} = 0.7, \ z_0 = 1.5, \ t_{-3} = 0.02, \ t_{-2} = 0.09, \ t_{-1} = -0.3 \) and \( t_0 = 0.8 \) for the difference system (3.1), see Figure 3.

**Example 3.4.** Figure 4 shows the periodicity behavior of the solution of the difference system (3.1) with the initial conditions \( x_{-3} = 2, \ x_{-2} = -0.5, \ x_{-1} = 1, \ x_0 = 4, \ y_{-3} = -5, \ y_{-2} = 10, \ y_{-1} = 2, \ y_0 = 0.1, \ z_{-3} = 15, \ z_{-2} = 0.6, \ z_{-1} = -0.7, \ z_0 = 1, \ t_{-3} = 2, \ t_{-2} = -4/3, \ t_{-1} = 1/6 \) and \( t_0 = 1/7 \).

The following theorems can be proved similarly.
4. Other systems

In this section, we investigate the solutions of the following systems of the difference equations

\[ x_{n+1} = \frac{x_{n-3}}{1 - t_n z_{n-1} y_{n-2} x_{n-3}}, \]
\[ y_{n+1} = \frac{y_{n-3}}{1 - x_n t_{n-1} z_{n-2} y_{n-3}}, \]
\[ z_{n+1} = \frac{z_{n-3}}{1 - y_n x_n t_{n-2} z_{n-3}}, \]
\[ t_{n+1} = \frac{t_{n-3}}{1 - z_n y_{n-1} x_{n-2} t_{n-3}}, \]

(4.1)

where \( n \in \mathbb{N}_0 \) and the initial conditions are arbitrary real numbers.

**Theorem 4.1.** If \( \{x_n, y_n, z_n, t_n\} \) are solutions of difference equation system (4.1), then for \( n = 0, 1, 2, \ldots, \)

\[ x_{n+3} = x_3 \prod_{i=0}^{n-1} \left( \frac{1 + (4i+1)x_{3i-2}z_{3i-1}t_{3i}}{1 + (4i+1)x_{3i-2}z_{3i-1}t_{3i}x_{3i-1}t_{3i}} \right), \]
\[ x_{n+1} = x_1 \prod_{i=0}^{n-1} \left( \frac{1 + (4i+1)x_{3i-2}z_{3i-1}t_{3i}}{1 + (4i+1)x_{3i-2}z_{3i-1}t_{3i}x_{3i-1}t_{3i}} \right), \]
\[ y_{n+3} = y_3 \prod_{i=0}^{n-1} \left( \frac{1 + (4i+1)y_{3i-2}z_{3i-1}t_{3i}}{1 + (4i+1)y_{3i-2}z_{3i-1}t_{3i}y_{3i-1}t_{3i}} \right), \]
\[ y_{n+1} = y_1 \prod_{i=0}^{n-1} \left( \frac{1 + (4i+1)y_{3i-2}z_{3i-1}t_{3i}}{1 + (4i+1)y_{3i-2}z_{3i-1}t_{3i}y_{3i-1}t_{3i}} \right), \]
\[ z_{n+3} = z_3 \prod_{i=0}^{n-1} \left( \frac{1 + (4i+1)z_{3i-2}t_{3i-1}x_{3i}}{1 + (4i+1)z_{3i-2}t_{3i-1}x_{3i}z_{3i-1}t_{3i}} \right), \]
\[ z_{n+1} = z_1 \prod_{i=0}^{n-1} \left( \frac{1 + (4i+1)z_{3i-2}t_{3i-1}x_{3i}}{1 + (4i+1)z_{3i-2}t_{3i-1}x_{3i}z_{3i-1}t_{3i}} \right), \]
\[ t_{n+3} = t_3 \prod_{i=0}^{n-1} \left( \frac{1 + (4i+1)t_{3i-2}x_{3i-1}y_{3i}}{1 + (4i+1)t_{3i-2}x_{3i-1}y_{3i}t_{3i-1}x_{3i}} \right), \]
\[ t_{n+1} = t_1 \prod_{i=0}^{n-1} \left( \frac{1 + (4i+1)t_{3i-2}x_{3i-1}y_{3i}}{1 + (4i+1)t_{3i-2}x_{3i-1}y_{3i}t_{3i-1}x_{3i}} \right), \]

where \( \prod_{i=0}^{n-1} A_i = 1 \).
Lemma 4.2. If $x_i, y_i, z_i, t_i$ for $i = -3, -2, -1, 0$ arbitrary real numbers and let $\{x_n, y_n, z_n, t_n\}$ are solutions of system (4.1), then the following conditions hold:

(i) If $x_{-3} = 0$, then we have $x_{4n-3} = 0$ and $y_{4n-2} = y_{-2}$, $z_{4n-1} = z_{-1}$, $t_{4n} = t_0$.

(ii) If $x_{-2} = 0$, then we have $x_{4n-2} = 0$ and $y_{4n-1} = y_{-1}$, $z_{4n} = z_0$, $t_{4n-3} = t_{-3}$.

(iii) If $x_{-1} = 0$, then we have $x_{4n-1} = 0$ and $y_{4n} = y_0$, $z_{4n-3} = z_{-3}$, $t_{4n-2} = t_{-2}$.

(iv) If $x_0 = 0$, then we have $x_{4n} = 0$ and $y_{4n-3} = y_{-3}$, $z_{4n-2} = z_{-2}$, $t_{4n-1} = t_{-1}$.

(v) If $y_{-3} = 0$, then we have $y_{4n-3} = 0$ and $x_{4n} = x_0$, $z_{4n-2} = z_{-2}$, $t_{4n-1} = t_{-1}$.

(vi) If $y_{-2} = 0$, then we have $y_{4n-2} = 0$ and $x_{4n-3} = x_{-3}$, $z_{4n-1} = z_{-1}$, $t_{4n} = t_0$.

(vii) If $y_{-1} = 0$, then we have $y_{4n-1} = 0$ and $x_{4n-2} = x_{-2}$, $z_{4n} = z_0$, $t_{4n-3} = t_{-3}$.

(viii) If $y_0 = 0$, then we have $y_{4n} = 0$ and $x_{4n-1} = x_{-1}$, $z_{4n-3} = z_{-3}$, $t_{4n-2} = t_{-2}$.

(ix) If $z_{-3} = 0$, then we have $z_{4n-3} = 0$ and $x_{4n-1} = x_{-1}$, $y_{4n} = y_0$, $t_{4n-2} = t_{-2}$.

(x) If $z_{-2} = 0$, then we have $z_{4n-2} = 0$ and $x_{4n} = x_0$, $y_{4n-3} = y_{-3}$, $t_{4n-1} = t_{-1}$.

(xi) If $z_{-1} = 0$, then we have $z_{4n-1} = 0$ and $x_{4n-3} = x_{-3}$, $y_{4n-2} = y_{-2}$, $t_{4n} = t_0$.

(xii) If $z_0 = 0$, then we have $z_{4n} = 0$ and $x_{4n-2} = x_{-2}$, $y_{4n-1} = y_{-1}$, $t_{4n-3} = t_{-3}$.

(xiii) If $t_{-3} = 0$, then we have $t_{4n-3} = 0$ and $x_{4n-2} = x_{-2}$, $y_{4n-1} = y_{-1}$, $z_{4n} = z_0$.

(xiv) If $t_{-2} = 0$, then we have $t_{4n-2} = 0$ and $x_{4n-1} = x_{-1}$, $y_{4n} = y_0$, $z_{4n-3} = z_{-3}$.

(xv) If $t_{-1} = 0$, then we have $t_{4n-1} = 0$ and $x_{4n} = x_0$, $y_{4n-3} = y_{-3}$, $z_{4n-2} = z_{-2}$.

(xvi) If $t_0 = 0$, then we have $t_{4n} = 0$ and $x_{4n-3} = x_{-3}$, $y_{4n-2} = y_{-2}$, $z_{4n-1} = z_{-1}$.

Theorem 4.3. The form of the solutions of system (4.2) are given by the following formula:

\[
x_{4n-3} = \frac{x_{-3}}{(-1 + x_{-3}y_{-2}z_{-1}t_{0})^n}, \\
x_{4n-1} = \frac{x_{-1}}{(-1 + z_{-3}t_{-2}x_{-1}y_{0})^n}, \\
y_{4n-3} = \frac{y_{-3}}{(-1 + y_{-3}z_{-2}x_{-1}t_{0})^n}, \\
y_{4n-1} = \frac{y_{-1}}{(-1 + y_{-1}x_{-2}z_{-1}t_{0})^n}, \\
z_{4n-3} = \frac{z_{-3}}{(1 + z_{-3}t_{-2}y_{-1}t_{0})^n}, \\
z_{4n-1} = \frac{z_{-1}}{(1 + y_{-1}x_{-2}z_{-1}t_{0})^n}, \\
t_{4n-3} = \frac{t_{-3}}{(-1 + t_{-3}x_{-2}z_{-1}t_{0})^n}, \\
t_{4n-1} = \frac{t_{-1}}{(-1 + t_{-1}y_{-2}z_{-1}t_{0})^n},
\]

where $x_{-3}y_{-2}z_{-1}t_{0} \neq \pm 1, z_{-3}t_{-2}x_{-1}y_{0} \neq \pm 1, t_{-3}x_{-2}y_{-1}z_{0} \neq -1, t_{-3}x_{-2}y_{-1}z_{0} \neq -\frac{1}{2}, y_{-3}z_{-2}t_{-1}x_{0} \neq 1$ and $y_{-3}z_{-2}t_{-1}x_{0} \neq \frac{1}{2}$.

Theorem 4.4. Let $\{x_n, y_n, z_n, t_n\}$ are solutions of difference equation system (4.3) with $x_{-3}y_{-2}z_{-1}t_{0} \neq -1, y_{-3}z_{-2}t_{-1}x_{0} \neq 1, t_{-3}x_{-2}y_{-1}z_{0} \neq -1$ and $z_{-3}t_{-2}x_{-1}y_{0} \neq -1$, then for $n = 0, 1, 2, ...,$

\[
x_{4n-3} = \frac{x_{-3}}{(1 + x_{-3}y_{-2}z_{-1}t_{0})^n}, \\
x_{4n-1} = \frac{x_{-1}}{(1 + z_{-3}t_{-2}x_{-1}y_{0})^n}, \\
y_{4n-3} = \frac{y_{-3}}{(-1 + y_{-3}z_{-2}t_{-1}x_{0})^n}, \\
y_{4n-1} = \frac{y_{-1}}{(1 - 2y_{-1}z_{-2}t_{-1}x_{0})^n}, \\
z_{4n-3} = \frac{z_{-3}}{(1 + z_{-3}t_{-2}y_{-1}t_{0})^n}, \\
z_{4n-1} = \frac{z_{-1}}{(1 + y_{-1}x_{-2}z_{-1}t_{0})^n}, \\
t_{4n-3} = \frac{t_{-3}}{(1 + t_{-3}x_{-2}z_{-1}t_{0})^n}, \\
t_{4n-1} = \frac{t_{-1}}{(-1 + t_{-1}y_{-2}z_{-1}t_{0})^n},
\]

where $x_{-3}y_{-2}z_{-1}t_{0} \neq -1, x_{-3}y_{-2}z_{-1}t_{0} \neq -1, y_{-3}z_{-2}t_{-1}x_{0} \neq 1$ and $z_{-3}t_{-2}x_{-1}y_{0} \neq -1$. 
Theorem 4.5. Suppose that the initial conditions of the system (4.4) are arbitrary real numbers satisfies $x_{-3}y_{-2}z_{-1}t_0 \neq \pm 1$, $y_{-3}z_{-2}t_{-1}x_0 \neq -1$, $y_{-3}z_{-2}t_{-1}x_0 \neq -\frac{1}{2}$, $t_{-3}x_{-2}y_{-1}z_0 \neq 1$, $t_{-3}x_{-2}y_{-1}z_0 \neq \frac{1}{2}$, $z_{-3}t_{-2}x_{-1}y_0 \neq \pm 1$, and if $\{x_n, y_n, z_n, t_n\}$ are solutions of system (4.4), then for $n = 0, 1, 2, \ldots$, \begin{align*}
x_{4n-3} &= (-1)^n x_{-3} \left(1 - \frac{1}{n+1} \right), \\
x_{4n-1} &= (-1)^n x_{-1} \left(1 - \frac{1}{n+1} \right), \\
y_{4n-3} &= \left(1 + z_{-3}t_{-2}x_{-1}y_0 \right)^n, \\
y_{4n-1} &= \left(1 + y_{-3}z_{-2}t_{-1}x_0 \right)^n, \\
z_{4n-3} &= \left(1 + z_{-3}t_{-2}x_{-1}y_0 \right)^n, \\
z_{4n-1} &= \left(1 + z_{-3}t_{-2}x_{-1}y_0 \right)^n, \\
t_{4n-3} &= \left(1 + t_{-3}x_{-2}y_{-1}z_0 \right)^n, \\
t_{4n-1} &= \left(1 + t_{-3}x_{-2}y_{-1}z_0 \right)^n.
\end{align*}

Theorem 4.6. Assume that $\{x_n, y_n, z_n, t_n\}$ are solutions of the system (4.5), with $x_{-3}y_{-2}z_{-1}t_0 \neq 1$, $x_{-3}y_{-2}z_{-1}t_0 \neq 1$, $y_{-3}z_{-2}t_{-1}x_0 \neq 1$, $t_{-3}x_{-2}y_{-1}z_0 \neq 1$, $x_{-3}y_{-2}z_{-1}t_0 \neq 2$, $z_{-3}t_{-2}x_{-1}y_0 \neq 2$, $y_{-3}z_{-2}t_{-1}x_0 \neq 2$ and $t_{-3}x_{-2}y_{-1}z_0 \neq -2$, then for $n = 0, 1, 2, \ldots$ \begin{align*}
x_{4n-3} &= \left(1 + x_{-3}y_{-2}z_{-1}t_0 \right)^n, \\
x_{4n-1} &= \left(1 + x_{-3}y_{-2}z_{-1}t_0 \right)^n, \\
y_{4n-3} &= \left(1 + y_{-3}z_{-2}t_{-1}x_0 \right)^n, \\
y_{4n-1} &= \left(1 + y_{-3}z_{-2}t_{-1}x_0 \right)^n, \\
z_{4n-3} &= \left(1 + z_{-3}t_{-2}x_{-1}y_0 \right)^n, \\
z_{4n-1} &= \left(1 + z_{-3}t_{-2}x_{-1}y_0 \right)^n, \\
t_{4n-3} &= \left(1 + t_{-3}x_{-2}y_{-1}z_0 \right)^n, \\
t_{4n-1} &= \left(1 + t_{-3}x_{-2}y_{-1}z_0 \right)^n.
\end{align*}
Theorem 4.7. If the sequences \( \{x_n, y_n, z_n, t_n\} \) are solutions of difference equation system \([4.5]\) such that
\[
x_{-3} y_{-2} z_{-1} t_0 = z_{-3} t_{-2} x_{-1} y_0 = y_{-3} z_{-2} t_{-1} x_0 = 2, \quad t_{-3} x_{-2} y_{-1} z_0 = -2,
\]
then \( \{x_n, y_n\} \) are periodic with period four and \( \{z_n, t_n\} \) are periodic with period eight and take the form
\[
x_{4n-3} = x_{-3}, \quad x_{4n-2} = x_{-2}, \quad x_{4n-1} = x_{-1}, \quad x_{4n} = x_0,
\]
\[
y_{4n-3} = y_{-3}, \quad y_{4n-2} = y_{-2}, \quad y_{4n-1} = y_{-1}, \quad y_{4n} = y_0,
\]
\[
z_{4n-3} = (-1)^n z_{-3}, \quad z_{4n-2} = (-1)^n z_{-2}, \quad z_{4n-1} = (-1)^n z_{-1}, \quad z_{4n} = (-1)^n z_0,
\]
\[
t_{4n-3} = (-1)^n t_{-3}, \quad t_{4n-2} = (-1)^n t_{-2}, \quad t_{4n-1} = (-1)^n t_{-1}, \quad t_{4n} = (-1)^n t_0.
\]

Example 4.8. Figure 5 shows the periodicity behavior of the solution of the difference system \([4.5]\) with the initial conditions \(x_{-3} = 2, \ x_{-2} = -0.5, \ x_{-1} = 1, \ x_0 = 4, \ y_{-3} = -5, \ y_{-2} = 3, \ y_{-1} = -2, \ y_0 = 0.1, \ z_{-3} = 5, \ z_{-2} = 0.6, \ z_{-1} = -0.1, \ z_0 = 1, \ t_{-3} = -2, \ t_{-2} = 4, \ t_{-1} = -1/6 \) and \( t_0 = -10/3 \).

![Figure 5](image_url)

Figure 5: Plot the periodicity of the solution of system \([4.5]\).

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