On common fixed points for \((\alpha, \psi)\)-contractions and generalized cyclic contractions in \(b\)-metric-like spaces and consequences

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Abstract

In this paper, using the concept of \(\alpha\)-admissible pairs of mappings, we prove several common fixed point results in the setting of \(b\)-metric-like spaces. We also introduce the notion of generalized cyclic contraction pairs and establish some common fixed results for such pairs in \(b\)-metric-like spaces. Some examples are presented making effective the new concepts and results. Moreover, as consequences we prove some common fixed point results for generalized contraction pairs in partially ordered \(b\)-metric-like spaces. ©2016 All rights reserved.

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1. Introduction and Preliminaries

The concept of \(b\)-metric spaces and related fixed point theorems have been investigated by a number of authors; see for example \[5, 8, 11, 12, 14, 15, 23, 28\]. In 2013, Alghamdi et al. \[2\] generalized the notion of a \(b\)-metric by introduction of the concept of a \(b\)-metric-like and proved some related fixed point results. After that, Chen et al. \[13\] and Hussain et al. \[16\] proved some fixed point theorems in the setting of \(b\)-metric-like spaces.

First, we recall some basic concepts and notations on \(b\)-metric-like concept.

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Definition 1.1. Let $X$ be a non-empty and $s \geq 1$. Let $d : X \times X \to [0, \infty)$ be a function such that:

(d1) $d(x, y) = 0$ implies $x = y$,

(d2) $d(x, y) = d(y, x)$,

(d3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then, $d$ is called a $b$-metric-like and the pair $(X, d)$ is called a $b$-metric-like space. The number $s$ is called the coefficient of $(X, d)$.

In the following, some examples of a $b$-metric-like which is nor a $b$-metric neither a metric-like.

Example 1.2. Let $X = \{0, 1, 2\}$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(0, 0) = 0, \quad d(1, 1) = d(2, 2) = 2,$$

$$d(0, 1) = 4, \quad d(1, 2) = 1 \quad \text{and} \quad d(2, 0) = 2,$$

with $d(x, y) = d(y, x)$ for all $x, y \in X$. Then, $(X, d)$ is a $b-$metric-like space with coefficient $s = 2$, but is nor a $b$-metric, neither a metric-like since $d(0, 1) = 4 > 3 = d(0, 2) + d(2, 1) = 2 + 1$.

Example 1.3. Let $X = \mathbb{R}$ and $p > 1$ be a real number. Define the function $d : X \times X \to [0, \infty)$ by

$$d(x, y) = (|x| + |y|)^p \quad \forall x, y \in X.$$ 

Then, $(X, d)$ is a $b$-metric-like space with coefficient $s = 2^{p-1}$, but is neither a $b$-metric space since $d(1, 1) = 2^p$ nor a metric-like space since $d(-1, 1) = 2^p > 2 = 1 + 1 = d(-1, 0) + d(0, 1)$.

Example 1.4. Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(x, y) = (x^3 + y^3)^2, \quad \forall x, y \in X.$$ 

Then $(X, d)$ is a $b$-metric-like space with coefficient $s = 2$, but is nor a $b$-metric space since $d(1, 1) = 4$ neither a metric-like space since $d(1, 2) = 81 > 65 = 1 + 64 = d(1, 0) + d(0, 2)$.

Definition 1.5. Let $(X, d)$ be a $b$-metric-like space, $\{x_n\}$ be a sequence in $X$, and $x \in X$. The sequence $\{x_n\}$ converges to $x$ if and only if

$$\lim_{n \to \infty} d(x_n, x) = d(x, x). \quad (1.1)$$

Remark 1.6. In a $b$-metric-like space, the limit for a convergent sequence is not unique in general.

Definition 1.7. Let $(X, d)$ be a $b$-metric-like space and $\{x_n\}$ be a sequence in $X$. We say that $\{x_n\}$ is Cauchy if and only if $\lim_{m \to \infty} d(x_n, x_m)$ exists and is finite.

Definition 1.8. Let $(X, d)$ be a $b$-metric-like space. We say that $(X, d)$ is complete if and only if each Cauchy sequence in $X$ is convergent.

Lemma 1.9. Let $(X, d)$ be a $b$-metric-like space and $\{x_n\}$ be a sequence that converges to $u$ with $d(u, u) = 0$. Then, for each $z \in X$ one has

$$\frac{1}{s} d(u, z) \leq \liminf_{n \to \infty} d(x_n, z) \leq \limsup_{n \to \infty} d(x_n, z) \leq sd(u, z).$$

Lemma 1.10. Let $(X, d)$ be a $b$-metric-like space and $T : X \to X$ be a given mapping. Suppose that $T$ is continuous at $u \in X$. Then, for all sequence $\{x_n\}$ in $X$ such that $x_n \to u$, we have $Tx_n \to Tu$, that is,

$$\lim_{n \to \infty} d(Tx_n, Tu) = d(Tu, Tu).$$

Let $(X, d)$ be a $b$-metric-like space. We need in the sequel the following trivial inequality:

$$d(x, x) \leq 2sd(x, y), \quad \text{for all} \ x, y \in X. \quad (1.2)$$

In 2012, Samet et al. [27] introduced the concept of $\alpha$-admissible maps.
Definition 1.11. For a nonempty set \( X \), let \( T : X \to X \) and \( \alpha : X \times X \to [0, \infty) \) be mappings. We say that the self-mapping \( T \) on \( X \) is \( \alpha \)-admissible if for all \( x, y \in X \), we have,

\[
\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.
\] (1.3)

Many papers dealing with above notion have been considered to prove some (common) fixed point results, for example see [1, 3, 6, 9, 17, 18, 20, 21, 24, 26].

Very recently, Aydi [4] generalized Definition 1.11 to a pair of mappings.

Definition 1.12. For a nonempty set \( X \), let \( A, B : X \to X \) and \( \alpha : X \times X \to [0, \infty) \) be mappings. We say that \( (A, B) \) is an \( \alpha \)-admissible pair if for all \( x, y \in X \), we have

\[
\alpha(x, y) \geq 1 \implies \alpha(Ax, By) \geq 1 \quad \text{and} \quad \alpha(By, Ax) \geq 1.
\]

The following examples illustrate Definition 1.12.

Example 1.13. Let \( X = \mathbb{R} \) and \( \alpha : X \times X \to [0, \infty) \) be defined by

\[
\alpha(x, y) = \begin{cases} 
1 & \text{if } x, y \in [0, 1], \\
0 & \text{otherwise}.
\end{cases}
\]

Consider the mappings \( A, B : X \to X \) given by

\[
Ax = \frac{x}{2} \quad \text{and} \quad Bx = x^2, \quad \forall x \in X.
\]

Then, \( (A, B) \) is an \( \alpha \)-admissible pair. In fact, let \( x, y \in X \) such that \( \alpha(x, y) \geq 1 \). By definition of \( \alpha \), this implies that \( x, y \in [0, 1] \). Thus,

\[
\alpha(Ax, By) = \alpha\left(\frac{x}{2}, y^2\right) = 1 \quad \text{and} \quad \alpha(By, Ax) = \alpha(y^2, \frac{x}{2}) = 1.
\]

Then, \( (A, B) \) is an \( \alpha \)-admissible pair.

Example 1.14. Let \( X = \mathbb{R} \) and \( \alpha : X \times X \to [0, \infty) \) be defined by

\[
\alpha(x, y) = e^{xy} \quad \forall x, y \in X.
\]

Consider the mappings \( A, B : X \to X \) given by

\[
Ax = x^3 \quad \text{and} \quad Bx = x^5, \quad \forall x \in X.
\]

Then, \( (A, B) \) is an \( \alpha \)-admissible pair. In fact, let \( x, y \in X \) such that \( \alpha(x, y) \geq 1 \). By definition of \( \alpha \), this implies that \( xy \geq 0 \). Thus,

\[
\alpha(Ax, By) = \alpha(By, Ax) = e^{x^3y^5} \geq 1,
\]

because \( x^3y^5 = x^2y^4xy \geq 0 \). Then, \( (A, B) \) is an \( \alpha \)-admissible pair.

Remark 1.15. It is easy to see that if \( \psi \in \Psi_s \), then \( \psi(t) < t \) for any \( t > 0 \).

In this paper, we provide some common fixed point results for generalized contractions (including cyclic contractions and contractions with a partial order) via \( \alpha \)-admissible pair of mappings on \( b \)-metric-like spaces. As consequences of our obtained results, we prove some existing known fixed point results on metric-like spaces and on \( b \)-metric spaces. Our results will be illustrated by some concrete examples.
2. Fixed Point Theorems for \((α, ψ)\)-contractions

First, we introduce the concept of \(α\)-contractive pair of mappings in the setting of \(b\)-metric-like spaces.

**Definition 2.1.** Let \((X, d)\) be a \(b\)-metric-like space, \(ψ ∈ Ψ_s\) and \(α : X × X → [0, ∞)\). A pair \(A, B : X → X\) is called an \((α, ψ)\)-contraction pair if

\[
d(Ax, By) ≤ ψ(M(x, y)),
\]

for all \(x, y ∈ X\) satisfying \(α(x, y) ≥ 1\), where

\[
M(x, y) = \max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4s}\}.
\]

Our first main result is

**Theorem 2.2.** Let \((X, d)\) be a complete \(b\)-metric-like space and \(A, B : X → X\) be an \((α, ψ)\)-contraction pair. Suppose that

(i) \((A, B)\) is an \(α\)-admissible pair;

(ii) there exists \(x_0 ∈ X\) such that \(\min\{α(x_0, Ax_0), α(Ax_0, x_0)\} ≥ 1\);

(iii) \(A\) and \(B\) are continuous on \((X, d)\);

(iv) \(α(z, z) ≥ 1\) for every \(z\) satisfying the conditions

\[
d(z, z) = 0, \ d(z, Az) ≤ sd(Az, Az) ≤ s^2d(z, Az) \text{ and } d(z, Bz) ≤ sd(Bz, Bz) ≤ s^2d(z, Bz);
\]

(v) \(ψ(t) < \frac{1}{2s}\) for each \(t > 0\).

Then, \(A\) and \(B\) admit a common fixed point, i.e. there exists \(u ∈ X\) such that

\[
Au = u = Bu.
\]

**Proof.** Choose \(x_1 = Ax_0\) and \(x_2 = Bx_1\). By induction, we can construct a sequence \(\{x_n\}\) in \(X\) such that

\[
x_{2n+1} = Ax_{2n}, \text{ and } x_{2n+2} = Bx_{2n+1},
\]

for all \(n ≥ 0\). We split the proof into several steps.

**Step 1:** \(α(x_n, x_{n+1}) ≥ 1\) and \(α(x_{n+1}, x_n) ≥ 1\) for all \(n ≥ 0\).

By condition (ii) and the fact that the pair \((A, B)\) is \(α\)-admissible,

\[
α(x_0, x_1) ≥ 1 \Rightarrow \begin{cases} α(x_1, x_2) = α(Ax_0, Bx_1) ≥ 1 \text{ and} \\ α(x_2, x_1) = α(Bx_1, Ax_0) ≥ 1. \end{cases}
\]

Again

\[
α(x_2, x_1) ≥ 1 \Rightarrow \begin{cases} α(x_3, x_2) = α(Ax_2, Bx_1) ≥ 1 \text{ and} \\ α(x_2, x_3) = α(Bx_1, Ax_2) ≥ 1. \end{cases}
\]

By induction, we may obtain \(α(x_n, x_{n+1}) ≥ 1\) and \(α(x_{n+1}, x_n) ≥ 1\) for all \(n ≥ 0\).

**Step 2:** We will show that

if for some \(n\), \(d(x_{2n}, x_{2n+1}) = 0\), then \(Ax_{2n} = x_{2n} = Bx_{2n}\)

and

\[
\text{if for some } n, \ d(x_{2n+1}, x_{2n+2}) = 0, \text{ then } Ax_{2n+1} = x_{2n+1} = Bx_{2n+1}.
\]
Suppose for some \( n \) that \( d(x_{2n}, x_{2n+1}) = 0 \). We shall prove that \( d(x_{2n+1}, x_{2n+2}) = 0 \). We argue by contradiction. For this, assume that

\[
d(x_{2n+1}, x_{2n+2}) > 0.
\]

Then, by Step 1 and \((2.1)\),

\[
d(x_{2n+1}, x_{2n+2}) = d(Ax_{2n}, Bx_{2n+1}) \leq \psi(M(x_{2n}, x_{2n+1})),
\]

where

\[
M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Ax_{2n}), d(x_{2n+1}, Bx_{2n+1}),
\]

\[
\frac{d(x_{2n}, Bx_{2n+1}) + d(x_{2n+1}, Ax_{2n})}{4s}
\]

\[
= \max\{0, d(x_{2n+1}, x_{2n+2}), \frac{1}{4s} (d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}))\}
\]

\[
= d(x_{2n+1}, x_{2n+2}),
\]

because

\[
d(x_{2n+1}, x_{2n+1}) \leq 2sd(x_{2n+1}, x_{2n+2}) \quad \text{and} \quad d(x_{2n}, x_{2n+2}) \leq sd(x_{2n}, x_{2n+1}) + sd(x_{2n+1}, x_{2n+2}) = sd(x_{2n+1}, x_{2n+2}).
\]

Consequently,

\[
d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n+1}, x_{2n+2})).
\]

Since \( \psi(t) < t \), so we get

\[
d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}),
\]

a contradiction. Thus, if \( d(x_{2n}, x_{2n+1}) = 0 \), then \( d(x_{2n+1}, x_{2n+2}) = 0 \). We deduce that \( x_{2n} = x_{2n+1} = x_{2n+2} \), so that

\[
x_{2n} = x_{2n+1} = Ax_{2n} \quad \text{and} \quad x_{2n} = x_{2n+2} = Bx_{2n+1} = B(Ax_{2n}) = Bx_{2n},
\]

that is \( x_{2n} \) is a common fixed point of \( A \) and \( B \).

Similarly, one shows that

\[
d(x_{2n+1}, x_{2n+2}) = 0 \Rightarrow d(x_{2n+2}, x_{2n+3}) = 0,
\]

and so \( x_{2n+1} = x_{2n+2} = x_{2n+3} \), which implies

\[
x_{2n+1} = x_{2n+2} = Bx_{2n+1} \quad \text{and} \quad x_{2n+1} = x_{2n+3} = Ax_{2n+2} = A(Bx_{2n+1}) = Ax_{2n+1},
\]

that is \( x_{2n+1} \) is a common fixed point of \( A \) and \( B \).

By \((2.6)\) and \((2.7)\), the proof is completed in the case when \( d(x_k, x_{k+1}) = 0 \) for some \( k \geq 0 \). From now on, we assume that

\[
d(x_n, x_{n+1}) > 0, \quad \forall n \geq 0.
\]
Step 3. We will show that
\[ d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \quad \text{for all } n \geq 0. \] (2.9)

By Step 1, \( \alpha(x_{2n}, x_{2n-1}) \geq 1 \), then
\[ d(x_{2n+1}, x_{2n}) = d(Ax_{2n}, Bx_{2n-1}) \leq \psi(M(x_{2n}, x_{2n-1})) \]
where
\[ M(x_{2n}, x_{2n-1}) = \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \]
\[ \frac{d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})}{4s} \}
\[ = \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \frac{1}{4s}(d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n}))\} \]
\[ = \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\}, \]
because
\[ d(x_{2n}, x_{2n}) \leq 2sd(x_{2n}, x_{2n+1}) \quad \text{and} \]
\[ d(x_{2n-1}, x_{2n+1}) \leq sd(x_{2n-1}, x_{2n}) + sd(x_{2n}, x_{2n+1}). \]

If \( \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n+1}) \) for some \( n \geq 1 \), then
\[ 0 < d(x_{2n+1}, x_{2n}) \leq \psi(d(x_{2n}, x_{2n+1})). \]

Taking into account \( \psi(t) < t \), one obtains a contradiction. It follows that
\[ \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n-1}) \]
for all \( n \geq 1 \). Then
\[ d(x_{2n}, x_{2n+1}) \leq \psi(d(x_{2n}, x_{2n-1})). \] (2.10)

A similar reasoning shows that
\[ d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n}, x_{2n+1})). \] (2.11)

Consequently, by (2.10) and (2.11),
\[ d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \quad \forall n \geq 1. \] (2.12)

Therefore
\[ d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)), \quad \forall n \geq 1. \]

Step 4. We shall show that \( \{x_n\} \) is a Cauchy sequence. Using (d3), we have
\[ d(x_n, x_{n+2}) \leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) \]
\[ \leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}). \]

Similarly,
\[ d(x_n, x_{n+3}) \leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+3}) \]
\[ \leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_{n+3}). \]
By induction, we get for all \(m > n\)

\[
\begin{align*}
d(x_n, x_m) & \leq \sum_{i=n}^{m-1} s^{i-n+1}d(x_i, x_{i+1}) \\
& \leq \sum_{i=n}^{m-1} s^{i}d(x_i, x_{i+1}) \\
& \leq \sum_{i=n}^{\infty} s^{i}\psi^i(d(x_0, x_1)) \rightarrow 0 \quad \text{as } n \to \infty,
\end{align*}
\]

which leads to

\[
\lim_{n,m \to \infty} d(x_n, x_m) = 0,
\]

that is, \(\{x_n\}\) is a Cauchy sequence. Since \((X, d)\) is a complete \(b\)-metric-like space, then there exists \(u \in X\) such that

\[
\lim_{n \to \infty} d(x_n, u) = d(u, u) = \lim_{n,m \to \infty} d(x_n, x_m) = 0.
\]

**Step 5.** \(u\) satisfies the condition (2.3).

By the continuity of \(A\), we have \(Ax_n \to Au\) in \((X, d)\), that is \(\lim_{n \to \infty} d(x_n, Au) = d(Au, Au)\), so that

\[
\lim_{n \to \infty} d(x_{2n+1}, Au) = \lim_{n \to \infty} d(Ax_{2n}, Au) = d(Au, Au).
\]

On the other side, \(\lim_{n \to \infty} d(x_n, u) = 0 = d(u, u)\) and so by Lemma 1.9,

\[
\frac{1}{s}d(u, Au) \leq \lim_{n \to \infty} d(x_{2n+1}, Au) \leq sd(u, Au).
\]

This yields that

\[
\frac{1}{s}d(u, Au) \leq d(Au, Au) \leq sd(u, Au).
\]

Similarly, one shows that

\[
\frac{1}{s}d(u, Bu) \leq d(Bu, Bu) \leq sd(u, Bu).
\]

**Step 6.** \(u\) is a common fixed point of \(A\) and \(B\).

Suppose by contradiction that \(d(Au, Bu) > 0\). Since \(u\) satisfies (2.3), it follows from (iv) that \(\alpha(u, u) \geq 1\), so by (2.1),

\[
d(Au, Bu) \leq \psi(M(u, u)),
\]

where

\[
M(u, u) = \max\{d(u, u), d(u, Au), d(u, Bu), \frac{d(u, Bu) + d(u, Au)}{4s}\}
\]

\[
= \max\{0, d(u, Au), d(u, Bu), \frac{d(u, Bu) + d(u, Au)}{4s}\}
\]

\[
= \max\{d(u, Au), d(u, Bu)\}.
\]

By using (2.15) and (2.16), we get

\[
M(u, u) \leq \max\{2s^2d(Au, Bu), 2s^2d(Au, Bu)\} = 2s^2d(Au, Bu).
\]

Again, by condition (v), we have

\[
d(Au, Bu) \leq \psi(2s^2d(Au, Bu)) < d(Au, Bu),
\]

which is a contradiction. Thus, \(d(Au, Bu) = 0\). In this case, the fact that \(d(u, Au) \leq sd(Au, Au)\) implies

\[
0 \leq d(u, Au) \leq sd(Au, Au) \leq 2s^2d(Au, Bu) = 0,
\]

and so \(Au = u\). Therefore, \(Bu = Au = u\). The proof is completed.
In the following, we state some consequences and corollaries of our obtained result.

**Corollary 2.3.** Let \((X, d)\) be a complete \(b\)-metric-like space, \(\psi \in \Psi_s\) and \(A, B : X \to X\) be given mappings. Suppose there exists a function \(\alpha : X \times X \to [0, \infty)\) such that
\[
\alpha(x, y)d(Ax, By) \leq \psi(M(x, y)),
\] for all \(x, y \in X\), where \(M(x, y)\) is defined by (2.2).

Also, suppose that

(i) \((A, B)\) is an \(\alpha\)–admissible pair;

(ii) there exists \(x_0 \in X\) such that \(\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1\);

(iii) \(A\) and \(B\) are continuous on \((X, d)\);

(iv) \(\alpha(z, z) \geq 1\) for every \(z\) satisfying the conditions
\[
d(z, z) = 0, \ d(z, Az) \leq s d(Az, Az) \text{ and } d(z, Bz) \leq s d(Bz, Bz) \leq s^2 d(z, Bz);
\] (2.18)

(v) \(\psi(t) < \frac{t^2}{s^2}\), for each \(t > 0\).

Then, \(A\) and \(B\) have a common fixed point.

**Proof.** Let \(x, y \in X\) such that \(\alpha(x, y) \geq 1\). Then, if (2.17) holds, we have
\[
d(Ax, By) \leq \alpha(x, y)d(Ax, By) \leq \psi(M(x, y)).
\]

Then, the proof is concluded by Theorem 2.2.

**Corollary 2.4.** Let \((X, d)\) be a complete \(b\)-metric-like space, \(\psi \in \Psi_s\) and \(A, B : X \to X\) be continuous mappings satisfying
\[
d(Ax, By) \leq \psi(M(x, y)),
\] for all \(x, y \in X\), where \(M(x, y)\) is defined by (2.2).

If \(\psi(t) < \frac{t^2}{s^2}\) for each \(t > 0\), then \(A\) and \(B\) have a common fixed point.

**Proof.** It suffices to take \(\alpha(x, y) = 1\) in Corollary 2.3.

**Corollary 2.5.** Let \((X, d)\) be a complete \(b\)-metric-like space and \(A, B : X \to X\) be continuous mappings. Suppose there exists \(k \in [0, \frac{1}{2s^2})\) such that
\[
d(Ax, By) \leq kd(x, y),
\] for all \(x, y \in X\), where \(M(x, y)\) is defined by (2.2). Then, \(A\) and \(B\) have a common fixed point.

**Proof.** It suffices to take \(\psi(t) = kt\) for all \(t \geq 0\) in Corollary 2.4.

**Corollary 2.6.** Let \((X, d)\) be a complete \(b\)-metric-like space and \(A, B : X \to X\) be continuous mappings. Suppose there exists \(k \in [0, \frac{1}{2s^2})\) such that
\[
d(Ax, By) \leq kd(x, y),
\] for all \(x, y \in X\). Then, \(A\) and \(B\) have a common fixed point.

In the setting of \(b\)-metric spaces, we have,
Corollary 2.7. Let \((X,d)\) be a complete b-metric space, \(\psi \in \Psi\), and \(A,B : X \rightarrow X\) be given mappings. Suppose there exists a function \(\alpha : X \times X \rightarrow [0,\infty)\) such that
\[
\alpha(x,y)d(Ax,By) \leq \psi(M(x,y)),
\] (2.22)
for all \(x,y \in X\), where \(M(x,y)\) is defined by \(\frac{\alpha(x,y)}{\alpha(x,y)}\).

Also, Suppose that
(i) \((A,B)\) is an \(\alpha\)-admissible pair;
(ii) there exists \(x_0 \in X\) such that \(\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1\);
(iii) \(A\) and \(B\) are continuous on \((X,d)\).

Then, \(A\) and \(B\) have a common fixed point.

Proof. Following the proof of Theorem 2.2, we know that the sequence \(\{x_n\}\) is Cauchy in \((X,d)\) and converges to some \(u \in X\). We show that \(u\) is a common fixed point of \(A\) and \(B\). Using the continuity of \(A\) and \(B\) and Lemma \(1.9\), we obtain \(Au = Bu = u\). \(\square\)

In metric-like spaces (the case \(s = 1\)), we may state the following result.

Corollary 2.8. Let \((X,d)\) be a complete metric-like space, \(\psi \in \Psi_1\) and \(A,B : X \rightarrow X\) such that
\[
d(Ax,By) \leq \psi(\max\{d(x,y),d(x,Ax),d(y,By),\frac{d(x,By) + d(y,Ax)}{4}\}),
\]
for all \(x,y \in X\) satisfying \(\alpha(x,y) \geq 1\).

Also, Suppose that
(i) \((A,B)\) is an \(\alpha\)-admissible pair;
(ii) there exists \(x_0 \in X\) such that \(\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1\);
(iii) \(A\) and \(B\) are continuous on \((X,d)\);
(iv) \(\alpha(z,z) \geq 1\) for every \(z\) satisfying the conditions
\[
d(z,z) = 0, d(z,Az) = d(Az,Az) \text{ and } d(z,Bz) = d(Bz,Bz);
\] (2.23)
(v) \(\psi(t) < \frac{1}{2}\) for each \(t > 0\).

Then, \(A\) and \(B\) have a common fixed point.

Theorem 2.2 remains true if we replace the continuity hypothesis by the following property:

\((H)\) If \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) and \(\alpha(x_{n+1}, x_n) \geq 1\) for all \(n\) and \(x_n \rightarrow x \in X\) as \(n \rightarrow \infty\), then there exists a subsequence \(\{x_{n(k)}\}\) of \(\{x_n\}\) such that \(\alpha(x_{n(k)}, x) \geq 1\) and \(\alpha(x, x_{n(k)}) \geq 1\) for all \(k\).

The statement is given as follows.

Theorem 2.9. Let \((X,d)\) be a complete b-metric-like space and \(A,B : X \rightarrow X\) an \((\alpha,\psi)\)-contraction pair. Suppose that
(i) \((A,B)\) is an \(\alpha\)-admissible pair;
(ii) there exists \(x_0 \in X\) such that \(\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1\);
(iii) \((H)\) holds;
(iv) \(\psi(t) < \frac{1}{2}\) for each \(t > 0\).

Then, \(A\) and \(B\) admit a common fixed point.
Proof. Following the proof of Theorem 2.2, we know that the sequence \( \{x_n\} \) is Cauchy in \((X, d)\) and converges to some \( u \in X \). We show that \( u \) is a common fixed point of \( A \) and \( B \).

Suppose on the contrary that \( Au \neq u \) or \( Bu \neq u \). Assume that \( d(u, Au) > 0 \).

By assumption (iii) (that is, \( \alpha(u, x_{2n(k)} - 1) \geq 1 \)), we have

\[
d(Au, x_{2n(k)}) = d(Au, Bx_{2n(k)-1}) \leq \psi(M(u, x_{2n(k)-1})),
\]

where

\[
M(u, x_{2n(k)-1}) = \max \{d(u, x_{2n(k)-1}), d(u, Au), d(x_{2n(k)-1}, x_{2n(k)}),
\frac{d(u, Bx_{2n(k)-1}) + d(x_{2n(k)-1}, Au)}{4s}\}
\]

\[
= \max \{d(u, x_{2n(k)-1}), d(u, Au), d(x_{2n(k)-1}, x_{2n(k)}),
\frac{d(u, x_{2n(k)}) + d(x_{2n(k)-1}, Au)}{4s}\}
\]

\[
\leq \max \{d(u, x_{2n(k)-1}), d(u, Au), d(x_{2n(k)-1}, x_{2n(k)}),
\frac{d(u, x_{2n(k)}) + sd(x_{2n(k)-1}, u) + sd(u, Au)}{4s}\}.
\]

We know that

\[
\lim_{n \to \infty} d(u, x_{2n(k)-1}) = \lim_{n \to \infty} d(x_{2n(k)-1}, x_{2n(k)}) = \lim_{n \to \infty} d(u, x_{2n(k)}) = 0.
\]

Then, there exists \( N \in \mathbb{N} \) such that for all \( k \geq N \),

\[
M(u, x_{2n(k)-1}) \leq d(u, Au).
\]

Then, by \((\psi)_1\), we obtain for all \( k \geq N \),

\[
d(Au, x_{2n(k)}) \leq \psi(d(u, Au)). \tag{2.24}
\]

On the other hand, we have

\[
d(Au, u) \leq sd(Au, x_{2n(k)}) + sd(x_{2n(k)}, u), \quad \forall k \geq 0. \tag{2.25}
\]

Combining (2.24) and (2.25), we get for all \( k \geq N \),

\[
d(Au, u) \leq s\psi(d(u, Au)) + sd(x_{2n(k)}, u). \tag{2.26}
\]

Having in mind \( \psi(t) < \frac{t}{s} \), so letting \( k \to \infty \) in (2.26), we get

\[
0 < d(u, Au) \leq s\psi(d(u, Au)) < d(u, Au),
\]

which is a contradiction. Similarly, if \( d(u, Bu) > 0 \) we get a contradiction. Hence, \( Au = u = Bu \) and so \( u \) is a common fixed point of \( A \) and \( B \).

Analogously, we can derive the following results.

**Corollary 2.10.** Let \((X, d)\) be a complete \( b \)-metric-like space, \( \psi \in \Psi_s \) and \( A, B : X \to X \) be given mappings. Suppose there exists a function \( \alpha : X \times X \to [0, \infty) \) such that

\[
\alpha(x, y)d(Ax, By) \leq \psi(M(x, y)), \tag{2.27}
\]

for all \( x, y \in X \), where \( M(x, y) \) is defined by (2.2).

Also, Suppose that
(i) \((A, B)\) is an \(\alpha\)-admissible pair;
(ii) \(\exists x_0 \in X\) such that \(\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1\);
(iii) \((H)\) holds;
(iv) \(\psi(t) < \frac{t}{s}\) for each \(t > 0\).

Then, \(A\) and \(B\) have a common fixed point.

**Corollary 2.11.** Let \((X, d)\) be a complete \(b\)-metric-like space, \(\psi \in \Psi\), and \(A, B : X \to X\) be given mappings. Suppose that
\[
d(Ax, By) \leq \psi(M(x, y)),
\]
for all \(x, y \in X\), where \(M(x, y)\) is defined by (2.2).

If \(\psi(t) < \frac{t}{s}\) for each \(t > 0\), then \(A\) and \(B\) have a common fixed point.

**Corollary 2.12.** Let \((X, d)\) be a complete \(b\)-metric-like space and \(A, B : X \to X\) be given mappings. Suppose there exists \(k \in [0, \frac{1}{s})\) such that
\[
d(Ax, By) \leq kM(x, y),
\]
for all \(x, y \in X\), where \(M(x, y)\) is defined by (2.2). Then, \(A\) and \(B\) have a common fixed point.

In the case \(s = 1\), we have the two following corollaries.

**Corollary 2.13.** Let \((X, d)\) be a complete metric-like space, \(\psi \in \Psi_1\) and \(A, B : X \to X\) such that
\[
d(Ax, By) \leq \psi(\max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4}\}),
\]
for all \(x, y \in X\) satisfying \(\alpha(x, y) \geq 1\).

Also, Suppose that
(i) \((A, B)\) is an \(\alpha\)-admissible pair;
(ii) there exists \(x_0 \in X\) such that \(\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1\);
(iii) \((H)\) holds.

Then, \(A\) and \(B\) have a common fixed point.

**Corollary 2.14.** Let \((X, d)\) be a complete metric-like space, \(\psi \in \Psi_1\) and \(A, B : X \to X\) such that
\[
d(Ax, By) \leq \psi(\max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4}\}),
\]
for all \(x, y \in X\). Then, \(A\) and \(B\) have a common fixed point.

We provide the following example.

**Example 2.15.** Take \(X = [0, \infty)\) endowed with the complete \(b\)-metric-like \(d(x, y) = (x^3 + y^3)^{\frac{1}{3}}\). Consider the mappings \(A, B : X \to X\) given by
\[
Ax = \begin{cases} \frac{x}{\sqrt[3]{3}} & \text{if } x \in [0, 1] \\ 2x - 2 & \text{if } x > 1 \end{cases}, \quad Bx = \begin{cases} \frac{x}{\sqrt[3]{3}} & \text{if } x \in [0, 1] \\ x & \text{if } x > 1. \end{cases}
\]

Define the mapping \(\alpha : X \times X \to [0, \infty)\) by
\[
\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}
\]
Let \( \psi(t) = \frac{1}{2}t \). Note that \((A, B)\) is an \( \alpha \)-admissible pair. In fact, let \( x, y \in X \) such that \( \alpha(x, y) \geq 1 \). By definition of \( \alpha \), this implies that \( x, y \in [0, 1] \). Thus, 

\[
\alpha(Ax, By) = \alpha\left(\frac{x}{\sqrt{3}}, \frac{y}{\sqrt{3}}\right) = 1 \quad \text{and} \quad \alpha(By, Ax) = \alpha\left(\frac{y}{\sqrt{3}}, \frac{x}{\sqrt{3}}\right) = 1.
\]

Then, \((A, B)\) is an \( \alpha \)-admissible pair.

Now, we show that \((A, B)\) is an \((\alpha, \psi)\)-contraction. Let \( x, y \in X \) such that \( \alpha(x, y) \geq 1 \). So, \( x, y \in [0, 1] \). We have

\[
d(Ax, By) = ((Ax)^3 + (By)^3)^2 = \left(\frac{x}{\sqrt{3}}\right)^3 + \left(\frac{y}{\sqrt{3}}\right)^3\}
= \left(\frac{1}{\sqrt{3}}\right)^6(x^3 + y^3)^2 = \psi(d(x, y)) \leq \psi(M(x, y)).
\]

Now, we show that \((H)\) is verified. Let \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) and \( \alpha(x_{n+1}, x_n) \geq 1 \) for all \( n \) and \( x_n \to u \) in \( (X, d) \). Then, \( \{x_n\} \subset [0, 1] \) and \( x_n \to u \) in \( (X, \|\cdot\|) \), where \( \|\cdot\| \) is the standard metric on \( X \). Thus, \( x_n, u \in [0, 1] \) and so \( \alpha(x_n, u) = \alpha(u, x_n) = 1 \) for all \( n \). Moreover, there exists \( x_0 \in X \) such that \( \alpha(x_0, Ax_0) \geq 1 \) and \( \alpha(Ax_0, x_0) \geq 1 \). In fact, for \( x_0 = 1 \), we have \( \alpha(1, A1) = \alpha(1, \frac{1}{\sqrt{3}}) = 1 \) and \( \alpha(A1, 1) = \alpha(\frac{1}{\sqrt{3}}, 1) = 1 \).

Thus, all hypotheses of Theorem 2.9 are verified. Here, \( \{0, 2\} \) is the set of common fixed points of \( A \) and \( B \).

The mappings considered in above example have two common fixed points which are 0 and 2. Note that \( \alpha(0, 2) = 0 \), which is not greater than 1. So for the uniqueness, we need the following additional condition.

\((U)\) For all \( x, y \in CF(A, B) \), we have \( \alpha(x, y) \geq 1 \), where \( CF(A, B) \) denotes the set of common fixed points of \( A \) and \( B \).

**Theorem 2.16.** Adding condition \((U)\) to the hypotheses of Theorem 2.2 (resp. Theorem 2.9 with \( \psi(t) < \frac{1}{2s} \) for all \( t > 0 \)), we obtain that \( u \) is the unique common fixed point of \( A \) and \( B \).

**Proof.** In Theorem 2.2, mention that \( \psi(t) < \frac{1}{2s^2} \) implies \( \psi(t) < \frac{1}{2s} \). We argue by contradiction, that is, there exist \( u, v \in X \) such that \( u = Au = Bu \) and \( v = Av = Bv \) with \( u \neq v \). By assumption \((U)\), we have \( \alpha(u, v) \geq 1 \). So by (2.1) and since \( \psi(t) < \frac{1}{2s} \), we have

\[
d(u, v) = d(Au, Bv) \leq \psi(M(u, v))) \leq \psi(\max\{d(u, v), d(u, u), d(v, v), \frac{d(u, v)}{2s} \})
= \psi(\max\{d(u, v), d(u, u), d(v, v)\})
\leq \psi(\max\{d(u, v), 2s d(u, v)\}) = \psi(2s d(u, v)) < d(u, v),
\]

which is a contradiction. Hence, \( u = v \). \( \square \)

**Corollary 2.17.** Let \((X, d)\) be a complete \( b \)-metric-like space, \( \psi \in \Psi_s \) and \( A, B : X \to X \) be given mappings. Suppose that 

\[
d(Ax, By) \leq \psi(M(x, y)), \tag{2.30}
\]

for all \( x, y \in X \), where \( M(x, y) \) is defined by (2.2). If \( \psi(t) < \frac{1}{2s} \) for all \( t > 0 \), then \( A \) and \( B \) have a unique common fixed point.

**Proof.** It suffices to take \( \alpha(x, y) = 1 \) in Corollary 2.11 The uniqueness of \( u \) follows from Theorem 2.16. \( \square \)
Corollary 2.18. Let \((X, d)\) be a complete b-metric-like space and \(A, B : X \to X\) be given mappings. Suppose there exists \(k \in [0, \frac{1}{s})\) such that
\[
d(Ax, By) \leq kM(x, y),
\]
(2.31)
for all \(x, y \in X\), where \(M(x, y)\) is defined by (2.2). Then, \(A\) and \(B\) have a unique common fixed point.

Proof. It suffices to take \(\psi(t) = kt\) in Corollary 2.17. The uniqueness of \(u\) follows from Theorem 2.16. \(\Box\)

The following example illustrates Theorem 2.2 where \(A\) and \(B\) have a unique common fixed point.

Example 2.19. Take \(X = [0, \frac{2}{3}]\) endowed with the complete b-metric-like \(d(x, y) = x^2 + y^2 + (x - y)^2\) with \(s = 2\). Consider the mappings \(A, B : X \to X\) given by
\[
Ax = \begin{cases} \ln(1 + \frac{x}{3}) & \text{if } x \in [0, 1] \\ x - 1 + \ln \frac{4}{3} & \text{if } x \in (1, \frac{2}{3}] \end{cases}, \quad Bx = \begin{cases} \ln(1 + \frac{x}{3}) & \text{if } x \in [0, 1] \\ x + \ln(1 + \frac{4}{3}) - 1 & \text{if } x \in (1, \frac{2}{3}] \end{cases}.
\]

Define the mapping \(\alpha : X \times X \to [0, \infty)\) by
\[
\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise}. \end{cases}
\]

Let \(\psi(t) = \frac{1}{2}t\). It is obvious that
(i) \((A, B)\) is an \(\alpha\)-admissible pair;
(ii) there exists \(x_0 \in X\) such that \(\alpha(x_0, Ax_0) \geq 1\) and \(\alpha(Ax_0, x_0) \geq 1\);
(iii) \(A\) and \(B\) are continuous on \((X, d)\);
(iv) \(\psi(t) < \frac{t}{2}\).

Now, we shall show that \((A, B)\) is an \((\alpha, \psi)\)-contraction. Let \(x, y \in X\) such that \(\alpha(x, y) \geq 1\). So, \(x, y \in [0, 1]\).

We have
\[
d(Ax, By) = (Ax)^2 + (By)^2 + (Ax - By)^2
\]
\[
= (\ln(1 + \frac{x}{3}))^2 + (\ln(1 + \frac{y}{3}))^2 + (\ln(1 + \frac{x}{3}) - \ln(1 + \frac{y}{3}))^2
\]
\[
\leq \left(\frac{x^2}{3} + \frac{y^2}{3} + \frac{1}{9}(x - y)^2\right) = \frac{1}{9}[x^2 + y^2 + (x - y)^2] = \frac{1}{9}d(x, y) \leq \psi(M(x, y)).
\]

Thus, all hypotheses of Theorem 2.2 are verified. Here, 0 is the unique common fixed points of \(A\) and \(B\).

3. Fixed Point Theorems for generalized cyclic contractions

In 2003, Kirk et al. \cite{22} introduced the concepts of cyclic mappings and cyclic contractions. For papers dealing with cyclic contractions, see \cite{7, 10, 25}. We recall some definitions from \cite{22}.

Definition 3.1 (\cite{22}). Let \(F\) and \(G\) be nonempty subsets of a space \(X\). A mapping \(T : F \cup G \to F \cup G\) is called cyclic if \(T(F) \subseteq G\) and \(T(G) \subseteq F\).

Definition 3.2 (\cite{22}). Let \(F\) and \(G\) be nonempty subsets of a metric space \((X, d)\). A mapping \(T : F \cup G \to F \cup G\) is called a cyclic contraction if there exists \(k \in [0, 1)\) such that
\[
d(Tx, Ty) \leq kd(x, y),
\]
(3.1)
for all \(x \in F\) and \(y \in G\).
Now, we introduce the concept of new generalized cyclic contractive pairs in the setting of $b$-metric-like spaces.

**Definition 3.3.** Let $F$ and $G$ be nonempty closed subsets of a $b$-metric-like space $(X,d)$, $\alpha : X \times X \rightarrow [0,\infty)$, $\psi \in \Psi_s$ and $A, B : X \rightarrow X$ be mappings. The pair $(A,B)$ is called a cyclic $(\alpha,\psi,F,G)$-contraction pair if

(i) $F \cup G$ has a cyclic representation w.r.t. the pair $(A,B)$, that is, $A(F) \subset G$ and $B(G) \subset F$;

(ii) $d(Ax, By) \leq \psi(M(x,y))$, \hspace{1cm} (3.2)

for all $x \in F$ and $y \in G$ satisfying $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$, where

$M(x,y) = \max\{d(x,y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4s}\}$.

Now, we state and prove the following results.

**Theorem 3.4.** Let $(X,d)$ be a complete $b$-metric-like space and $F$ and $G$ be nonempty closed subsets of $X$. Suppose that $A, B : X \rightarrow X$ is a cyclic $(\alpha,\psi,F,G)$-contraction pair and the following conditions hold:

(i) $\alpha(Ax, B Ax) \geq 1$ for all $x \in F$ and $\alpha(Bx, A Bx) \geq 1$ for all $x \in G$;

(ii) $A$ or $B$ is continuous on $(X,d)$;

(iii) if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \rightarrow z$ as $n \rightarrow \infty$, then $\alpha(z,z) \geq 1$;

(iv) $\psi(t) < \frac{t}{2^{n+1}}$ for each $t > 0$.

Then, $A$ and $B$ have a common fixed point in $F \cap G$.

**Proof.** Let $x_0 \in F$ and $x_1 = Ax_0$. Since $A(F) \subset G$, then $x_1 \in G$. Also, let $x_2 = Bx_1 = B Ax_0$. Since $B(G) \subset F$, then $x_2 \in F$. Continuing in this fashion, we can construct a sequence $\{x_n\}$ in $X$ such that

$x_{2n+2} = B x_{2n+1} \in F$, \hspace{0.5cm} x_{2n+1} = A x_{2n} \in G$, \hspace{0.5cm} \forall n \geq 0.$

By condition (i), we have $\alpha(x_1, x_2) = \alpha(Ax_0, B Ax_0) \geq 1$ and $\alpha(x_2, x_3) = \alpha(Bx_1, A Bx_1) \geq 1$. Continuing this process, we get

$\alpha(x_n, x_{n+1}) \geq 1$, \hspace{0.5cm} \forall n \geq 0.$

Following the proof of Theorem 2.2, we know that the sequence $\{x_n\}$ is Cauchy in $(X,d)$ and converges to some $u \in X$ with $d(u, u) = 0$. We shall show that $u$ is a common fixed point of $A$ and $B$ in $F \cap G$. Since $\{x_{2n}\}$ is a sequence in the closed set $F$ and $\{x_{2n}\}$ converges to $u$, then $u \in F$. Also, $\{x_{2n+1}\}$ is a sequence in the closed set $G$ and $\{x_{2n+1}\}$ converges to $u$, then $u \in G$. We deduce that $u \in F \cap G$.

First, assume that $A$ is continuous on $(X,d)$. Since $\{x_{2n}\}$ converges to $u$, so $\{x_{2n+1} = A x_{2n}\}$ converges to $Au$.

On the other hand, \hspace{0.5cm} $\lim_{n \rightarrow \infty} d(x_n, u) = 0 = d(u, u)$ and by Lemma 1.9, we have

$\frac{1}{s}d(u, Au) \leq d(Au, Au) \leq sd(u, Au).$

If $d(Au, Bu) = 0$, then $Au = Bu$. Moreover, the fact that $d(u, Au) \leq sd(Au, Au)$ implies

$0 \leq d(u, Au) \leq sd(Au, Au) \leq 2s^2d(Au, Bu) = 0,$

and so $Au = u$. Then, $Bu = Au = u$ and so $u$ is a common fixed point of $A$ and $B$.

Suppose by contradiction that $d(Au, Bu) > 0$. Since $u \in F \cap G$ and by (iii), it follows that $\alpha(u, u) \geq 1$, so that

$d(Au, Bu) \leq \psi(M(u, u)),$
where
\[ M(u, u) = \max\{d(u, u), d(u, Au), d(u, Bu), \frac{d(u, Bu) + d(u, Au)}{4s}\} \]
\[ = \max\{0, d(u, Au), \frac{d(u, Bu) + d(u, Au)}{4s}\} \]
\[ = \max\{d(u, Au), d(u, Bu)\} \leq \max\{d(u, Au), sd(u, Au) + sd(Au, Bu)\} \]
\[ = sd(u, Au) + sd(Au, Bu) \leq 2s^3d(Au, Bu) + sd(Au, Bu) = (2s^3 + s)d(Au, Bu). \]

Then
\[ d(Au, Bu) \leq \psi((2s^3 + s)d(Au, Bu)) < d(Au, Bu), \]
which is a contradiction.

The proof is similar when \( B \) is assumed to be continuous on \((X, d)\). \( \square \)

**Theorem 3.5.** Let \((X, d)\) be a complete \(b\)-metric-like space and \(F\) and \(G\) be nonempty closed subsets of \(X\). Suppose that \(A, B : X \to X\) is a cyclic \((\alpha, \psi, F, G)\)-contraction pair and the following conditions hold:

(i) \( \alpha(Ax, BAx) \geq 1\) for all \( x \in F \) and \( \alpha(Bx, ABx) \geq 1\) for all \( x \in G\);
(ii) \(A\) and \(B\) are continuous on \((X, d)\);
(iii) if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \( n \geq 0\) and \(x_n \to z\) as \(n \to \infty\), then \(\alpha(z, z) \geq 1\);
(iv) \(\psi(t) < \frac{1}{2s}\) for each \( t > 0 \).

Then, \(A\) and \(B\) have a common fixed point in \(F \cap G\).

**Proof.** The proof is similar to the proofs of Theorem 3.4 and Theorem 2.2. \( \square \)

Theorem 3.4 and Theorem 3.5 can be proved without assuming the continuity of \(A\) or the continuity of \(B\). For this instance, we suppose that \(X\) has the following property:

(R) If \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \( n \) and \(x_n \to x\) in \(X\) as \(n \to \infty\), then there exists a subsequence \(\{x_{n(k)}\}\) of \(\{x_n\}\) such that \(\alpha(x_{n(k)}), x) \geq 1\) for all \(k\).

This statement is given as follows.

**Theorem 3.6.** Let \((X, d)\) be a complete \(b\)-metric-like space and \(F\) and \(G\) be nonempty closed subsets of \(X\). Suppose that \(A, B : X \to X\) is a cyclic \((\alpha, \psi, F, G)\)-contraction pair and the following conditions hold:

(i) \( \alpha(Ax, BAx) \geq 1\) for all \( x \in F \) and \( \alpha(Bx, ABx) \geq 1\) for all \( x \in G\);
(ii) \( (R)\) holds;
(iii) if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \( n \geq 0\) and \(x_n \to z\) as \(n \to \infty\), then \(\alpha(z, z) \geq 1\);
(iv) \(\psi(t) < \frac{1}{2s}\) for each \( t > 0 \).

Then, \(A\) and \(B\) have a common fixed point in \(F \cap G\).

**Proof.** The proof is similar to that of Theorem 3.4 and Theorem 2.9. \( \square \)

Taking \( A = B \) in Theorem 3.5 and Theorem 3.6 we state the followings results.
Corollary 3.7. Let \((X,d)\) be a complete \(b\)-metric-like space and \(F\) and \(G\) be nonempty closed subsets of \(X\). Suppose that \(\psi \in \Psi_b\), \(\alpha : X \times X \to X\) and \(A : X \to X\) such that
\[
d(Ax, Ay) \leq \psi(\max\{d(x, y), d(x, Ax), d(y, Ay), \frac{d(Ax, y) + d(x, Ay)}{4s}\})
\]
for all \(x \in F\) and \(y \in G\) satisfying \(\alpha(x, y) \geq 1\) or \(\alpha(y, x) \geq 1\).

Also, suppose the following conditions hold:

(i) \(\alpha(Ax, AAx) \geq 1\) for all \(x \in F \cap G\);
(ii) \(A\) is a cyclic mapping;
(iii) \(A\) is continuous on \((X,d)\);
(iv) \(\psi(t) < \frac{t}{s^2}\) for each \(t > 0\).

Then, \(A\) has a fixed point in \(F \cap G\).

Corollary 3.8. Let \((X,d)\) be a complete \(b\)-metric-like space and \(F\) and \(G\) be nonempty closed subsets of \(X\). Suppose that \(\psi \in \Psi_b\), \(\alpha : X \times X \to X\) and \(A : X \to X\) a mapping such that
\[
d(Ax, Ay) \leq \psi(\max\{d(x, y), d(x, Ax), d(y, Ay), \frac{d(Ax, y) + d(x, Ay)}{4s}\}),
\]
for all \(x \in F\) and \(y \in G\) satisfying \(\alpha(x, y) \geq 1\) or \(\alpha(y, x) \geq 1\).

Also, suppose the following conditions hold:

(i) \(\alpha(Ax, AAx) \geq 1\) for all \(x \in F \cap G\);
(ii) \(A\) is a cyclic mapping;
(iii) \(\alpha \) is a cyclic mapping;
(iv) \(\psi(t) < \frac{t}{s^2}\) for each \(t > 0\).

Then, \(A\) has a fixed point in \(F \cap G\).

Now, we give an example to illustrate Theorem 3.6.

Example 3.9. Let \(X = \{0, 1, 2\}\) and \(d : X \times X \to [0, \infty)\) defined by
\[
d(0, 0) = 9,\ \ d(1, 1) = 0,\ \ d(2, 2) = 0,\ \ d(0, 1) = d(1, 0) = 16,\ \ d(0, 2) = d(2, 0) = 9\ \text{and}\ \ d(1, 2) = d(2, 1) = 49.
\]

Then, \((X,d)\) is a complete \(b\)-metric-like space with coefficient \(s = 2\). Let \(F = \{0, 1\}\) and \(G = \{1, 2\}\). Note that \(F\) and \(G\) are nonempty closed subsets of \(X\). Consider the mappings \(A, B : X \to X\) and \(\alpha : X \times X \to X\) as follows:
\[
A0 = 2,\ \ A1 = 1,\ \ A2 = 0,\ \ B0 = 0,\ \ B1 = 1\ \text{and}\ \ B2 = 1
\]
and
\[
\begin{cases}
\alpha(1, 1) = \alpha(2, 1) = 1; \\
\alpha(x, y) = 0\ \text{otherwise}.
\end{cases}
\]

Now, we show that all the conditions of Theorem 3.6 are satisfied.
We show that condition (i) of Theorem 3.6 is verified. Let \( x \in F \), then
\[
\alpha(Ax, BAx) = \begin{cases} 
\alpha(2, 1) = 1 & \text{if } x = 0; \\
\alpha(1, 1) = 1 & \text{if } x = 1. 
\end{cases}
\]
Also, let \( x \in G \), then
\[
\alpha(Bx, ABx) = \begin{cases} 
\alpha(1, 1) = 1 & \text{if } x = 1; \\
\alpha(1, 1) = 1 & \text{if } x = 2. 
\end{cases}
\]
Then, \( \alpha(Ax, BAx) \geq 1 \) for all \( x \in F \) and \( \alpha(Bx, ABx) \geq 1 \) for all \( x \in G \).

It is clear that \( A(F) \subset G \) and \( B(G) \subset F \).

Now, we sow that \((A, B)\) is a cyclic \((\alpha, \psi, F, G)\)-contraction pair.

Let \( x \in F \) and \( y \in G \) such that \( \alpha(x, y) \geq 1 \) or \( \alpha(y, x) \geq 1 \). It follows from definition of \( \alpha \) that \((x = y = 1)\) or \((x = 1, y = 2)\). We have for \((x = y = 1)\) or \((x = 1, y = 2)\)
\[
d(Ax, By) = d(1, 1) = 0 \leq \psi(M(x, y)),
\]
for all \( \psi \in \Psi_s \) such that \( \psi(t) < \frac{t}{2} \) for all \( t > 0 \). Then, \((A, B)\) is a cyclic \((\alpha, \psi, F, G)\)-contraction pair.

It is easy to show that \( X \) satisfies the property \((R)\). Moreover, condition (iii) of Theorem 3.6 holds. Hence, all conditions of Theorem 3.6 are verified. Here, 1 is the unique common fixed point of \( A \) and \( B \).

4. Fixed Point Theorems for generalized contractions in partially ordered \( b \)-metric-like spaces

Now, we give some fixed points results on partially ordered \( b \)-metric-like spaces as consequences of our results presented in the last section.

**Definition 4.1.** Let \( X \) be a nonempty set. We say that \((X, d, \preceq)\) is a partially ordered \( b \)-metric-like space if \((X, d)\) is a \( b \)-metric-like space and \((X, \preceq)\) is a partially ordered set.

**Definition 4.2.** Let \( F \) and \( G \) be nonempty closed subsets of a partially ordered \( b \)-metric-like space \((X, d, \preceq)\), \( \psi \in \Psi_s \) and \( A, B : X \rightarrow X \) be mappings. The pair \((A, B)\) is called a cyclic \((\psi, F, G)\)-contraction pair if

(i) \( F \cup G \) has a cyclic representation w.r.t. the pair \((A, B)\);

(ii) \[
d(Ax, By) \leq \psi(M(x, y)),
\]
for all \( x \in F \) and \( y \in G \) satisfying \( x \preceq y \) or \( y \preceq x \), where
\[
M(x, y) = \max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4s}\}.
\]

**Definition 4.3.** Let \((X, d, \preceq)\) a partially ordered \( b \)-metric-like space and \( F, G \) be nonempty closed subsets of \( X \) with \( X = F \cup G \). Let \( A, B : X \rightarrow X \) be mappings. We say that the pair \((A, B)\) is \((F, G)\)-weakly increasing if \( Ax \preceq BAx \) for all \( x \in F \) and \( Bx \preceq ABx \) for all \( x \in G \).

Now, we state and prove the following results.

**Theorem 4.4.** \((X, d, \preceq)\) be a complete partially ordered \( b \)-metric-like space and \( F, G \) be nonempty closed subsets of \( X \). Suppose that \( A, B : X \rightarrow X \) is a cyclic \((\psi, F, G)\)-contraction pair and the following conditions hold:

(i) \((A, B)\) is \((F, G)\)-weakly increasing;

(ii) \( A \) or \( B \) is continuous on \((X, d)\);

(iii) \( \psi(t) < \frac{t}{2\sqrt{s} + s} \) for each \( t > 0 \).
Then, $A$ and $B$ have a common fixed point in $F \cap G$.

**Proof.** Let the function $\alpha : X \times X \to X$ such that

$$\alpha(x, y) = \begin{cases} 
1 & \text{if } x \preceq y; \\
0 & \text{otherwise.}
\end{cases}$$

Then, all hypotheses of Theorem 3.4 are satisfied and hence $A$ and $B$ have a common fixed point in $F \cap G$.

Also, by using the same technique, we have the following results.

**Theorem 4.5.** $(X, d, \preceq)$ be a complete partially ordered $b$-metric-like space and $F, G$ be nonempty closed subsets of $X$. Suppose that $A, B : X \to X$ is a cyclic $(\psi, F, G)$-contraction pair and the following conditions hold:

(i) $(A, B)$ is $(F, G)$-weakly increasing;

(ii) $A$ and $B$ are continuous on $(X, d)$;

(iii) $\psi(t) < \frac{t^2}{s^2}$ for each $t > 0$.

Then, $A$ and $B$ have a common fixed point in $F \cap G$.

**Theorem 4.6.** $(X, d, \preceq)$ be a complete partially ordered $b$-metric-like space and $F, G$ be nonempty closed subsets of $X$. Suppose that $A, B : X \to X$ is a cyclic $(\psi, F, G)$-contraction pair and the following conditions hold:

(i) $(A, B)$ is $(F, G)$-weakly increasing;

(ii) for a sequence $\{x_n\} \subset X$ with $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to z$ in $(X, d)$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq z$, for all $k \in \mathbb{N}$;

(iii) $\psi(t) < \frac{t}{s}$ for each $t > 0$.

Then, $A$ and $B$ have a common fixed point in $F \cap G$.

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**References**


