A new kind of generalized fuzzy integrals

Cuilian You*, Hongyan Ma, Huae Huo

College of Mathematics and Information Science, Hebei University, Baoding 071002, China.

Communicated by Y. J. Cho

Abstract

Fuzzy integral is an important tool to study fuzzy differential equations. Under normal circumstances, there are two basic limitations: limited of integral interval and boundedness of integrand. However, in practical problems, it is difficult to calculate when integral interval is not common interval. Then fuzzy integral on infinite interval is taken into consideration. In this paper, definition of a kind of generalized Liu integral is given. Moreover, properties and theorems of this kind of generalized fuzzy integral are obtained.

Keywords: Fuzzy variable, fuzzy process, Liu process, generalized fuzzy integral.

1. Introduction

In real world, there exist many fuzzy phenomena. The uncertainty of fuzzy phenomenon is a basic type of subjective uncertainty which is characterized by membership function given by experts. To describe a set without definite boundary, fuzzy set was initiated by Zadeh [20] in 1965, and a possibility measure was presented by Zadeh [21] in 1978. However, possibility measure has no self duality. Then credibility measure, a self-duality measure was introduced by Liu and Liu [11] in 2002. A sufficient and necessary condition for credibility measure was given by Li and Liu [6] in 2006. Credibility theory, founded by Liu [7] in 2004 and refined by Liu [9] in 2007, is a branch of mathematics for studying the behavior of fuzzy phenomena. A survey of credibility theory can be found in Liu [8], and interested reader may consult the book [9].

There are many types of fuzzy integrals in literatures, such as Choquet fuzzy integral and Sugeno fuzzy integral (see[5], [15] and [16]). However, these fuzzy integrals are all integrals with respect to variable,
which have any relationship with fuzzy process. To describe dynamic fuzzy phenomena, a fuzzy process (Liu process), a differential formula (Liu formula) and a fuzzy integral (Liu integral) were introduced by Liu [10] in 2008. Here the fuzzy integral is the integral of fuzzy process with respect to Liu process. As for Liu process, some researches concerning have been done. You, Huo and Wang [17] extended Liu process, Liu integral and Liu formula to the case of multi-dimensional. Complex Liu process was studied by Qin and Wen [14]. Dai [2] and Dai [3] proposed Lipschitz continuity and reflection principle of Liu process. Some properties of Liu integral were studied by You and Wang [18]. You, Wang and Huo [19] discussed existence and uniqueness theorems for some special fuzzy differential equations. Chen and Qin [1] studied a new existence and uniqueness theorem for fuzzy differential equations, which is a general case. Liu process has also been applied to stock model and fuzzy finance. A basic stock model was proposed by Liu [10], which is called Liu’s stock model. Since then, fuzzy calculus was widely used in finance. Assumed that stock price is modeled by geometric Liu process, Qin and Li [13] first deduced option pricing formula for European option. Most results concerning fuzzy finance were studied by Gao [4], Peng [12]. Fuzzy process is also used in control fields by Zhu [22].

The purpose of this paper is to discuss generalized fuzzy integral based on credibility theory. The structure of this paper is as follows: In Section 2 of this paper, some concepts and results of Liu integral will be given as preliminaries. The definitions and properties of infinite Liu integral will be discussed in Section 3. In the end, a brief summary is given in Section 4.

2. Preliminaries

In the setting of credibility theory, let $T$ be an index set, $\Theta$ an empty set, $\mathcal{P}$ the power set of $\Theta$ and $\text{Cr}$ a credibility measure. Then $(\Theta, \mathcal{P}, \text{Cr})$ is called a credibility space. A fuzzy process $X_t(\theta)$ is defined as a function from $T \times (\Theta, \mathcal{P}, \text{Cr})$ to the set of real numbers, where $t$ is time and $\theta$ is a point in credibility space $(\Theta, \mathcal{P}, \text{Cr})$. In other words, $X_t(\theta)$ is a fuzzy variable for each $t$; $X_t(\theta)$ is a function of $t$ for any given $\theta \in \Theta$, such a function is called a sample path of $X_t(\theta)$. For simplicity, we use the symbol $X_t$ to replace $X_t(\theta)$ in the following sections.

A fuzzy process $X_t$ is called continuous if the sample paths of $X_t$ are all continuous functions of $t$ for almost all $\theta \in \Theta$.

**Definition 2.1** ([10]). A fuzzy process $C_t$ is said to be a Liu process if

(i) $C_0 = 0$,

(ii) $C_t$ has stationary and independent increments,

(iii) every increment $C_{t+s} - C_s$ is a normally distributed fuzzy variable with expected value $et$ and variance $\sigma^2t^2$, whose membership function is

$$
\mu(x) = 2 \left(1 + \exp \left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \quad -\infty < x < +\infty.
$$

The Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

**Definition 2.2.** (Liu integral, [10]) Let $X_t$ be a fuzzy process and let $C_t$ be a standard Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$
\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.
$$

Then the Liu integral of $X_t$ with respect to $C_t$ is defined as follows,

$$
\int_a^b X_t \, dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})
$$
provided that the limitation exists almost surely and is a fuzzy variable. In this case, $X_t$ is called Liu integrable.

**Theorem 2.3 (Liu Formula, [10]).** Let $C_t$ be a standard Liu process and let $h(t,x)$ be a continuously differentiable function. If fuzzy process $X_t$ is given by $dX_t = u_t dt + v_t dC_t$, where $u_t, v_t$ are absolutely integrable fuzzy process and Liu integrable fuzzy process, respectively. Define $Y_t = h(t, X_t)$. Then

$$dY_t = \frac{\partial h}{\partial t}(t, X_t) dt + \frac{\partial h}{\partial x}(t, X_t) dX_t,$$

which is called Liu formula.

**Theorem 2.4 ([18]).** Let $a < k < b$. If fuzzy process $X_t$ is Liu integrable on any closed interval $[a,k]$ and $[k,b]$, then $X_t$ is Liu integral for closed interval $[a,b]$, and

$$\int_a^b X_t dC_t = \int_a^k X_t dC_t + \int_k^b X_t dC_t.$$

**Theorem 2.5 ([18]).** Let fuzzy process $X_t$ and $Y_t$ be Liu integrable on closed interval $[a,b]$. Then

$$\int_a^b (k_1 X_t + k_2 Y_t) dC_t = k_1 \int_a^b X_t dC_t + k_2 \int_a^b Y_t dC_t$$

for any real numbers $k_1$ and $k_2$.

3. Infinite Liu Integral

This section aims to give the definition of infinite Liu integral and discuss some properties of infinite Liu integral.

**Definition 3.1.** Let $X_t$ be a fuzzy process and let $C_t$ be a standard Liu process. Suppose $X_t$ is defined on interval $(a, +\infty)$ and integrable on any finite closed interval $[a, u]$ with respect to $C_t$. If the limitation

$$\lim_{u \to +\infty} \int_a^u X_t dC_t = J(\theta)$$

exists almost surely and is a fuzzy variable, then the limitation $J(\theta)$ is called infinite Liu integral of fuzzy process $X_t$ on interval $(a, +\infty)$ (For short, infinite Liu integral). Denote

$$\int_a^{+\infty} X_t dC_t = J(\theta).$$

In this case, $\int_a^{+\infty} X_t dC_t$ is called convergent almost surely to $J(\theta)$.

On the contrary, if the limitation $\lim_{u \to +\infty} \int_a^u X_t dC_t$ does not exist, $\int_a^{+\infty} X_t dC_t$ is called divergent.

**Example 3.2.** Let $C_t$ be a standard Liu process and $C_t \to +\infty$ when $t \to +\infty$. Discuss the convergence of infinite Liu integral $\int_0^{+\infty} \exp(-C_t) dC_t$.

By using Liu formula, we have

$$d\exp(-C_t) = \exp(-C_t) d(-C_t),$$

that is

$$\int_0^{+\infty} \exp(-C_t) dC_t = - \int_0^{+\infty} \exp(-C_t) d(-C_t)$$

$$= - \int_0^{+\infty} \exp(-C_t) = - \lim_{u \to +\infty} \exp(-C_u) + 1 = 1.$$

Thus infinite Liu integral $\int_0^{+\infty} \exp(-C_t) dC_t$ is convergent almost surely.
Example 3.3. Let $C_t$ be a standard Liu process. If $t \to +\infty$, $C_t \to +\infty$, discuss the convergence of the following infinite Liu integral $\int_0^{+\infty} \frac{1}{1 + C_t^2} dC_t$.

It follows from Liu formula that

$$\int_0^{+\infty} \frac{1}{1 + C_t^2} dC_t = \lim_{u \to +\infty} \arctan C_t,$$

then

$$\int_0^{+\infty} \frac{1}{1 + C_t^2} dC_t = \int_0^{+\infty} \arctan C_t = \lim_{u \to +\infty} \arctan C_u = \frac{\pi}{2},$$

thus $\int_0^{+\infty} \frac{1}{1 + C_t^2} dC_t$ is convergent almost surely.

The definition shows that the convergence or divergence of infinite Liu integral $\int_a^{+\infty} X_t dC_t$ is determined by the existence of limitation of Liu integral $\lim_{u \to +\infty} \int_a^u X_t dC_t$.

Next, some properties of infinite Liu integral will be derived.

Theorem 3.4. If infinite Liu integral $\int_a^{+\infty} X_t dC_t$ is convergent almost surely, then there exists a fuzzy event $A$ with $\text{Cr}\{A\} = 1$ and a real number $G \geq a$ such that for every $\varepsilon(\theta) > 0$, we have

$$\left| \int_a^{u_2} X_t dC_t - \int_a^{u_1} X_t dC_t \right| = \left| \int_{u_1}^{u_2} X_t dC_t \right| < \varepsilon(\theta),$$

if $u_1, u_2 > G$, for each $\theta \in A$.

Proof. Since $\int_a^{+\infty} X_t dC_t$ is convergent almost surely, denoting the limitation by $J(\theta)$, we know there exists a fuzzy event $A$ with $\text{Cr}\{A\} = 1$ such that $\lim_{G \to +\infty} \int_a^G X_t dC_t = J(\theta)$ for each $\theta \in A$.

Fix $\varepsilon(\theta) > 0$, by the definition of limitation, there exists $G \geq a$ such that

$$\left| \int_a^{u_1} X_t dC_t - J(\theta) \right| < \varepsilon(\theta), \quad \left| \int_a^{u_2} X_t dC_t - J(\theta) \right| < \varepsilon(\theta),$$

when $u_1 > G, u_2 > G$, for each $\theta \in A$.

According to Theorem 2.4, we have

$$\left| \int_{u_1}^{u_2} X_t dC_t \right| = \left| \int_a^{u_2} X_t dC_t - \int_a^{u_1} X_t dC_t \right|$$

$$= \left| \int_a^{u_1} X_t dC_t - J(\theta) - \int_a^{u_2} X_t dC_t + J(\theta) \right|$$

$$\leq \left| \int_a^{u_1} X_t dC_t - J(\theta) \right| + \left| \int_a^{u_2} X_t dC_t - J(\theta) \right|$$

$$< 2\varepsilon(\theta).$$

The theorem is proved.

Theorem 3.5. Let $C_t$ be a standard Liu process. If infinite Liu integral $\int_a^{+\infty} X_t dC_t$ and $\int_a^{+\infty} Y_t dC_t$ are both convergent almost surely, then infinite Liu integral $\int_a^{+\infty} (k_1 X_t + k_2 Y_t) dC_t$ is convergent, and

$$\int_a^{+\infty} (k_1 X_t + k_2 Y_t) dC_t = k_1 \int_a^{+\infty} X_t dC_t + k_2 \int_a^{+\infty} Y_t dC_t$$

for any constant $k_1$ and $k_2$. 

Proof.
Proof. Since infinite Liu integrals $\int_{a}^{+\infty} X_t dC_t$ and $\int_{a}^{+\infty} Y_t dC_t$ are both convergent almost surely, then $\lim_{u \to +\infty} \int_{a}^{u} X_t dC_t$ and $\lim_{u \to +\infty} \int_{a}^{u} Y_t dC_t$ exist. Let

$$\lim_{u \to +\infty} \int_{a}^{u} X_t dC_t = J(\theta), \quad \lim_{u \to +\infty} \int_{a}^{u} X_t dC_t = K(\theta).$$

It follows from Theorem 2.4 that

$$\int_{a}^{+\infty} (k_1 X_t + k_2 Y_t) dC_t = \lim_{u \to +\infty} \int_{a}^{+\infty} k_1 X_t dC_t + \lim_{u \to +\infty} k_2 \int_{a}^{+\infty} Y_t dC_t = k_1 \lim_{u \to +\infty} \int_{a}^{+\infty} X_t dC_t + k_2 \lim_{u \to +\infty} \int_{a}^{+\infty} Y_t dC_t = k_1 J(\theta) + k_2 K(\theta).$$

Hence

$$\int_{a}^{+\infty} (k_1 X_t + k_2 Y_t) dC_t = k_1 \int_{a}^{+\infty} X_t dC_t + k_2 \int_{a}^{+\infty} Y_t dC_t.$$

The theorem is proved. \qed

**Theorem 3.6.** Let $X_t$ be Liu integrable fuzzy process on any finite closed interval $[a, b]$, and $a < b$. Then infinite Liu integral $\int_{a}^{+\infty} X_t dC_t$ and $\int_{b}^{+\infty} X_t dC_t$ are convergent or divergent at the same time and

$$\int_{a}^{+\infty} X_t dC_t = \int_{a}^{b} X_t dC_t + \int_{b}^{+\infty} X_t dC_t.$$

**Proof.** It follows from the definition of infinite Liu integral and Theorem 2.4 that

$$\int_{a}^{+\infty} X_t dC_t = \lim_{c \to +\infty} \int_{a}^{c} X_t dC_t = \lim_{c \to +\infty} \left( \int_{a}^{b} X_t dC_t + \int_{b}^{c} X_t dC_t \right) = \int_{a}^{b} X_t dC_t + \lim_{c \to +\infty} \int_{b}^{c} X_t dC_t = \int_{a}^{b} X_t dC_t + \int_{b}^{+\infty} X_t dC_t.$$

The theorem is proved. \qed

**Theorem 3.7.** Let $C_t$ be a standard Liu process and $F(t)$ be an absolutely continuous function. If $\lim_{t \to +\infty} F(t)$ and $\lim_{t \to +\infty} C_t$ exist, then

$$\int_{0}^{+\infty} F(t) dC_t = \lim_{t \to +\infty} F(t) C_t - \int_{0}^{+\infty} C_t dF(t).$$

**Proof.** Taking $h(t, C_t) = F(t) dC_t$, it follows from Liu Formula that

$$d(F(t)C_t) = C_t dF(t) + F(t) dC_t.$$

Thus

$$\lim_{t \to +\infty} F(t) C_t = \int_{0}^{+\infty} d(F(t)C_t) = \int_{0}^{+\infty} C_t dF(t) + \int_{0}^{+\infty} F(t) dC_t,$$

that is

$$\int_{0}^{+\infty} F(t) dC_t = \lim_{t \to +\infty} F(t) C_t - \int_{0}^{+\infty} C_t dF(t).$$

The theorem is proved. \qed
4. Conclusions

This paper was mainly to extend Liu integral to a kind of generalized Liu integral, that is Liu integral on infinite interval. The results of this paper can be summarized as follows: (a) the definition of infinite Liu integral was presented; (b) some properties of infinite Liu integral were given, which include linear properties, the additivity of integral interval, the formula of integration by parts and etc..

Acknowledgement

This work was supported by Natural Science Foundation of China Grant No. 11201110, 61374184, and 11401157, and Outstanding Youth Science Fund of the Education Department of Hebei Province No. Y2012021.

References