Numerical solution of $n$'th order fuzzy initial value problems by six stages

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Abstract

The purpose of this paper is to present a numerical approach to solve fuzzy initial value problems (FIVPs) involving $n$-th order ordinary differential equations. The idea is based on the formulation of the six stages Runge-Kutta method of order five (RKM56) from crisp environment to fuzzy environment followed by the stability definitions and the convergence proof. It is shown that the $n$-th order FIVP can be solved by RKM56 by transforming the original problem into a system of first-order FIVPs. The results indicate that the method is very effective and simple to apply. An efficient procedure is proposed of RKM56 on the basis of the principles and definitions of fuzzy sets theory and the capability of the method is illustrated by solving second-order linear FIVP involving a circuit model problem. ©2016 All rights reserved.

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1. Introduction

Many dynamical real life problems may be formulated as a mathematical model. These problems can be formulated either as a system of ordinary or partial differential equations [6, 15, 24]. Fuzzy differential
equations (FDEs) are a useful tool to model a dynamical system when information about its behavior is inadequate. Fuzzy Initial value problem (FIVP) appears when the modeling of these problems was imperfect and its nature is under uncertainty. Fuzzy ordinary differential equations are suitable mathematical models to model dynamical systems in which there exist uncertainties or vagueness. These models are used in various applications including, population models [26, 30] mathematical physics [9], and medicine [3, 7]. An ordinary differential equation involving linear FIVP can be considered to be the simplest case to test the effectiveness of proposed methods for solving FDE. The explicit Runge-Kutta methods have been widely used in the solution of FIVPs. Even in the early days of these methods, the conditions for order seemed to have been thoroughly understood and methods of order 4 were successfully used to solve both single FIVPs [27] and systems of FIVPs [2]. The efficiency of Runge-Kutta methods depends on increasing the number of function evaluations per step [19]. In the last ducat various types of Runge Kuttte methods have been used to solve n-th order FIVPs including, second order Runge Kutta method [28], third order Runge Kutta methods [12, 17, 21], fourth order Runge Kutta methods [2, 14], also fifth order Runge Kutta method of five stages in [18] and other methods [22, 31].

Our main motivation is to develop and analyze the use of RKM56 in order to obtain the numerical solution of FIVPs involving fuzzy linear second-order differential equation. To the best of our knowledge, this is the first attempt at solving n-th order FIVP by using RKM56. The structure of this paper is as follows. We will begin in Section 2 with some preliminary concepts about fuzzy numbers. In Section 3 we define the defuzzification procedure of n-th order FIVP. In Section 4 we analyze and formulate RKM56 for solving n-th order FIVP. In Section 5 we present the convergence and error analysis of RKM56 in detail. In Section 6 we employ RKM56 on test example involving second order linear FIVP model and finally, in Section 7 we give the conclusion of this study.

2. Preliminaries

Definition 2.1. Fuzzy numbers are a subset of the real numbers set, and represent uncertain values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. A fuzzy number is called a triangular fuzzy number [8] if defined by three numbers $\alpha < \beta < \gamma$ where the graph of $\mu(x)$ is a triangle with the base on the interval $[\alpha, \beta]$ and vertex at $x = \beta$ and its membership function has the following form:

$$
\mu(x; \alpha, \beta, \gamma) = \begin{cases} 
0, & \text{if } x < \alpha, \\
\frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \leq x \leq \beta, \\
\frac{\gamma-x}{\gamma-\beta}, & \text{if } \beta \leq x \leq \gamma, \\
0, & \text{if } x > \gamma
\end{cases}
$$

Figure 1: Triangular Fuzzy Number
and its \( r \)-level is: [\( \mu _r = [\alpha + r (\beta - \alpha ) , \gamma - r (\gamma - \beta )] \), \( r \in [0, 1] \).]

In this paper the class of all fuzzy subsets of \( \mathbb{R} \) will be denoted by \( \tilde{E} \) and satisfy the following properties [8 23]:

1. \( \mu (t) \) is normal, i.e \( \exists t_0 \in \mathbb{R} \) with \( \mu (t_0) = 1 \).
2. \( \mu (t) \) is convex fuzzy set, i.e. \( \mu (\lambda t + (1 - \lambda ) s) \geq \min \{ \mu (t) , \mu (s) \} \) \( \forall t , s \in \mathbb{R} , \lambda [0, 1] \).
3. \( \mu \) is upper semi-continuous on \( \mathbb{R} \).
4. \( \{ t \in \mathbb{R} : \mu (t) > 0 \} \) is compact.

\( \tilde{E} \) is called the space of fuzzy numbers and \( \mathbb{R} \) is a proper subset of \( \tilde{E} \).

Define the \( - \)-level set \( x \in \mathbb{R} , [\mu ]_r = \{ x \setminus \mu (x) \geq r \} , \) \( 0 \leq r \leq 1 \), where \( [\mu ]_0 = \{ x \setminus \mu (x) > 0 \} \) is compact [30] which is a closed bounded interval and denoted by \( [\mu ]_r = (\mu (t) , \pi (t)) \). In the parametric form [20], a fuzzy number is represented by an ordered pair of functions \( (\mu (t) , \pi (t)) , r \in [0, 1] \) which satisfies:

1. \( \mu (t) \) is a bounded left continuous non-decreasing function over \( [0, 1] \).
2. \( \pi (t) \) is a bounded left continuous non-increasing function over \( [0, 1] \).
3. \( \mu (t) \leq \pi (t) , r \in [0, 1] \). A crisp number \( r \) is simply represented by \( \mu (r) = \pi (r) = r \), \( r \in [0, 1] \).

**Definition 2.2 (21).** If \( \tilde{E} \) be the set of all fuzzy numbers, we say that \( f (t) \) is a fuzzy function if \( f : \mathbb{R} \to \tilde{E} \).

**Definition 2.3 (13).** A mapping \( f : T \to \tilde{E} \) for some interval \( T \subseteq \tilde{E} \) is called a fuzzy function process and we denote \( r \)-level set by:

\[
[f(t)]_r = \{ f(t;r), f(t;r) \} , t \in T , r \in [0, 1].
\]

The \( r \)-level sets of a fuzzy number are much more effective as representation forms of fuzzy sets than the above. Fuzzy sets can be defined by the families of their \( r \)-level sets based on the resolution identity theorem.

**Definition 2.4 (33 24).** Each function \( f : X \to Y \) induces another function \( \tilde{f} : F(X) \to F(Y) \) defined for each fuzzy interval \( U \) in \( X \) by:

\[
\tilde{f} (U) (y) = \left\{ \begin{array}{ll}
\sup_{x \in f^{-1}(y)} U (x) , & \text{if } y \in \text{range} \left( f \right) , \\
0 & \text{if } y \notin \text{range} \left( f \right) .
\end{array} \right.
\]

This is called the Zadeh extension principle.

**Definition 2.5 (29).** Consider \( \tilde{x} , \tilde{y} \in \tilde{E} \). If there exists \( \tilde{z} \in \tilde{E} \) such that \( \tilde{x} = \tilde{y} + \tilde{z} \), then \( z \) is called the H-difference (Hukuhara difference) of \( x \) and \( y \) and is denoted by \( \tilde{z} = \tilde{x} \ominus \tilde{y} \).

**Definition 2.6 (29).** If \( \tilde{f} : I \to \tilde{E} \) and \( y_0 \in I \), where \( I \subseteq [t_0 , T] \). We say that \( \tilde{f} \) Hukuhara Differentiable at \( y_0 \), if there exists an element \( \left[ \tilde{f}^h \right]_r \in \tilde{E} \) such that for all \( h > 0 \) sufficiently small (near to 0), exists \( \tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r) , \tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r) \) and the limits are taken in the metric(\( \tilde{E} , \mathcal{D} \))

\[
\lim_{h \to 0^+} \frac{\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)}{h} = \lim_{h \to 0^+} \frac{\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)}{h}.
\]

The fuzzy set \( \left[ \tilde{f}^h (y_0) \right]_r \) is called the Hukuhara derivative of \( \left[ \tilde{f}^h \right]_r \) at \( y_0 \).

These limits are taken in the space (\( \tilde{E} , \mathcal{D} \)) if \( t_0 \) or \( T \), then we consider the corresponding one-side derivation. Recall that \( \tilde{x} \ominus \tilde{y} = \tilde{z} \) are defined on \( r \)-level set, where \( [\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r , \forall r \in [0, 1] \) By consideration of definition of the metric \( \mathcal{D} \) all the \( r \)-level set \( \left[ \tilde{f}(0) \right]_r \) are Hukuhara differentiable at \( y_0 \), with
Hukuhara derivatives \( \left[ \tilde{f}'(y_0) \right]_r \), when \( \tilde{f} : I \to \tilde{E} \) is Hukuhara differentiable at \( y_0 \) with Hukuhara derivative \( \left[ \tilde{f}'(y_0) \right]_r \) it’ lead to that \( \tilde{f} \) is Hukuhara differentiable for all \( r \in [0, 1] \) which satisfies the above limits i.e. if \( f \) is differentiable at \( t_0 \in [t_0 + \alpha, T] \) then all its \( r \)-levels \( \left[ \tilde{f}'(t) \right]_r \) are Hukuhara differentiable at \( t_0 \).

**Definition 2.7** (23). Define the mapping \( \tilde{f}' : I \to \tilde{E} \) and \( y_0 \in I \), where \( I \in [t_0, T] \). We say that \( \tilde{f}' \) Hukuhara differentiable \( t \in E \), if there exists an element \( \left[ \tilde{f}'(n) \right]_r \in \tilde{E} \) such that for all \( h > 0 \) sufficiently small (near to 0), exists such that \( \tilde{f}'(n-1)(y_0 + h; r) \cap \tilde{f}'(n-1)(y_0; r), \tilde{f}'(n-1)(y_0; r) \cap \tilde{f}'(n-1)(y_0 - h; r) \) and the limits are taken in the metric \((E, D)\)

\[
\lim_{h \to 0^+} \frac{\tilde{f}'(n-1)(y_0 + h; r) \cap \tilde{f}'(n-1)(y_0; r)}{h} = \lim_{h \to 0^+} \frac{\tilde{f}'(n-1)(y_0; r) \cap \tilde{f}'(n-1)(y_0 - h; r)}{h}
\]

exists and equal to \( \tilde{f}'(n) \) and for \( n = 2 \) we have second order Hukuhara derivative.

**Theorem 2.8** (29). Let \( \tilde{f} : [t_0 + \alpha, T] \to \tilde{E} \) be Hukuhara differentiable and denote by

\[
\left[ \tilde{f}'(t) \right]_r = \left[ f'(t), \tilde{f}'(t) \right]_r = \left[ f'(t; r), \tilde{f}'(t; r) \right].
\]

Then the boundary functions \( f'(t; r), \tilde{f}'(t; r) \) are differentiable

\[
\left[ \tilde{f}'(t) \right]_r = \left[ (f(t; r))', (\tilde{f}(t; r))' \right], \forall r \in [0, 1].
\]

**Theorem 2.9** (23). Let \( \tilde{f} : [t_0 + \alpha, T] \to \tilde{E} \) be Hukuhara differentiable and denote

\[
\left[ \tilde{f}'(t) \right]_r = \left[ f'(t), \tilde{f}'(t) \right]_r = \left[ f'(t; r), \tilde{f}'(t; r) \right].
\]

Then the boundary functions

\[
f'(t; r), \tilde{f}'(t; r)
\]

are differentiable we can write for \( n^{th} \) order fuzzy derivative

\[
\left[ \tilde{f}'(n)(t) \right]_r = \left[ \left( f'(n)(t; r) \right)', \left( \tilde{f}'(n)(t; r) \right)' \right], \forall r \in [0, 1].
\]

### 3. Defuzzification of \( n \)-th order FIVP

Consider the \( n \)-th order FIVP

\[
\ddot{\tilde{y}}^{(n)}(t) = f \left( t, \dot{\tilde{y}}(t), \ddot{\tilde{y}}(t), \ldots, \dddot{\tilde{y}}^{(n-1)}(t) \right) + \tilde{w}(t), \ t \in [t_0, T]. \tag{3.1}
\]

Subject to the initial conditions

\[
\tilde{y}(t_0) = \tilde{y}_0, \dot{\tilde{y}}(t_0) = \dot{\tilde{y}}_0, \ldots, \dddot{\tilde{y}}^{(n-1)}(t_0) = \dddot{\tilde{y}}^{(n-1)}_0 , \tag{3.2}
\]

where \( \tilde{y} \) is a fuzzy function of the crisp variable \( t \) with \( \tilde{f} \) being a fuzzy function of the crisp variable \( t \), the fuzzy variable \( \tilde{y} \) and the fuzzy Hukuhara-derivatives \( \dot{\tilde{y}}(t), \ddot{\tilde{y}}(t), \ldots, \dddot{\tilde{y}}^{(n-1)}(t) \). Here \( \tilde{y}^{(n)}(t) \) is the fuzzy \( n^{th} \) order Hukuhara-derivative and \( \dddot{\tilde{y}}(t_0), \dddot{\tilde{y}}^{(n-1)}(t_0) \) are triangular fuzzy numbers. We denote the fuzzy function \( y \) by \( \tilde{y} = [\underline{y}, \bar{y}] \) for \( t \in [t_0, T] \) and \( r \in [0, 1] \). It means that the \( r \)-level sets of \( \tilde{y}(t) \) can be defined as:

\[
[\tilde{y}(t)]_r = [\underline{y}(t; r), \bar{y}(t; r)],
\]
\[
\begin{align*}
\tilde{y}'(t) &= \left[y'(t), y'(t_0), \cdots, y^{(n-1)}(t) \right]_r = \left[y^{(n-1)}(t), y^{(n-1)}(t) \right], \\
\tilde{y}(t_0) &= \left[y(t_0), y(t_0), \cdots, \tilde{y}(t_0) \right], \\
\tilde{y}'(t_0) &= \left[y'(t_0), y'(t_0), \cdots, \tilde{y}'(t_0) \right]_r = \left[y^{(n-1)}(t_0), y^{(n-1)}(t_0) \right],
\end{align*}
\]

where \( w(t) \) is crisp or fuzzy inhomogeneous term such that \( \tilde{w}(t) = [w(t), \overline{w}(t)] \). Since
\[
y^{(n)}(t) = f(t, y(t), y'(t), y''(t), \ldots, y^{(n-1)}(t)) + w(t).
\]
Let
\[
\mathcal{Y}(t) = y(t), y'(t), y''(t), \ldots, y^{(n-1)}(t),
\]
such that
\[
\tilde{\mathcal{Y}}(t) = \left[\mathcal{Y}(t), \mathcal{Y}(t) \right] = \left[y(t), y(t), \cdots, y^{(n-1)}(t), \mathcal{Y}(t), \mathcal{Y}(t), \cdots, \mathcal{Y}(n-1)(t) \right].
\]
Also, we can write
\[
\left[\tilde{f}(t, \tilde{\mathcal{Y}})\right]_r = \left[f(t, \tilde{\mathcal{Y}}), f(t, \tilde{\mathcal{Y}}) \right],
\]
By using Zadeh extension principles, we have
\[
\tilde{f}(t, \tilde{\mathcal{Y}}(t)) = \left[f(t, \tilde{\mathcal{Y}}(t)), f(t, \tilde{\mathcal{Y}}(t)) \right],
\]
such that
\[
\tilde{f}(t, \tilde{\mathcal{Y}}(t)) = \mathcal{F}(t, \mathcal{Y}(t), \mathcal{Y}(t)) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t)), \\
\tilde{\mathcal{F}}(t, \tilde{\mathcal{Y}}(t)) = \mathcal{G}(t, \mathcal{Y}(t), \mathcal{Y}(t)) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t)).
\]
Then we have
\[
y^{(n)}(t) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t)) + w(t) \quad \text{(3.3)}
\]
\[
\overline{y}^{(n)}(t) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t)) + \overline{w}(t) \quad \text{(3.4)}
\]
where the membership function of
\[
\mathcal{F}(t, \tilde{\mathcal{Y}}(t)) + w(t) \]
and
\[
\mathcal{G}(t, \tilde{\mathcal{Y}}(t)) + \overline{w}(t)
\]
can be defined as follows:
\[
\mathcal{F}(t, \tilde{\mathcal{Y}}(t)) + w(t) = \min \{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu | \mu \in [\tilde{\mathcal{Y}}(t), r] \}, \\
\mathcal{G}(t, \tilde{\mathcal{Y}}(t)) + \overline{w}(t) = \max \{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu | \mu \in [\tilde{\mathcal{Y}}(t), r] \}.
\]

4. Analysis of RKM56 for \( n \)-th order FIVP

In this section we consider the six order Runge-Kutta method of order five as our proposed method to obtain a numerical solution of \( n \)th order FIVP. The structure of this method for solving crisp first order initial value problem have been introduced in \( \text{[10][11][25]} \). Also Jameel et al. \( \text{[16]} \) proposed RK56 to solve
first order linear FIVPs In order to solve \( n \)-th order FIVPs, we need to fuzzify RKM56 then defuzzify it. We consider the \( n \)-th order FIVP \((3.1)–(3.2)\) in Section \(3\) and since RKM56 is numerical method we need to reduce Eq. \((3.1)\) in to system of first order FIVPs for all \( r \in [0,1] \) as follows

\[
\begin{align*}
\dot{y}_1 (t;r) &= \tilde{f}_1 (\tilde{y}_1 (t;r), \ldots, \tilde{y}_n (t;r)) \\
\vdots \\
\dot{y}_n (t;r) &= \tilde{f}_n (\tilde{y}_1 (t;r), \ldots, \tilde{y}_n (t;r)) \\
\end{align*}
\tag{4.1}
\]

where \( t \in [t_0, T] \) and \( a_1, \ldots, a_n \) are fuzzy numbers that defined Section \( 2 \) with \( r \)-level intervals

\[
\tilde{y}_i (0; r) = [\underline{y}_{i,0} (r), \overline{y}_{i,0} (r)] = [a_i, \overline{a}_i]
\]

for \( i = 1, 2, \ldots, n \) and \( r \in [0,1] \). We let

\[
\tilde{y} (t;r) = [\tilde{y}_1 (t;r), \tilde{y}_2 (t;r), \ldots, \tilde{y}_n (t;r)]^t
\]

be the fuzzy solution of Eq. \((3.1)\) on \([t_0, T]\) if

\[
\begin{align*}
\underline{y}' (t;r) &= \min \left\{ \tilde{f}_i (\tilde{y}_1 (t;r), \ldots, \tilde{y}_n (t;r)) : \tilde{y}_i (t;r) \in [\underline{y}_i (t;r), \overline{y}_i (t;r)] \right\} \\
\overline{y}' (t;r) &= \max \left\{ \tilde{f}_i (\tilde{y}_1 (t;r), \ldots, \tilde{y}_n (t;r)) : \tilde{y}_i (t;r) \in [\underline{y}_i (t;r), \overline{y}_i (t;r)] \right\}
\end{align*}
\tag{4.2}
\]

and

\[
\begin{align*}
\left[ \underline{f}_i (t; \tilde{y} (t;r)) \right]^t &= \mathcal{F} (t, \underline{\tilde{y}} (t;r)) \\
\left[ \overline{f}_i (t; \tilde{y} (t;r)) \right]^t &= \mathcal{F} (t, \overline{\tilde{y}} (t;r)) \\
\left[ \underline{f}_i (t; \tilde{y} (t;r)) \right]^t &= \mathcal{G} (t, \underline{\tilde{y}} (t;r)) \\
\left[ \overline{f}_i (t; \tilde{y} (t;r)) \right]^t &= \mathcal{G} (t, \overline{\tilde{y}} (t;r)).
\end{align*}
\tag{4.4}
\]

With respect to the above mentioned indicators, system \((4.1)\) can be written as with assumptions for the

\[
\begin{align*}
\underline{y}' (t;0) &= \mathcal{F} (t, \underline{\tilde{y}} (t;r)) \\
\overline{y}' (t;0) &= \mathcal{F} (t, \overline{\tilde{y}} (t;r)) \\
\end{align*}
\tag{4.5}
\]

where

\[
\begin{align*}
\underline{y} (t;0) &= [\underline{y}_{\ast,0} (r)]^t \\
\overline{y}' (t;0) &= [\overline{y}' (t;r)]^t
\end{align*}
\]

and

\[
\begin{align*}
\underline{y} (t;0) &= [\underline{y}_{\ast,0} (r)]^t \\
\overline{y}' (t;0) &= [\overline{y}' (t;r)]^t
\end{align*}
\tag{4.6}
\]

where

\[
\begin{align*}
\underline{y} (t;0) &= [\underline{y}_{\ast,0} (r)]^t \\
\overline{y}' (t;0) &= [\overline{y}' (t;r)]^t
\end{align*}
\]

Now we show that under the assumption for the fuzzy functions \( \tilde{f}_i \), for \( i = 1, 2, \ldots, n \) we can obtain a unique fuzzy solution for system \((4.1)\) for each \( r \)-level set (see \([2]\)). For finding an approximate solution for \((4.1)\) with RKM56, we first define the approximate numerical solution of systems \((4.1)\) by

\[
\tilde{y} (t;r) = [\tilde{y}_1 (t;r), \tilde{y}_2 (t;r), \ldots, \tilde{y}_n (t;r)]^t
\]
such that $\bar{s}_i = \tilde{y}_i (t; r), \bar{s}_2 = \tilde{y}_2 (t; r), \ldots, \bar{s}_n = \tilde{y}_n (t; r)$, where $\bar{s}_j \in [\bar{y}_j, \bar{y}_j]$ for $1 \leq i, j \leq n$

$$
K_{i1} (t; \bar{y} (t; r)) = \min \left\{ f_i (t, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\bar{y}_j (t; r), \bar{y}_j (t; r)] \right\}
$$

$$
\overline{K}_{i1} (t; \bar{y} (t; r)) = \max \left\{ f_i (t, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\bar{y}_j (t; r), \bar{y}_j (t; r)] \right\}
$$

$$
K_{i2} (t; \bar{y} (t; r)) = \min \left\{ f_i (t + \frac{h}{2}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j1} (t, \bar{y} (t; r), h), \tilde{z}_{j1} (t, \bar{y} (t; r), h)] \right\}
$$

$$
\overline{K}_{i2} (t; \bar{y} (t; r)) = \max \left\{ f_i (t + \frac{h}{2}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j1} (t, \bar{y} (t; r), h), \tilde{z}_{j1} (t, \bar{y} (t; r), h)] \right\}
$$

$$
K_{i3} (t; \bar{y} (t; r)) = \min \left\{ f_i (t + \frac{h}{4}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j2} (t, \bar{y} (t; r), h), \tilde{z}_{j2} (t, \bar{y} (t; r), h)] \right\}
$$

$$
\overline{K}_{i3} (t; \bar{y} (t; r)) = \max \left\{ f_i (t + \frac{h}{4}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j2} (t, \bar{y} (t; r), h), \tilde{z}_{j2} (t, \bar{y} (t; r), h)] \right\}
$$

$$
K_{i4} (t; \bar{y} (t; r)) = \min \left\{ f_i (t + \frac{h}{2}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j3} (t, \bar{y} (t; r), h), \tilde{z}_{j3} (t, \bar{y} (t; r), h)] \right\}
$$

$$
\overline{K}_{i4} (t; \bar{y} (t; r)) = \max \left\{ f_i (t + \frac{h}{2}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j3} (t, \bar{y} (t; r), h), \tilde{z}_{j3} (t, \bar{y} (t; r), h)] \right\}
$$

$$
K_{i5} (t; \bar{y} (t; r)) = \min \left\{ f_i (t + \frac{3h}{4}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j4} (t, \bar{y} (t; r), h), \tilde{z}_{j4} (t, \bar{y} (t; r), h)] \right\}
$$

$$
\overline{K}_{i5} (t; \bar{y} (t; r)) = \max \left\{ f_i (t + \frac{3h}{4}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j4} (t, \bar{y} (t; r), h), \tilde{z}_{j4} (t, \bar{y} (t; r), h)] \right\}
$$

$$
K_{i6} (t; \bar{y} (t; r)) = \min \left\{ f_i (t + h, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j5} (t, \bar{y} (t; r), h), \tilde{z}_{j5} (t, \bar{y} (t; r), h)] \right\}
$$

$$
\overline{K}_{i6} (t; \bar{y} (t; r)) = \max \left\{ f_i (t + h, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n); \bar{s}_j \in [\tilde{z}_{j5} (t, \bar{y} (t; r), h), \tilde{z}_{j5} (t, \bar{y} (t; r), h)] \right\},
$$

where

$$
\tilde{z}_{j1} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{2} K_{i1} (r), \tilde{z}_{j1} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{2} \overline{K}_{i1} (r)
$$

$$
\tilde{z}_{j2} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{16} (3 K_{i1} (r) + K_{i2} (r))
$$

$$
\tilde{z}_{j2} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{16} (3 \overline{K}_{i2} (r) + K_{i2} (r))
$$

$$
\tilde{z}_{j3} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{2} K_{i3} (r), \tilde{z}_{j3} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{2} \overline{K}_{i3} (r)
$$

$$
\tilde{z}_{j4} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{16} (-3 K_{i2} (r) + 6 K_{i3} (r) + 9 K_{i4} (r))
$$

$$
\tilde{z}_{j4} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{16} (-3 \overline{K}_{i2} (r) + 6 \overline{K}_{i3} (r) + 9 \overline{K}_{i4} (r))
$$

$$
\tilde{z}_{j5} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{7} (K_{i1} (r) + 4 K_{i2} (r) + 6 K_{i3} (r) - 12 K_{i4} (r) + 8 K_{i5} (r))
$$

$$
\tilde{z}_{j5} (t, \bar{y} (t; r), h) = \bar{y}_j (t; r) + \frac{h}{7} (K_{i1} (r) + 4 \overline{K}_{i2} (r) + 6 \overline{K}_{i3} (r) - 12 \overline{K}_{i4} (r) + 8 \overline{K}_{i5} (r)),
$$

where

$$
K_{ij} (t; \bar{y} (t; r), h) = K_{ij} (r)
$$

and

$$
\overline{K}_{ij} (t; \bar{y} (t; r), h) = \overline{K}_{ij} (r).
$$
Now if we can consider the following

\[ F_i (t, \vec{y} (t; r), h) = 7K_{i1} (r) + 32K_{i3} (r) + 12K_{i4} (r) + 32K_{i5} (r) + 7K_{i6} (r) \]

\[ G_i (t, \vec{y} (t; r), h) = 7K_{i1} (r) + 32K_{i3} (r) + 12K_{i4} (r) + 32K_{i5} (r) + 7K_{i6} (r), \]

where \( t \in [t_0, T] \) and \( r \in [0, 1] \). If the exact and approximate solution in the \( i \)-th \( r \) cut at \( t_m \) for \( 0 \leq m \leq N \) are denoted by

\[ \tilde{Y}_i (t; r) = \left[ \mathcal{Y}_i (t; r), \tilde{Y}_i (t; r) \right] \]

\[ \tilde{y}_i (t; r) = \left[ y_i (t; r), \tilde{y}_i (t; r) \right] \]

respectively, then the numerical method for solution approximation in the \( i \)-th coordinate \( r \) level set, with RKM56 is

\[ \tilde{y}^{[m+1]}_i (t; r) = \tilde{y}^{[m]}_i (t; r) + \frac{1}{90} F_i \left( t_m, \tilde{y}^{[m]}_i (t; r), h \right), \tilde{y}^{[0]}_i (r) = \mathcal{Y}^{[0]}_i (r) \]

\[ \tilde{y}^{[m+1]}_i (t; r) = \tilde{y}^{[m]}_i (t; r) + \frac{1}{90} G_i \left( t_m, \tilde{y}^{[m]}_i (t; r), h \right), \tilde{y}^{[0]}_i (r) = \mathcal{Y}^{[0]}_i (r), \]

where

\[ \tilde{y}^{[m]}_i (t; r) = \left[ \tilde{y}^{[m]}_i (t; r), \tilde{y}^{[m]}_i (t; r) \right], \tilde{y}^{[m]}_i (r) = \left[ \tilde{y}^{[m]}_1 (r), \tilde{y}^{[m]}_2 (r), \ldots, \tilde{y}^{[m]}_n (r) \right]^t. \]

Now we let

\[ \mathcal{F} \left( t, \tilde{y}^{[m]} (r), h \right) = \frac{1}{90} \left( F_1 \left( t, \tilde{y}^{[m]} (r), h \right), \ldots, F_n \left( t, \tilde{y}^{[m]} (r), h \right) \right)^t \]

\[ \mathcal{G} \left( t, \tilde{y}^{[m]} (r), h \right) = \frac{1}{90} \left( G_1 \left( t, \tilde{y}^{[m]} (r), h \right), \ldots, G_n \left( t, \tilde{y}^{[m]} (r), h \right) \right)^t. \]

The RK5 for solutions approximation \( r \)-level sets of Eqs. (4.7)–(4.8) are as follows

\[ \tilde{y}^{[m+1]}_i (r) = \tilde{y}^{[m]}_i (r) + h \mathcal{F} \left( t, \tilde{y}^{[m]} (r), h \right), \tilde{y}^{[0]}_i (r) = \mathcal{Y}^{[0]}_i (r) \]

\[ \tilde{y}^{[m+1]}_i (r) = \tilde{y}^{[m]}_i (r) + h \mathcal{G} \left( t, \tilde{y}^{[m]} (r), h \right), \tilde{y}^{[0]}_i (r) = \mathcal{Y}^{[0]}_i (r) \]

and

\[ \mathcal{F} \left( t, \tilde{y}^{[m]} (r), h \right) = \frac{1}{90} \left[ 7K_{i1} (r) + 32K_{i3} (r) + 12K_{i4} (r) + 32K_{i5} (r) + 7K_{i6} (r) \right] \]

\[ \mathcal{G} \left( t, \tilde{y}^{[m]} (r), h \right) = \frac{1}{90} \left[ 7K_{i1} (r) + 32K_{i3} (r) + 12K_{i4} (r) + 32K_{i5} (r) + 7K_{i6} (r) \right], \]

where in Eqs. (4.13)–(4.14)

\[ \tilde{K}_{ij} (r) = \tilde{K}_{ij} \left( t_m, \tilde{y}^{[m]} (r), h \right) = \left( \tilde{K}_{1j} \left( t_m, \tilde{y}^{[m]} (r), h \right), \ldots, \tilde{K}_{1j} \left( t_m, \tilde{y}^{[m]} (r), h \right) \right)^t. \]

5. Stability, convergence and error analysis

According the definition given in [2] we can define the following:

**Definition 5.1.** A one-step method for approximating the solution of differential Eqs. (4.5) and (4.6) is a method which can be written in the form
where the increment function $\phi$ is determined by $F$ in Eq. (4.5). Also from Eq. (4.6) we have

$$\left[\hat{w}^{[n+1]}\right]_r = \left[\hat{w}^{[n]}\right]_r + h\phi(t, [\hat{w}^{[n]}]_r; h; r),$$

(5.1)

where the increment function $\hat{\phi}$ is determined by $G$ in Eq. (4.5).

According to Theorems 5.2 and 5.3 in [18] we can define the Theorems 5.2 and 5.3 as follows:

**Theorem 5.2.** If $\phi(t, \hat{w}^{[n]}|_r; h; r)$ and $\hat{\phi}(t, [\hat{w}^{[n]}]_r; h; r)$ satisfies a Lipschitz condition in $\hat{y}(t; r)$ then the method given by Eqs. (5.1)–(5.2) is stable.

**Theorem 5.3.** From Eqs. (10-11), if $\hat{F}(t, \hat{y}(t; r))$ and $G(t, \hat{y}(t; r))$ satisfies a Lipschitz condition in $\hat{y}(t; r)$ then the method given by Eqs. (4.7)–(4.8) is stable.

**Lemma 5.4 ([4]).** If the sequence of non-negative numbers $\{W_n\}_{n=0}^N$ satisfy

$$|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N - 1,$$

for the given positive constants $A$ and $B$, then

$$|W_n| \leq A^n |W_0| + B\frac{A^n - 1}{A - 1}, \quad 0 \leq n \leq N.$$

**Lemma 5.5 ([4]).** If the sequence of non-negative numbers $\{W_n\}_{n=0}^N$ and $\{V_n\}_{n=0}^N$ satisfy

$$|W_{n+1}| \leq |W_n| + A\max\{|W_n|, |V_n|\} + B$$

$$|V_{n+1}| \leq |V_n| + A\max\{|W_n|, |V_n|\} + B$$

for the given positive constants $A$ and $B$, then

$$|U_n| = |W_n| + |V_n|, \quad 0 \leq n \leq N,$$

then we have

$$U_n \leq \hat{A}^n U_0 + \hat{B}\frac{\hat{A}^n - 1}{\hat{A} - 1}, \quad 0 \leq n \leq N,$$

where $\hat{A} = 1 + 2A$ and $\hat{B} = 2B$.

Now let

$$\hat{y}(t; r) = [\hat{y}(t; r); \hat{y}(t; r)] = [U, V]$$

such that

$$\hat{F}(t, \hat{Y}(t; r); h) = \hat{F}(t, U, V; h), \quad \hat{G}(t, \hat{y}(t; r); h) = \hat{G}(t, U, V; h),$$

then the domain of $\hat{F}(t, U, V; h)$ and $\hat{G}(t, \hat{y}(t; r); h)$ is give by the set all nonempty fuzzy sets $RF$ such that

$$\hat{E}\{t, U, V|_r \in [0, T], -\infty < V < \infty, -\infty < U \leq V\}.$$
Proof. It is sufficient to show 
\[ \lim_{h \to \infty} \overline{y}(T; r) = \overline{Y}(T; r), \lim_{h \to \infty} \underline{y}(T; r) = \underline{Y}(T; r) \]
for \( 0 \leq i \leq N \). Using the Taylor theorem in formulas \((4.7)-(4.8)\) with RKM65 formulas \((4.13)-(4.14)\) we get
\[
\begin{cases}
\underline{Y}(t_{i+1}; r) = \underline{Y}(t_i; r) + h \hat{F} \left( t_i, \tilde{Y}(t_i; r); h \right) + \frac{h^6}{6!} \underline{Y}^{(6)} (\theta_{i,1}; r) \\
\overline{Y}(t_{i+1}; r) = \overline{Y}(t_i; r) + h \hat{G} \left( t_i, \tilde{Y}(t_i; r); h \right) + \frac{h^6}{6!} \overline{Y}^{(6)} (\theta_{i,2}; r),
\end{cases}
\]
where \( \theta_{i,1}, \theta_{i,2} \in (t_i, t_{i+1}) \). Denoting
\[ W_i = \underline{Y}(t_i; r) - \underline{y}(t_i; r), \overline{Z}_i = \overline{Y}(t_i; r) - \overline{y}(t_i; r) , \]
from \((4.7)-(4.8)\) and \((5.3)\) we have
\[
\begin{align*}
W_{i+1} &= W_i + h \left\{ \hat{F} \left( t_i, \tilde{Y}(t_i; r); h \right) - \hat{F} \left( t_i, \tilde{y}(t_i; r); h \right) \right\} + \frac{h^6}{6!} \underline{Y}^{(6)} (\theta_{i,1}; r), \\
\overline{Z}_{i+1} &= \overline{Z}_i + h \left\{ \hat{G} \left( t_i, \tilde{Y}(t_i; r); h \right) - \hat{G} \left( t_i, \tilde{y}(t_i; r); h \right) \right\} + \frac{h^6}{6!} \overline{Y}^{(6)} (\theta_{i,2}; r).
\end{align*}
\]
Hence
\[
\begin{align*}
|W_{i+1}| &\leq |W_i| + 2Lh \max \{|W_i|, |Z_i|\} + \frac{h^6}{6!}M, \\
|\overline{Z}_{i+1}| &\leq |Z_i| + 2Lh \max \{|W_i|, |Z_i|\} + \frac{h^6}{6!}M,
\end{align*}
\]
where \( M = \max \{ \underline{M}, \overline{M} \} \) for \( t \in [t_0, T] \) such that
\[ \underline{M} = \min \left| \underline{Y}^{(6)}(t; r), \overline{Y}^{(6)}(t; r) \right|, \overline{M} = \max \left| \underline{Y}^{(6)}(t; r), \overline{Y}^{(6)}(t; r) \right| \]
and \( L > 0 \) is a bound for the partial derivatives of \( \hat{F} \) and \( \hat{G} \). Therefore, from Lemma \([5.4, 5.5]\) we obtain
\[
\begin{align*}
|W_i| &\leq (1 + 4Lh)^i |\underline{U}_0| + \frac{2h^6}{6!}M \frac{(1 + 4Lh)^i - 1}{4Lh}, \\
|Z_i| &\leq (1 + 4Lh)^i |\underline{U}_0| + \frac{2h^6}{6!}M \frac{(1 + 4Lh)^i - 1}{4Lh},
\end{align*}
\]
where \( |\underline{U}_0| = |W_0| + |Z_0| \). In particular, according to Lemma 1.2 in \([32]\) we have
\[
\begin{align*}
|W_N| &\leq (1 + 4Lh)^N |\underline{U}_0| + \frac{h^6}{6!}M \frac{(1 + 4Lh)^T - 1}{2Lh}, \\
|Z_N| &\leq (1 + 4Lh)^N |\underline{U}_0| + \frac{h^6}{6!}M \frac{(1 + 4Lh)^T - 1}{2Lh}.
\end{align*}
\]
Since \( W_0 = Z_0 = 0 \) and from Proposition 4 in \([27]\) we have
\[
|W_N| \leq \frac{M e^{ALT} - 1}{1440L} h^6, |Z_N| \leq \frac{M e^{ALT} - 1}{1440L} h^6.
\]
Thus, if \( h \to 0 \), we get \( W_N \to 0 \) and \( Z_N \to 0 \). Thus absolute error bound of RKM65 for the numerical approximate solution of the FIVP \((3.1)-(3.2)\) can be defined as:
\[
\begin{align*}
|E_{i+1}^r| &= |Y(t_{i+1}; r) - \underline{y}(t_{i+1}; r)| \leq \xi, \\
|E_{i+1}^r| &= |Y(t_{i+1}; r) - \overline{y}(t_{i+1}; r)| \leq \zeta,
\end{align*}
\]
where \( \xi \approx \frac{M e^{ALT} - 1}{1440L} h^6, \zeta \approx \frac{M e^{ALT} - 1}{1440L} h^6 \) for \( \theta = i = N \) and \( t \in [0, T] \), with
\[ M = \max \left| \underline{Y}^{(6)}(t; r), \overline{Y}^{(6)}(t; r) \right|, \overline{M} = \max \left| \underline{Y}^{(6)}(t; r), \overline{Y}^{(6)}(t; r) \right|, \]
which completes the proof. \( \square \)
6. Numerical example

Consider the circuit model problem [23] shown in Figure 2 where $L = 1h$, $R = 2\Omega$, $C = 0.25f$ and $E(t) = 50\cos t$. If $Q(t)$ is the charge on the capacitor at time $t > 0$.

![Electric circuit model](image)

Thus the FIVP of this model is given as follows:

$$\ddot{Q}(t; r) + 2\dot{Q}(t; r) + 4Q(t; r) = 50\cos t, \quad t > 0$$

(6.1)

$$\dot{Q}(0; r) = [4 + r, 6 - r]$$

$$\dot{Q}'(0; r) = [r, 2 - r], \quad r \in [0, 1].$$

According to Section 2, Eq. (6.1) can be written into first order system as follows

$$\begin{align*}
\dot{Q}'_1(t; r) &= \dot{Q}_2(t; r) \\
Q_1(t; r) &= [4 + r, 6 - r] \\
\dot{Q}'_2(t; r) &= 50\cos t - 4\dot{Q}_1(t; r) - 2\dot{Q}_2(t; r) \\
Q_2(t; r) &= [r, 2 - r].
\end{align*}$$

The exact analytical solution of Eq. (6.1) was given in [5]. According to RK56 in Section 4, we have:

$$U_1\left(t, \dot{Q}'_1(t; r), \dot{Q}_2(t; r)\right) = U_1\left(t, \dot{Q}_1, \dot{Q}_2\right) = \dot{Q}_2(t; r)$$

$$U_2\left(t, \dot{Q}'_1(t; r), \dot{Q}_2(t; r)\right) = U_2\left(t, \dot{Q}_1, \dot{Q}_2\right) = 50\cos t - 4\dot{Q}_1(t; r) - 2\dot{Q}_2(t; r)$$

then

$$\begin{align*}
\bar{K}_1(r) &= hU_1\left(t_i, \bar{Q}_1, \bar{Q}_2\right) \\
\bar{D}_1(r) &= hU_2\left(t_i, \bar{Q}_1, \bar{Q}_2\right) \\
\bar{K}_2(r) &= hU_1\left(t_i + \frac{h}{2}, \bar{Q}_1 + \frac{\bar{K}_1(r)}{2}, \bar{Q}_2 + \frac{\bar{D}_1(r)}{2}\right) \\
\bar{D}_2(r) &= hU_2\left(t_i + \frac{h}{2}, \bar{Q}_1 + \frac{\bar{K}_1(r)}{2}, \bar{Q}_2 + \frac{\bar{D}_1(r)}{2}\right)
\end{align*}$$
\[ \begin{align*}
\tilde{K}_3 (r) &= hU_1 \left( t_i + \frac{h}{4} \tilde{Q}_1 + \frac{3\tilde{K}_1 (r) + \tilde{K}_2 (r)}{2}, \tilde{Q}_2 + \frac{3\tilde{D}_1 (r) + \tilde{D}_2 (r)}{2} \right) \\
\tilde{D}_3 (r) &= hU_2 \left( t_i + \frac{h}{4} \tilde{Q}_1 + \frac{3\tilde{K}_1 (r) + \tilde{K}_2 (r)}{16}, \tilde{Q}_2 + \frac{3\tilde{D}_1 (r) + \tilde{D}_2 (r)}{16} \right) \\
\tilde{K}_4 (r) &= hU_1 \left( t_i + \frac{h}{2} \tilde{Q}_1 + \frac{\tilde{K}_4 (r)}{2}, \tilde{Q}_2 + \frac{\tilde{D}_4 (r)}{2} \right) \\
\tilde{D}_4 (r) &= hU_2 \left( t_i + \frac{h}{2} \tilde{Q}_1 + \frac{\tilde{K}_3 (r)}{2}, \tilde{Q}_2 + \frac{\tilde{D}_3 (r)}{2} \right) \\
\tilde{K}_5 (r) &= hU_1 \left( t_i + \frac{3h}{4} \tilde{Q}_1 + \frac{-3\tilde{K}_1 (r) + 6\tilde{K}_3 (r) + 9\tilde{K}_4 (r)}{16}, \tilde{Q}_2 + \frac{-3\tilde{D}_1 (r) + 6\tilde{D}_3 (r) + 9\tilde{D}_4 (r)}{16} \right) \\
\tilde{D}_5 (r) &= hU_2 \left( t_i + \frac{3h}{4} \tilde{Q}_1 + \frac{-3\tilde{K}_1 (r) + 6\tilde{K}_3 (r) + 9\tilde{K}_4 (r)}{16}, \tilde{Q}_2 + \frac{-3\tilde{D}_1 (r) + 6\tilde{D}_3 (r) + 9\tilde{D}_4 (r)}{16} \right) \\
\tilde{K}_6 (r) &= hU_1 (t_i + h, A_2, A_3) \\
\tilde{D}_6 (r) &= hU_2 (t_i + h, A_4, A_5),
\end{align*}\]

where

\[ \begin{align*}
A_2 &= \tilde{Q}_1 + \frac{\tilde{K}_1 (r) + 4\tilde{K}_2 (r) + 6\tilde{K}_3 (r) - 12\tilde{K}_4 (r) + 9\tilde{K}_8 (r)}{7} \\
A_3 &= \tilde{Q}_2 + \frac{\tilde{D}_1 (r) + 4\tilde{D}_2 (r) + 6\tilde{D}_3 (r) - 12\tilde{D}_4 (r) + 9\tilde{D}_5 (r)}{7} \\
A_4 &= \tilde{Q}_1 + \frac{\tilde{K}_1 (r) + 4\tilde{K}_2 (r) + 6\tilde{K}_3 (r) - 12\tilde{K}_4 (r) + 9\tilde{K}_8 (r)}{7} \\
A_5 &= \tilde{Q}_2 + \frac{\tilde{D}_1 (r) + 4\tilde{D}_2 (r) + 6\tilde{D}_3 (r) - 12\tilde{D}_4 (r) + 9\tilde{D}_5 (r)}{7}
\end{align*}\]

then

\[ \begin{align*}
\tilde{Q}_1 (t_{i+1}; r) &= \tilde{Q}_1 (t_i; r) + \frac{7\tilde{K}_1 (r) + 32\tilde{K}_3 (r) + 12\tilde{K}_4 (r) + 32\tilde{K}_5 (r) + 7\tilde{K}_6 (r)}{90} \\
\tilde{Q}_2 (t_{i+1}; r) &= \tilde{Q}_2 (t_i; r) + \frac{7\tilde{D}_1 (r) + 32\tilde{D}_3 (r) + 12\tilde{D}_4 (r) + 32\tilde{D}_5 (r) + 7\tilde{D}_6 (r)}{90},
\end{align*}\] (6.2) (6.3)

where

\[ \begin{align*}
\tilde{Q}_1 (t_i; r) &= \left[ Q_1 (t_i; r), \bar{Q}_1 (t_i; r) \right], \\
U_1 (t_i, \tilde{Q}_1, \tilde{Q}_2) &= \left[ U_1 (t_i, Q_1, Q_2), U_1 (t_i, \bar{Q}_1, \bar{Q}_2) \right], \\
U_2 (t_i, \tilde{Q}_1, \tilde{Q}_2) &= \left[ U_2 (t_i, Q_1, Q_2), U_2 (t_i, \bar{Q}_1, \bar{Q}_2) \right], \\
K_j (r) &= \left[ K_j (r), \bar{K}_j (r) \right], \quad \tilde{D}_j (r) = \left[ D_j (r), D_j (r) \right]
\end{align*}\]

and \( t_{i+1} = t_i + h \) for \( j = 1, \ldots, 6, i = 1, \ldots, n \) and \( r \in [0, 1] \). From Eqs. (6.2)–(6.3), the results obtained and represented in Table 1 and Figure 3. One can note that RKM56 approximate solution with 100 iterations at \( t=1 \) and for all \( 0=r=1 \) satisfy the fuzzy numbers properties in Section 2 by taking the triangular shape.
7. Conclusions

In this paper, we presented and developed a numerical method for solving fuzzy ordinary differential equations. The scheme is based on the six stages Runge-Kutta method of order five for solving \( n \)-th order FIVP. The stability, convergence and error analysis of RKM56 have been presented and proved. The numerical example involving second-order linear FIVP model showed that the RKM56 is a capable and accurate method for \( n \)-th order FIVP. Also the obtained results by RKM56 satisfy the properties of fuzzy numbers by taking triangular shape.

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