Hermite-Hadamard type inequalities for operator s-preinvex functions

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Abstract

In this paper, we introduce the concept of operator s-preinvex function, establish some new Hermite-Hadamard type inequalities for operator s-preinvex functions, and provide the estimates of both sides of Hermite-Hadamard type inequality in which some operator s-preinvex functions of positive selfadjoint operators in Hilbert spaces are involved. ©2015 All rights reserved.

Keywords: Hermite-Hadamard type inequality, operator s-convex function, operator preinvex function, operator s-preinvex function.


1. Introduction and Preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_0 = [0, \infty)$.

The following inequality holds for any convex function $f$ defined on $\mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

Both inequalities hold in the reversed direction if $f$ is concave on $[a, b]$. The inequality (1.1) is well known in the literature as Hermite-Hadamard’s inequality. We note that the Hermite-Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. The classical Hermite-Hadamard’s inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

In [10], Hudzik and Maligranda considered $s$-convex function in the second sense. This class is defined in the following way.

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Received 2015-03-12
**Definition 1.1** ([10]). For some fixed $s \in (0, 1]$, a function $f : \mathbb{R}_0 \to \mathbb{R}$ is said to be $s$-convex in the second sense if
\[
    f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)
\]
holds for all $x, y \in \mathbb{R}_0$ and $\lambda \in [0, 1]$. If the inequality (1.2) reverses, then $f$ is said to be $s$-concave in the second sense on $\mathbb{R}_0$.

In [9], Dragomir and Fitzpatrick proved the following variant of Hadamard’s inequality which holds for $s$-convex functions in the second sense.

**Theorem 1.2** ([3]). Suppose that $f : \mathbb{R}_0 \to \mathbb{R}$ is an $s$-convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in \mathbb{R}_0$ with $a < b$. If $f \in L([a, b])$, then the following inequality holds
\[
2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}. \tag{1.3}
\]
The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

In [2], the authors obtained the estimate of the left-hand side of Hermite-Hadamard’s inequality for $s$-convex functions.

**Theorem 1.3** ([2]). Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on $I^2$, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then the following inequality holds
\[
\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx\right| \leq \frac{2}{s+1} \left[|f'(a)|+2(s+1)|f'(a+\frac{a+b}{2})|+|f'(b)|\right]. \tag{1.4}
\]

In [12], Kirmaci et al. gave the estimate of the right-hand side of Hermite-Hadamard’s inequality for $s$-convex functions.

**Theorem 1.4** ([12]). Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on $I^2$ and $a, b \in I$ with $a < b$. If $f' \in L([a, b])$ and $|f'|$ is $s$-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then
\[
\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx\right| \leq \frac{2}{s+1}(2s+1)(s+2)\left[|f'(a)|+|f'(b)|\right]. \tag{1.5}
\]

Hermite-Hadamard’s inequality has several applications in nonlinear analysis and the geometry of Banach spaces, see [11]. In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [9].

Let $X$ be a vector space, $x, y \in X$, $x \neq y$. Define the segment
\[
[x, y] := (1-t)x + ty, \quad t \in [0, 1].
\]
We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function
\[
g(x, y) : [0, 1] \to \mathbb{R},
\]
\[
g(x, y)(t) := f((1-t)x + ty), \quad t \in [0, 1].
\]
Note that $f$ is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function defined on a segment $[x, y] \subseteq X$, we have the Hermite-Hadamard integral inequality (see [1], p.2 and [5], p.2)
\[
f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)dt \leq \frac{f(x) + f(y)}{2}, \tag{1.6}
\]
which can be derived from the classical Hermite-Hadamard inequality \((1.1)\) for the convex function 
\(g(x, y) : [0; 1] \to \mathbb{R}\).

Now we review the operator order in \(B(H)\) which is the set of all bounded linear operators on a Hilbert space \((H; \langle ., . \rangle)\), and the continuous functional calculus for a bounded self-adjoint operator. For self-adjoint operators \(A, B \in B(H)\), we write \(A \leq B\) if \(\langle Ax, x \rangle \leq \langle Bx, x \rangle\) for every vector \(x \in H\), we call it the operator order.

Let \(A\) be a bounded self-adjoint linear operator on a complex Hilbert space \((H; \langle ., . \rangle)\). The Gelfand map establishes a \(*\)-isometrically isomorphism \(\Phi\) between the set \(C(Sp(A))\) of all continuous complex-valued functions defined on the spectrum of \(A\), denoted \(Sp(A)\), the \(C^*\)-algebra \(C^*(A)\) generated by \(A\) and the identity operator \(1_H\) on \(H\) as follows (see for instance [6], p.3). For any \(f, g \in C(Sp(A))\) and any \(\alpha, \beta \in \mathbb{C}\), we have

(i) \(\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)\);

(ii) \(\Phi(fg) = \Phi(f)\Phi(g)\) and \(\Phi(f^*) = \Phi(f)^*\);

(iii) \(\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|\);

(iv) \(\Phi(f_0) = 1_H\) and \(\Phi(f_1) = A\), where \(f_0(t) = 1\) and \(f_1(t) = t\) for \(t \in Sp(A)\).

With this notation, we define

\[ f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A)) \]  

(1.7)

and we call it the continuous functional calculus for a bounded self-adjoint operator \(A\).

If \(A\) is a bounded self-adjoint operator and \(f\) is a real-valued continuous function on \(Sp(A)\), then \(f(t) \geq 0\) for any \(t \in Sp(A)\) implies that \(f(A) \geq 0\), i.e. \(f(A)\) is a positive operator on \(H\). Moreover, if both \(f\) and \(g\) are real-valued functions on \(Sp(A)\) such that \(f(t) \leq g(t)\) for any \(t \in Sp(A)\), then \(f(A) \leq g(A)\) in the operator order in \(B(H)\).

A real valued continuous function \(f\) on an interval \(I \subseteq \mathbb{R}\) is said to be operator convex (operator concave) if

\[ f((1 - \lambda)A + \lambda B) \leq (\geq) (1 - \lambda)f(A) + \lambda f(B) \]

in the operator order in \(B(H)\), for all \(\lambda \in [0, 1]\) and for every bounded self-adjoint operators \(A\) and \(B\) in \(B(H)\) whose spectra are contained in \(I\).

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [6] and the references therein.

In [7], Ghazanfari et al. gave the concept of operator preinvex function and obtained Hermite-Hadamard type inequality for operator preinvex function.

**Definition 1.5** ([7]). Let \(X\) be a real vector space, a set \(S \subseteq X\) is said to be invex with respect to the map \(\eta : S \times S \to X\), if for every \(x, y \in S\) and \(t \in [0, 1]\),

\[ x + t\eta(x, y) \in S. \]  

(1.8)

It is obvious that every convex set is invex with respect to the map \(\eta(x, y) = x - y\), but there exist invex sets which are not convex (see [11]).

Let \(S \subseteq X\) be an invex set with respect to \(\eta : S \times S \to X\). For every \(x, y \in S\), the \(\eta\)-path \(P_{xv}\) joining the points \(x\) and \(v := x + \eta(y, x)\) is defined as follows

\[ P_{xv} := \{z : z = x + t\eta(y, x), t \in [0, 1]\}. \]

The mapping \(\eta\) is said to satisfy the condition \((C)\) if for every \(x, y \in S\) and \(t \in [0, 1]\),

\[ \eta(y, y + t\eta(x, y)) = -t\eta(x, y), \quad \eta(x, y + t\eta(x, y)) = (1 - t)\eta(x, y). \]  

(C)
Note that for every \( x, y \in S \) and every \( t_1, t_2 \in [0, 1] \) from condition (C) we have
\[
\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y),
\]
see [13, 16] for details.

Let \( A \) be a \( C^* \)-algebra, denote by \( A_{sa} \) the set of all self-adjoint elements in \( A \).

**Definition 1.6 ([7]).** Let \( S \subseteq B(H)_{sa} \) be an invex set with respect to \( \eta : S \times S \to B(H)_{sa} \). Then, the continuous function \( f : \mathbb{R} \to \mathbb{R} \) is said to be operator preinvex with respect to \( \eta \) on \( S \), if for every \( A, B \in S \) and \( t \in [0, 1] \),
\[
f(A + t\eta(B, A)) \leq (1 - t)f(A) + tf(B)
\]
in the operator order in \( B(H) \).

Every operator convex function is operator preinvex with respect to the map \( \eta(A, B) = A - B \), but the converse does not hold (see [7]).

**Theorem 1.7 ([7]).** Let \( S \subseteq B(H)_{sa} \) be an invex set with respect to \( \eta : S \times S \to B(H)_{sa} \) and \( \eta \) satisfies condition (C). If for every \( A, B \in S \) and \( V = A + \eta(B, A) \) the function \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is operator preinvex with respect to \( \eta \) on \( \eta \)-path \( P_{AV} \) with spectra of \( A \) and spectra of \( V \) in the interval \( I \). Then we have the inequality
\[
f\left(\frac{A + V}{2}\right) \leq \int_0^1 f((A + t\eta(B, A)))dt \leq \frac{f(A) + f(B)}{2}.
\]

In [8], Ghazanfari defined the operator \( s \)-convex function and proved Hermite-Hadamard type inequality for operator \( s \)-convex function as follows.

We denote by \( B(H)^+ \) the set of all positive operators in \( B(H) \) and
\[
C(H) := \{ A \in B(H)^+ : AB + BA \geq 0 \text{ for all } B \in B(H)^+ \}.
\]
It is obvious that \( C(H) \) is a closed convex cone in \( B(H) \).

**Definition 1.8 ([8]).** Let \( I \) be an interval in \( \mathbb{R}_0 \) and \( S \) be a convex subset of \( B(H)^+ \). A continuous function \( f : I \to \mathbb{R} \) is said to be operator \( s \)-convex on \( I \) for operators in \( S \) if
\[
f(\lambda A + (1 - \lambda)B) \leq \lambda^sf(A) + (1 - \lambda)^sf(B)
\]
in the operator order in \( B(H) \), for all \( \lambda \in [0, 1] \) and for every positive operators \( A \) and \( B \) in \( S \) whose spectra are contained in \( I \) and for some fixed \( s \in (0, 1] \).

**Theorem 1.9 ([8]).** Let \( f : I \subseteq \mathbb{R}_0 \to \mathbb{R} \) be an operator \( s \)-convex function on the interval \( I \) for operators in \( S \subseteq B(H)^+ \) and for some fixed \( s \in (0, 1] \). Then for all positive operators \( A \) and \( B \) in \( S \) with spectra in \( I \) we have the inequality
\[
2^{s-1}f\left(\frac{A + B}{2}\right) \leq \int_0^1 f((1 - t)A + tB)dt \leq \frac{f(A) + f(B)}{s + 1}.
\]

Motivated by the above results we investigate in this paper the operator version of the Hermite-Hadamard inequality for operator \( s \)-preinvex functions.
2. Main results

In order to verify our main results, the following preliminary definition and lemmas are necessary.

**Definition 2.1.** Let $I$ be an interval in $\mathbb{R}_0$ and $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)^+_{sa}$. Then, the continuous function $f : I \to \mathbb{R}$ is said to be operator $s$-preinvex with respect to $\eta$ on $I$ for operators in $S$, if

$$f(A + t\eta(B, A)) \leq (1 - t)s f(A) + tf(B) \quad (2.1)$$

in the operator order in $B(H)$, for all $t \in [0, 1]$ and every positive operators $A$ and $B$ in $S$ whose spectra are contained in $I$ and for some fixed $s \in (0, 1]$.

It is obvious that every operator 1-preinvex function is operator preinvex, and every operator $s$-convex function is operator $s$-preinvex with respect to the map $\eta(A, B) = A - B$.

**Lemma 2.2.** Let $A, B \in B(H)^+$. Then $AB + BA$ is positive if and only if $f(AB) \leq f(A) + f(B)$ for all non-negative operator monotone functions $f$ on $\mathbb{R}_0$.

Now, we give an example of operator $s$-preinvex function.

**Example 2.3.** Suppose that $1_H$ is the identity operator on a Hilbert space $H$, and

$$S := (1_H, 5 \cdot 1_H) = \{A \in B(H)^+_{sa} : 1_H < A < 5 \cdot 1_H\}.$$ \(\tag{2.2}\)

The map $\eta : S \times S \to B(H)^+_{sa}$ is defined by $\eta(A, B) = A - B$ for all $A > B \geq 0$ in the operator order in $B(H)$. Clearly $\eta$ satisfies condition $(C)$ and $S$ is an invex set with respect to $\eta$. From Lemma 2.2 and (1.12), the continuous function $f(t) = t^s(0 < s \leq 1)$ is operator $s$-preinvex with respect to $\eta$ on $S$ for operators in $C(H)$.

The following lemma is a generalization of Proposition 1 in [7].

**Lemma 2.4.** Let $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)^+_{sa}$ and $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a continuous function on the interval $I$. Suppose that $\eta$ satisfies condition $(C)$ on $S$. Then for every $A, B \in S$ and $V = A + \eta(B, A)$ and for some fixed $s \in (0, 1]$, the function $f$ is operator $s$-preinvex with respect to $\eta$ on $\eta$-path $P_{AV}$ with spectra of $A$ and with spectra of $V$ in the interval $I$ if and only if the function $\varphi_{x, A, B} : [0, 1] \to \mathbb{R}$ defined by

$$\varphi_{x, A, B}(t) := \langle f(A + t\eta(B, A))x, x \rangle \quad (2.3)$$

is $s$-convex on $[0, 1]$ for every $x \in H$ with $\|x\| = 1$.

**Proof.** Suppose that $x \in H$ with $\|x\| = 1$ and $\varphi_{x, A, B} : [0, 1] \to \mathbb{R}$ is $s$-convex on $[0, 1]$ for some fixed $s \in (0, 1]$. For every $C_1 := A + t_1\eta(B, A) \in P_{AV}$, $C_2 := A + t_2\eta(B, A) \in P_{AV}$, fix $\lambda \in [0, 1]$, by (2.2), we have

$$\langle f(C_1 + \lambda\eta(C_2, C_1))x, x \rangle = \langle f(A + ((1 - \lambda)t_1 + \lambda t_2)\eta(B, A))x, x \rangle$$

$$= \varphi_{x, A, B}((1 - \lambda)t_1 + \lambda t_2)$$

$$\leq (1 - \lambda)s\varphi_{x, A, B}(t_1) + \lambda s\varphi_{x, A, B}(t_2)$$

$$= (1 - \lambda)s\varphi_{x, A, B}(t_1) + \lambda s\varphi_{x, A, B}(t_2).$$ \(\tag{2.3}\)

Hence, $f$ is operator $s$-preinvex with respect to $\eta$ on $\eta$-path $P_{AV}$.

Conversely, let $A, B \in S$ and $f$ be operator $s$-preinvex with respect to $\eta$ on $\eta$-path $P_{AV}$ for some fixed $s \in (0, 1]$. Suppose that $t_1, t_2 \in [0, 1]$. Then for every $\lambda \in [0, 1]$ and $x \in H$ with $\|x\| = 1$, we have

$$\varphi_{x, A, B}((1 - \lambda)t_1 + \lambda t_2) = \langle f(A + t_1\eta(B, A) + \lambda\eta(A + t_2\eta(B, A), A + t_1\eta(B, A)))x, x \rangle$$

$$\leq (1 - \lambda)s\langle f(A + t_1\eta(B, A))x, x \rangle + \lambda s\langle f(A + t_2\eta(B, A))x, x \rangle$$

$$= (1 - \lambda)s\varphi_{x, A, B}(t_1) + \lambda s\varphi_{x, A, B}(t_2).$$ \(\tag{2.4}\)

Therefore, $\varphi_{x, A, B}$ is $s$-convex on $[0, 1]$. The proof of Lemma 2.4 is complete. \(\square\)
The following theorem is the generalization of Hermite-Hadamard’s inequality for operator $s$-preinvex functions.

**Theorem 2.5.** Let $S \subseteq B(H)_{sa}^+$ be an invex set with respect to $\eta : S \times S \rightarrow B(H)_{sa}^+$ and $\eta$ satisfy condition (C) on $S$. If for every $A, B \in S$ and $V = A + \eta(B, A)$ and for some fixed $s \in (0, 1]$, the continuous function $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ is operator $s$-preinvex with respect to $\eta$ on $\eta$-path $P_{AV}$ with spectra of $A$ and with spectra of $V$ in the interval $I$. Then we have the inequality

$$2^{s-1}f\left(\frac{A + V}{2}\right) \leq \int_0^1 f(A + t\eta(B, A))dt \leq \frac{f(A) + f(B)}{s + 1}.$$  (2.5)

**Proof.** For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have

$$\langle (A + t\eta(B, A))x, x \rangle = \langle Ax, x \rangle + t\langle \eta(B, A)x, x \rangle \in I,$$  (2.6)

since $\langle Ax, x \rangle \in Sp(A) \subseteq I$ and $\langle Vx, x \rangle \in Sp(V) \subseteq I$.

Continuity of $f$ and (2.6) imply that the operator valued integral $\int_0^1 f(A + t\eta(B, A))dt$ exists.

Since $\eta$ satisfies condition (C) and $f$ is $s$-preinvex with respect to $\eta$, for every $t \in [0, 1]$, we have

$$f\left(A + \frac{1}{2}\eta(B, A)\right) \leq \frac{1}{2^s}f(A + t\eta(B, A)) + \frac{1}{2^s}f(A + (1-t)\eta(B, A))$$

$$\leq \frac{1}{2^s}[(1-t)^s + t^s][f(A) + f(B)].$$  (2.7)

Integrating the inequality (2.7) over $t \in [0, 1]$ and taking into account that

$$\int_0^1 f(A + t\eta(B, A))dt = \int_0^1 f(A + (1-t)\eta(B, A))dt,$$  (2.8)

we obtain the inequality (2.5), which completes the proof of Theorem 2.5. □

**Remark 2.6.** Choosing $s = 1$ and $\eta(B, A) = B - A$ respectively, we obtain Theorem 1.7 and Theorem 1.9.

Now we establish the estimates of both sides of Hermite-Hadamard type inequality in which some operator $s$-preinvex functions of selfadjoint operators in Hilbert spaces are involved.

**Theorem 2.7.** Let the function $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be continuous, $S \subseteq B(H)_{sa}^+$ be an invex set with respect to $\eta : S \times S \rightarrow B(H)_{sa}^+$, and $\eta$ satisfy condition (C) on $S$. If for every $A, B \in S$ and $V = A + \eta(B, A)$ and for some fixed $s \in (0, 1]$, the function $f$ is operator $s$-preinvex with respect to $\eta$ on $\eta$-path $P_{AV}$ with spectra of $A$ and with spectra of $V$ in the interval $I$. Then for every $a, b \in (0, 1)$ with $a < b$ and every $x \in H$ with $\|x\| = 1$, the following inequality holds,

$$\left|\int_0^{(a+b)/2} f(A + u\eta(B, A))du \cdot x, x\right| - \frac{1}{b - a} \int_a^b \left(\int_0^t f(A + u\eta(B, A))du \cdot x, x\right)dt$$

$$\leq \frac{b - a}{4(s + 1)(s + 2)} \left[\langle f(A + an(B, A))x, x \rangle + 2(s + 1)\langle f\left(A + \frac{b + a}{2}\eta(B, A)\right)x, x\rangle + \langle f(A + b\eta(B, A))x, x\rangle\right].$$  (2.9)

Moreover, we have

$$\left\|\int_0^{(a+b)/2} f(A + u\eta(B, A))du - \frac{1}{b - a} \int_a^b \int_0^t f(A + u\eta(B, A))du dt\right\|$$

$$\leq \frac{b - a}{2(s + 1)} \left[\|f(A + an(B, A))\| + 2(s + 1)\|f\left(A + \frac{b + a}{2}\eta(B, A)\right)\| + \|f(A + b\eta(B, A))\|\right].$$  (2.10)
Proof. Let \( A, B \in S \) and \( a, b \in (0, 1) \) with \( a < b \). For \( x \in H \) with \( \|x\| = 1 \), we define the function \( \varphi : [a, b] \subseteq [0, 1] \to \mathbb{R}_0 \) by
\[
\varphi(t) := \left\langle \int_0^t f(A + u\eta(B, A))du, x, x \right\rangle.
\]

Utilizing the continuity of \( f \), the continuity property of the inner product, and the properties of the integral of operator-valued functions, we have
\[
\left\langle \int_0^t f(A + u\eta(B, A))du, x, x \right\rangle = \int_0^t \langle f(A + u\eta(B, A))x, x \rangle du.
\]

Since \( f(A + u\eta(B, A)) \geq 0 \), \( \varphi(t) \geq 0 \) for all \( t \in [a, b] \). Obviously for every \( t \in [a, b] \), we have
\[
\varphi'(t) = \langle f(A + t\eta(B, A))x, x \rangle \geq 0,
\]
hence, \( |\varphi'(t)| = \varphi'(t) \).

Since \( f \) is operator \( s \)-preinexact with respect to \( \eta \) on \( \eta \)-path \( P_{AV} \) for some fixed \( s \in (0, 1) \), by Lemma 2.4 \( \varphi' \) is \( s \)-convex. Applying Theorem 1.3 to the function \( \varphi \) implies that
\[
|\varphi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \varphi(t)dt| \leq \frac{b-a}{4(s+1)(s+2)} \left[ \varphi'(a) + 2(s+1)\varphi\left(\frac{a+b}{2}\right) + \varphi'(b) \right],
\]
and we know that the inequality (2.9) holds. Taking supremum over both sides of inequality (2.9) for all \( x \) with \( \|x\| = 1 \), we deduce that the inequality (2.10) holds. Theorem 2.7 is thus proved.

**Corollary 2.8.** Under the assumptions of Theorem 2.7, it turns out that
\[
\left| \int_0^{(a+b)/2} f(A + u\eta(B, A))du \right| - \frac{1}{b-a} \int_a^b \left[ \int_0^t f(A + u\eta(B, A))du \right] dt \leq \frac{(2^{2-s} + 1)(b-a)}{2(s+1)(s+2)} \left[ \|f(A + an(B, A))\| + \|f(A + b\eta(B, A))\| \right].
\]

Furthermore, we have
\[
\left\| \int_0^{(a+b)/2} f(A + u\eta(B, A))du - \frac{1}{b-a} \int_a^b \int_0^t f(A + u\eta(B, A))dudt \right\| \leq \frac{(2^{2-s} + 1)(b-a)}{2(s+1)(s+2)} \left[ \|f(A + an(B, A))\| + \|f(A + b\eta(B, A))\| \right].
\]

**Proof.** As the proof of Theorem 2.7 employing \( s \)-convexity of \( \varphi \) and (2.11) yield the results of Corollary 2.8.

**Corollary 2.9.** With the conditions of Theorem 2.7, if \( s = 1 \), then
\[
\left| \int_0^{(a+b)/2} f(A + u\eta(B, A))du \right| - \frac{1}{b-a} \int_a^b \left[ \int_0^t f(A + u\eta(B, A))du \right] dt \leq \frac{b-a}{4} \left[ \langle f(A + an(B, A))x, x \rangle + 4\langle f(A + \frac{a+b}{2}\eta(B, A))x, x \rangle + \langle f(A + b\eta(B, A))x, x \rangle \right].
\]

In addition, we have
\[
\left\| \int_0^{(a+b)/2} f(A + u\eta(B, A))du - \frac{1}{b-a} \int_a^b \int_0^t f(A + u\eta(B, A))dudt \right\| \leq \frac{b-a}{4} \left[ \|f(A + an(B, A))\| + 4\|f(A + \frac{a+b}{2}\eta(B, A))\| + \|f(A + b\eta(B, A))\| \right].
\]
Corollary 2.10. Under the assumptions of Theorem 2.7, if $\eta(B, A) = B - A$, then
\[
\left| \frac{1}{2} \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| + \frac{1}{2} \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| dt \leq \frac{b - a}{2(s+1)} \left[ \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right] + \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| dt + \frac{1}{2} \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| dt \leq \frac{(b - a)(2^{2s+1} + 1)}{2^{s(s+1)}(s+2)} \left[ \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right] + \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| dt. \]

Furthermore, we have
\[
\left| \int_a^b f(A + u\eta(B, A))du \right| + \frac{1}{2} \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| dt \leq \frac{(b - a)(2^{2s+1} + 1)}{2^{s(s+1)}(s+2)} \left[ \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right] + \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| dt. \]

Proof. With the inequality (1.5) and the similar approach of the proof of Theorem 2.7, it is a simple verification. We omit the routine details. \qed

Corollary 2.13. With the conditions of Theorem 2.12, if $s = 1$, then
\[
\left| \frac{1}{2} \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| + \frac{1}{2} \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| dt \leq \frac{5(b - a)}{12} \left[ \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right] + \left( \int_a^b f(A + u\eta(B, A))du \right) x, x \right| dt. \]
Moreover, we have
\[
\left\| \frac{1}{2} \int_0^a f(A + u\eta(B, A))du + \frac{1}{2} \int_0^b f(A + u\eta(B, A))du - \frac{1}{b - a} \int_a^b \int_0^t f(A + u\eta(B, A))dudu \right\| 
\leq \frac{5(b - a)}{12} \left[ \frac{\|f(A + a\eta(B, A))\| + \|f(A + b\eta(B, A))\|}{2} \right].
\]  
\tag{2.21}

**Corollary 2.14.** Under the assumptions of Theorem 2.12 if \(\eta(B, A) = B - A\), then
\[
\left\| \frac{1}{2} \int_0^a f((1 - u)A + uB)du \ x, x \right\| + \left\| \frac{1}{2} \int_0^b f((1 - u)A + uB)du \ x, x \right\| 
\leq \frac{(b - a)(2^{s+1} + 1)}{2^s(s + 1)(s + 2)} \left[ \frac{\|f((1 - a)A + aB)x, x\| + \|f((1 - b)A + bB)x, x\|}{2} \right].
\]  
\tag{2.22}

In addition, we have
\[
\left\| \frac{1}{2} \int_0^a f((1 - u)A + uB)du + \frac{1}{2} \int_0^b f((1 - u)A + uB)du - \frac{1}{b - a} \int_a^b \int_0^t f((1 - u)A + uB)dudu \right\| 
\leq \frac{(b - a)(2^{s+1} + 1)}{2^s(s + 1)(s + 2)} \left[ \frac{\|f((1 - a)A + aB)\| + \|f((1 - b)A + bB)\|}{2} \right].
\]  
\tag{2.23}

**Remark 2.15.** Corollaries 2.13 and 2.14 are generalizations of Theorem 1.4 and Theorem 4 in [12], respectively.

In what follows, Hermite-Hadamard type inequalities for the product of two operator \(s\)-preinvex functions are established.

For some fixed \(s_1, s_2 \in (0, 1]\), let \(f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}\) be an operator \(s_1\)-preinvex function and \(g : I \rightarrow \mathbb{R}\) be an operator \(s_2\)-preinvex function on the interval \(I\). Then for all positive operators \(A, B\) on a Hilbert space \(H\) with spectra in \(I\), we define real functions \(M(A, B)\) and \(N(A, B)\) on \(H\) by
\[
M(A, B)(x) = \langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle, \quad x \in H,
\]
\[
N(A, B)(x) = \langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle, \quad x \in H.
\]  
\tag{2.24}

We note that, the Beta function is defined as follows:
\[
\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x > 0, y > 0.
\]  
\tag{2.25}

The following two theorems are the generalization of Theorem 3.1 and Theorem 3.2 in [8] respectively for operator \(s\)-preinvex functions.

**Theorem 2.16.** Let \(S \subseteq B(H)_{sa}^+\) be an invex set with respect to \(\eta : S \times S \rightarrow B(H)_{sa}^+\) and \(\eta\) satisfy condition (C) on \(S\). If for every \(A, B \in S\) and \(V = A + \eta(B, A)\) and for some fixed \(s_1, s_2 \in (0, 1]\), the continuous function \(f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}\) is an operator \(s_1\)-preinvex function and \(g : I \rightarrow \mathbb{R}\) is an operator \(s_2\)-preinvex function on the interval \(I\) with respect to \(\eta\) on \(\eta\)-path \(P_{AV}\) with spectra of \(A\) and with spectra of \(V\) in the interval \(I\). Then we have the inequality
\[
\int_0^1 \langle f(A + t\eta(B, A))x, x \rangle \langle g(A + t\eta(B, A))x, x \rangle dt 
\leq \frac{1}{s_1 + s_2 + 1} \left[ M(A, B)(x) + s_1 \beta(s_1, s_2 + 1) N(A, B)(x) \right]
\]  
\tag{2.26}
holds for any \(x \in H\) with \(|x| = 1\), where \(M(A, B)\) and \(N(A, B)\) are defined in (2.24), and the Beta function is defined in (2.25).
Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have
$$\langle (A + t\eta(B, A))x, x \rangle = \langle Ax, x \rangle + t\langle \eta(B, A)x, x \rangle \in I,$$

since $\langle Ax, x \rangle \in Sp(A) \subseteq I$ and $\langle Vx, x \rangle \in Sp(V) \subseteq I$.

From the continuity of $f$, $g$, it shows that the operator valued integral $\int_0^1 f(A + t\eta(B, A))dt$, $\int_0^1 g(A + t\eta(B, A))dt$, and $\int_0^1 (fg)(A + t\eta(B, A))dt$ exist.

Since $f : I \rightarrow \mathbb{R}$ is operator $s_1$-preinvex and $g : I \rightarrow \mathbb{R}$ is operator $s_2$-preinvex for some fixed $s_1, s_2 \in (0, 1]$, therefore for every $t \in [0, 1]$ we derive
$$\langle f(A + t\eta(B, A))x, x \rangle (g(A + t\eta(B, A))x, x)$$
$$\leq (1 - t)^{s_1+s_2} \langle f(A)x, x \rangle \langle g(A)x, x \rangle + (1 - t)^{s_1} t^{s_2} \langle f(A)x, x \rangle \langle g(B)x, x \rangle$$
$$+ t^{s_1} (1 - t)^{s_2} \langle f(B)x, x \rangle \langle g(A)x, x \rangle + t^{s_1+s_2} \langle f(B)x, x \rangle \langle g(B)x, x \rangle. \tag{2.27}$$

Integrating both sides of (2.27) over $t \in [0, 1]$, we get the required inequality (2.26). The proof of Theorem 2.16 is complete. \hfill \Box

Corollary 2.17. Under the assumptions of Theorem 2.16 if $s_1 = s_2 = s$, then
$$\int_0^1 \langle f(A + t\eta(B, A))x, x \rangle (g(A + t\eta(B, A))x, x) dt$$
$$\leq \frac{1}{2s + 1} [M(A, B)(x) + s\beta(s, s + 1)N(A, B)(x)]. \tag{2.28}$$

Specially, if $s_1 = s_2 = 1$, then
$$\int_0^1 \langle f(A + t\eta(B, A))x, x \rangle (g(A + t\eta(B, A))x, x) dt \leq \frac{2M(A, B)(x) + N(A, B)(x)}{6}. \tag{2.29}$$

Corollary 2.18. With the conditions of Theorem 2.16 if $\eta(B, A) = B - A$, then
$$\int_0^1 \langle f((1 - t)A + tB)x, x \rangle (g((1 - t)A + tB)x, x) dt$$
$$\leq \frac{1}{s_1 + s_2 + 1} [M(A, B)(x) + s_1\beta(s_1, s_2 + 1)N(A, B)(x)]. \tag{2.30}$$

Theorem 2.19. Let $S \subseteq B(H)_{sa}^+$ be an invex set with respect to $\eta : S \times S \rightarrow B(H)_{sa}^+$ and $\eta$ satisfy condition (C) on $S$. If for every $A, B \in S$ and $V = A + \eta(B, A)$ and for some fixed $s_1, s_2 \in (0, 1]$, the continuous function $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ is an operator $s_1$-preinvex function and $g : I \rightarrow \mathbb{R}$ is an operator $s_2$-preinvex function on the interval $I$ with respect to $\eta$ on $\eta$-path $P_{AV}$ with spectra of $A$ and with spectra of $V$ in the interval $I$. Then we have that the inequality
$$2^{s_1+s_2-1} \left\langle f \left( \frac{A + V}{2} \right) x, x \right\rangle \left\langle g \left( \frac{A + V}{2} \right) x, x \right\rangle \tag{2.31}$$
$$\leq \int_0^1 \langle f(A + t\eta(B, A))x, x \rangle (g(A + t\eta(B, A))x, x) dt$$
$$+ \frac{1}{s_1 + s_2 + 1} [M(A, B)(x) + s_1\beta(s_1, s_2 + 1)M(A, B)(x)] \tag{2.32}$$
holds for any $x \in H$ with $\|x\| = 1$, where $M(A, B)$ and $N(A, B)$ are defined on $H$ in (2.24) and the Beta function is defined in (2.25).
Proof. Since \( f : I \to \mathbb{R} \) is operator \( s_1 \)-preinvex and \( g : I \to \mathbb{R} \) be operator \( s_2 \)-preinvex for some fixed \( s_1, s_2 \in (0, 1] \), therefore for every \( t \in [0, 1] \) we have
\[
\begin{align*}
\left\langle f \left( \frac{A + V}{2} \right), x \right\rangle \left\langle g \left( \frac{A + V}{2} \right), x \right\rangle \\
\leq \frac{1}{2^{s_1}} \left[ f(A + t\eta(B, A)) + f(A + (1 - t)\eta(B, A)) \right] \left\langle \frac{A + V}{2}, x \right\rangle \\
+ \frac{1}{2^{s_2}} \left[ g(A + t\eta(B, A)) + g(A + (1 - t)\eta(B, A)) \right] \left\langle \frac{A + V}{2}, x \right\rangle \\
\leq \frac{1}{2^{s_1 + s_2}} \left[ \langle f(A + t\eta(B, A))x, x \rangle g(A + t\eta(B, A))x, x \right] \\
+ \langle f(A + (1 - t)\eta(B, A))x, x \rangle g(A + (1 - t)\eta(B, A))x, x \right] \\
+ \frac{1}{2^{s_1 + s_2}} \left[ t^{s_1 + s_2} + (1 - t)^{s_1 + s_2} \right] \left[ \langle f(A)x, x \rangle g(B)x, x \right] + \langle f(B)x, x \rangle g(A)x, x \right] \\
+ \langle f(B)x, x \rangle g(B)x, x \right] \right). \quad (2.33)
\end{align*}
\]
By integrating over \( t \in [0, 1] \) and taking into account that
\[
\int_0^1 \langle f(A + t\eta(B, A))x, x \rangle g(A + t\eta(B, A))x, x \rangle dt \\
= \int_0^1 \langle f(A + (1 - t)\eta(B, A))x, x \rangle g(A + (1 - t)\eta(B, A))x, x \rangle dt,
\]
we obtain the required inequality \((2.31)\). Theorem \(2.19\) is thus proved.

Corollary 2.20. Under the assumptions of Theorem 2.19, if \( s_1 = s_2 = s \), then
\[
2^{2s - 1} \left\langle f \left( \frac{A + V}{2} \right), x \right\rangle \left\langle g \left( \frac{A + V}{2} \right), x \right\rangle \\
\leq \int_0^1 \langle f(A + t\eta(B, A))x, x \rangle g(A + t\eta(B, A))x, x \rangle dt \\
+ \frac{1}{2^{s_1 + s_2}} \left( N(A, B)(x) + s\beta(s, s + 1)M(A, B)(x) \right). \quad (2.34)
\]
In particular, if \( s_1 = s_2 = 1 \), then
\[
2 \left\langle f \left( \frac{A + V}{2} \right), x \right\rangle \left\langle g \left( \frac{A + V}{2} \right), x \right\rangle \\
\leq \int_0^1 \langle f(A + t\eta(B, A))x, x \rangle g(A + t\eta(B, A))x, x \rangle dt \\
+ \frac{2N(A, B)(x) + M(A, B)(x)}{6}. \quad (2.35)
\]
Corollary 2.21. With the conditions of Theorem 2.19, if \( \eta(B, A) = B - A \), then
\[
2^{s_1 + s_2 - 1} \left\langle f \left( \frac{A + B}{2} \right), x \right\rangle \left\langle g \left( \frac{A + B}{2} \right), x \right\rangle \\
\leq \int_0^1 \langle f((1 - t)A + tB)x, x \rangle g((1 - t)A + tB)x, x \rangle dt \\
+ \frac{1}{s_1 + s_2 + 1} \left( N(A, B)(x) + s_1\beta(s_1, s_2 + 1)M(A, B)(x) \right). \quad (2.36)
\]
Corollary 2.22. With the assumptions of Theorem 2.16 and Theorem 2.19, we get

\[ 2^{s_1+s_2-1} \left\langle f\left( \frac{A+B}{2} \right) x, x \right\rangle \left\langle g\left( \frac{A+B}{2} \right) x, x \right\rangle - \frac{1}{s_1+s_2+1} \left[ N(A,B)(x) + s_1 \beta(s_1, s_2 + 1) M(A,B)(x) \right] \]

\[ \leq \int_0^1 \left\langle f((1-t)A + tB)x, x \right\rangle \left\langle g((1-t)A + tB)x, x \right\rangle dt \]

\[ \leq \frac{1}{s_1+s_2+1} \left[ M(A,B)(x) + s_1 \beta(s_1, s_2 + 1) N(A,B)(x) \right]. \]  

(2.37)

Acknowledgements

This work was supported in part by the Fundamental Research Funds for the Central Universities (No. DUT14ZD208) and by the NNSF of China (No. 61163034 and No. 61373067). The authors are thankful to the referee for giving valuable comments and suggestions which helped to improve the final version of this paper.

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