Fixed point theorems for multivalued $G$-contractions in Hausdorff $b$-Gauge spaces

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Communicated by Yonghong Yao

Abstract

In this paper, we extend gauge spaces in the setting of $b$ metric spaces and prove fixed point theorems for multivalued mappings in this new setting endowed with a graph. An example is constructed to substantiate our result. We also discuss possible application of our result for solving integral equations. ©2015 All rights reserved.

Keywords: Gauge space, graph, fixed point, nonlinear integral equation.

2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

Czerwik [11] introduced the notion of a $b$-metric space. Let $X$ be a nonempty set. A mapping $d: X \times X \to [0, \infty)$ is said to be a $b$-metric on $X$, if there exists $s \geq 1$ such that for each $x, y, z \in X$, we have

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

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Received 2015-5-1
The triplet $(X, d, s)$ is said to be a $b$-metric space.

Note that every metric space is a $b$-metric but converse is not true.

Convergence of a sequence in a $b$-metric space is defined in a similar fashion as in a metric space. A sequence $\{x_n\} \subset X$ is a Cauchy sequence in $(X, d, s)$, if for each $\epsilon > 0$ there exists a natural number $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for each $m, n \geq N(\epsilon)$. A $b$-metric space $(X, d, s)$ is a complete if each Cauchy sequence in $X$ converges to some point of $X$.

Czerwik [11] extended Banach contraction principle for self mappings on $b$ metric spaces; for recent research in this direction, please see: Phiangsungnoen et al. [20], Shatanawi et al [23]. Czerwik [12] further extended the notion of a $b$-metric space $(X, d, s)$ by defining Hausdorff metric for the space of all nonempty closed and bounded subsets of the $b$-metric space $(X, d, s)$.

Let $(X, d, s)$ be a $b$ metric space. For $x \in X$ and $A \subset X$, $d(x, A) = \inf\{d(x, a) : a \in A\}$. Denote by $CB(X)$ the class of all nonempty closed and bounded subsets of $X$ and by $CL(X)$ the class of all nonempty closed subsets of $X$. For $A, B \in CB(X)$, the function

$$H: CB(X) \times CB(X) \to [0, \infty), \quad H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

is said to be a Hausdorff $b$-metric induced by a $b$-metric space $(X, d, s)$. A Hausdorff $b$ metric space enjoys the same properties as a Hausdorff metric, except for triangular inequality which in Hausdorff $b$ metric spaces has the following form $H(A, B) \leq s[H(A, C) + H(C, B)]$. Czerwik [12] extended Nadler’s fixed point theorem in the setting of Hausdorff $b$ metric spaces.

Jachymski [15] introduced the notion of Banach $G$-contraction to extend the notion of Banach contraction, where $G$ is a graph in the metric space whose vertex set coincides with the metric space. He obtain some fixed point theorems for such mappings on complete metric space. Afterwards, many authors extended Banach $G$-contraction in single as well as multivalued case, see for examples: Tiammee and Suantai [24], Bojor [4, 5, 6], Nicolae et al. [19], Aleomraninejad et al. [2], Asl et al. [3].

One may characterize gauge spaces by the fact that the distance between two distinct points of the space may be zero. For details on gauge spaces, we refer the reader to [13]. Frigon [14] and Chis and Precup [10] generalized the Banach contraction principle on gauge spaces. Some interesting results are also been obtained by the authors: Agarwal et al. [1], Chifu and Petrusel [9], Cherichi et al. [8, 7], Lazara and Petrusel [18], Jleli et al. [16].

By using $b$ metric spaces, in this paper, we first introduce the notion of $b_s$-gauge spaces. Then we extend this notion to define $b_s$-gauge structure on the space of nonempty closed subsets of the $b$ metric space and prove some fixed point theorems for multivalued $G$ contractions. To substantiate our main result we have constructed an example. Moreover, a possible application of our result in solving an integral equation is also been discussed.

2. Main results

We begin this section by introducing the notion of a $b_s$-pseudo metric space.

**Definition 2.1.** Let $X$ be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called $b_s$-pseudo metric on $X$ if there exists $s \geq 1$ such that for each $x, y, z \in X$, we have

(i) $d(x, x) = 0$ for each $x \in X$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq s[d(x, y) + d(y, x)]$.

**Remark 2.2.** Every $b$-metric space $(X, d, s)$ is a $b_s$-pseudo metric space, but the converse is not true.
Example 2.3. Let \( X = C([0, \infty), \mathbb{R}) \). Define a function
\[
d : X \times X \to [0, \infty), \quad d(x(t), y(t)) = \max_{t \in [0,1]} (x(t) - y(t))^2.
\]
Then

(i) It is clear that \( d \) is not a metric on \( X \).

(ii) \( d \) is not a pseudo metric on \( X \). In this respect, consider \( x, y, z \in C([0, \infty), \mathbb{R}) \) be defined by
\[
x(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ t - 1 & \text{if } t > 1, \end{cases}
\]
y(t) = 3, for each \( t \geq 0 \), and \( z(t) = -3 \), for each \( t \geq 0 \). Then, we can see that \( d(y, z) = 36 \nless 18 = d(y, x) + d(x, z) \).

(iii) \( d \) is not a \( b \)-metric on \( X \). Since, if \( u, v \in C([0, \infty), \mathbb{R}) \) are defined by
\[
u(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ t - 1 & \text{if } t > 1, \end{cases}
\]
and
\[
u(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ 2t - 2 & \text{if } t > 1, \end{cases}
\]
then \( u \neq v \), but \( d(u, v) = 0 \).

(iv) \( d \) is \( b_2 \)-pseudo metric on \( X \) with \( s = 2 \).

In order to define gauge spaces in the setting of \( b_s \)-pseudo metrics we need to define following.

Definition 2.4. Let \( X \) be a nonempty set endowed with the \( b_s \)-pseudo metric \( d \). The \( d_s \)-ball of radius \( \epsilon > 0 \) centered at \( x \in X \) is the set
\[
B(x, d, \epsilon) = \{ y \in X : d(x, y) < \epsilon \}.
\]

Definition 2.5. A family \( F = \{ d_\nu : \nu \in \mathcal{A} \} \) of \( b_s \)-pseudo metrics is said to be separating if for each pair \( (x, y) \) with \( x \neq y \), there exists \( d_\nu \in F \) with \( d_\nu(x, y) \neq 0 \).

Definition 2.6. Let \( X \) be a nonempty set and \( F = \{ d_\nu : \nu \in \mathcal{A} \} \) be a family of \( b_s \)-pseudo metrics on \( X \). The topology \( T(F) \) having subbases the family
\[
B(F) = \{ B(x, d_\nu, \epsilon) : x \in X, d_\nu \in F \text{ and } \epsilon > 0 \}
\]
of balls is called topology induced by the family \( F \) of \( b_s \)-pseudo metrics. The pair \( (X, T(F)) \) is called a \( b_s \)-gauge space. Note that \( (X, T(F)) \) is Hausdorff if \( F \) is separating.

Definition 2.7. Let \((X, T(F))\) be a \( b_s \)-gauge space with respect to the family \( F = \{ d_\nu : \nu \in \mathcal{A} \} \) of \( b_s \)-pseudo metrics on \( X \) and \( \{x_n\} \) is a sequence in \( X \) and \( x \in X \). Then:

(i) The sequence \( \{x_n\} \) converges to \( x \) if for each \( \nu \in \mathcal{A} \) and \( \epsilon > 0 \), there exists \( N_0 \in \mathbb{N} \) such that \( d_\nu(x_n, x) < \epsilon \) for each \( n \geq N_0 \). We denote it as \( x_n \to_F x \);

(ii) The sequence \( \{x_n\} \) is a Cauchy sequence if for each \( \nu \in \mathcal{A} \) and \( \epsilon > 0 \), there exists \( N_0 \in \mathbb{N} \) such that \( d_\nu(x_n, x_m) < \epsilon \) for each \( n, m \geq N_0 \);

(iii) \((X, T(F))\) is complete if each Cauchy sequence in \((X, T(F))\) is convergent in \( X \);
A subset of $X$ is said to be closed if it contains the limit of each convergent sequence of its elements.

**Remark 2.8.** When $s = 1$, then all above definitions reduce to the corresponding definitions in a gauge space.

Subsequently, in this paper, $\mathcal{A}$ is directed set and $X$ is a nonempty set endowed with a separating complete $b_s$-gauge structure $\{d_\nu : \nu \in \mathcal{A}\}$. Further, $G = (V, E)$ is a directed graph in $X \times X$, where the set of its vertices $V$ is equal to $X$ and set of its edges $E$ contains $\{(x, x) : x \in V\}$. Furthermore, $G$ has no parallel edges. For each $d_\nu \in \mathcal{F}$, $CL_\nu(X)$ denote the set of all nonempty closed subsets of $X$ with respect to $d_\nu$. For each $\nu \in \mathcal{A}$ and $A, B \in CL_\nu(X)$, the function $H_\nu : CL_\nu(X) \times CL_\nu(X) \to [0, \infty)$ defined by

$$H_\nu(A, B) = \begin{cases} \max \{ \sup_{x \in A} d_\nu(x, B), \sup_{y \in B} d_\nu(y, A) \} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases}$$

is a generalized Hausdorff $b_s$-pseudo metric on $CL_\nu(X)$. We denote by $CL(X)$ the set of all nonempty closed subsets in the $b_s$-gauge space $(X, \mathcal{T}(\mathcal{F}))$.

**Theorem 2.9.** Let $T : X \to CL(X)$ be a mapping such that for each $\nu \in \mathcal{A}$, we have

$$H_\nu(Tx, Ty) \leq a_\nu d_\nu(x, y) + b_\nu d_\nu(x, Tx) + c_\nu d_\nu(y, Ty) + e_\nu d_\nu(x, Ty) + L_\nu d_\nu(y, Tx), \quad (2.1)$$

for all $(x, y) \in E$, where $a_\nu, b_\nu, c_\nu, e_\nu, L_\nu \geq 0$, and $s^2a_\nu + s^2b_\nu + s^2c_\nu + 2s^3e_\nu < 1$.

Assume that following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) if $(x, y) \in E$, for $u \in Tx$ and $v \in Ty$ such that $d_\nu(u, v) \leq d_\nu(x, y)$ for each $\nu \in \mathcal{A}$, then $(u, v) \in E$;

(iii) if $\{x_n\}$ is a sequence in $X$ such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$;

(iv) for each $\{q_\nu : q_\nu > 1\}_{\nu \in \mathcal{A}}$ and $x \in X$ there exists $y \in Tx$ such that

$$d_\nu(x, y) \leq q_\nu d_\nu(x, Tx) \quad \forall \nu \in \mathcal{A}.$$}

Then $T$ has a fixed point.

**Proof.** By hypothesis (i), there exist $x_0, x_1 \in X$ such that $x_1 \in Tx_0$ and $(x_0, x_1) \in E$. Now, it follows form (2.1) that

$$H_\nu(Tx_0, Tx_1) \leq a_\nu d_\nu(x_0, x_1) + b_\nu d_\nu(x_0, Tx_0) + c_\nu d_\nu(x_1, Tx_1) + e_\nu d_\nu(x_0, Tx_1) + L_\nu d_\nu(x_1, Tx_0), \quad (2.2)$$

for all $\nu \in \mathcal{A}$.

Since $d_\nu(x_1, Tx_1) \leq H_\nu(Tx_0, Tx_1)$ and $d_\nu(x_0, Tx_1) \leq d_\nu(x_0, x_1) + d_\nu(x_1, Tx_1)$, therefore from (2.2), we get

$$d_\nu(x_1, Tx_1) \leq \frac{1}{\xi_\nu} d_\nu(x_0, x_1) \quad (2.3)$$

where, $\xi_\nu = \frac{1 - c_\nu - e_\nu}{a_\nu + b_\nu + s\nu} > 1$. Using hypothesis (iv) there exists $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \leq \sqrt{\xi_\nu} d_\nu(x_1, Tx_1). \quad (2.4)$$

Combining (2.3) and (2.4), we get

$$d_\nu(x_1, x_2) \leq \frac{1}{\sqrt{\xi_\nu}} d_\nu(x_0, x_1) \quad \forall \nu \in \mathcal{A}. \quad (2.5)$$
Hypothesis (ii) and \[2.5\], implies that \((x_1, x_2) \in E\). Continuing in the same way, we get a sequence \(\{x_m\}\) in \(X\) such that \((x_m, x_{m+1}) \in E\) and

\[
d_\nu(x_m, x_{m+1}) \leq \left(\frac{1}{\sqrt{\kappa}}\right)^m d_\nu(x_0, x_1) \quad \forall \nu \in A.
\]

For convenience we assume that \(\eta_\nu = \frac{1}{\sqrt{\kappa}}\) for each \(\nu \in A\). Now we show that \(\{x_m\}\) is a Cauchy sequence. For each \(m, p \in \mathbb{N}\) and \(\nu \in A\), we have

\[
d_\nu(x_m, x_{m+p}) \leq \sum_{i=m}^{m+p-1} s^i d_\nu(x_i, x_{i+1})
\]

\[
\leq \sum_{i=m}^{m+p-1} s^i (\eta_\nu)^i d_\nu(x_0, x_1)
\]

\[
\leq \sum_{i=m}^{\infty} (s\eta_\nu)^i < \infty \quad (\text{since } s\eta_\nu < 1).
\]

This implies that \(\{x_m\}\) is a Cauchy sequence in \(X\). By completeness of \(X\), we have \(x^* \in X\) such that \(x_m \to x^*\) as \(m \to \infty\). By using hypothesis (iii), triangular inequality and \[2.1\], we have

\[
d_\nu(x^*, Tx^*) \leq sd_\nu(x^*, x_{m-1}) + sd_\nu(x_{m-1}, Tx^*)
\]

\[
\leq sd_\nu(x^*, x_{m-1}) + sH_\nu(Tx_m, Tx^*)
\]

\[
\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, Tx_m)
\]

\[
+ sc_\nu d_\nu(x^*, Tx^*) + se_\nu d_\nu(x_m, Tx^*) + sL_\nu d_\nu(x^*, Tx_m)
\]

\[
\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, x_{m+1})
\]

\[
+ sc_\nu d_\nu(x^*, Tx^*) + se_\nu d_\nu(x_m, Tx^*) + sL_\nu d_\nu(x^*, x_{m+1}) \quad \forall \nu \in A.
\]

Letting \(m \to \infty\), we get

\[
d_\nu(x^*, Tx^*) \leq (sc_\nu + se_\nu)d_\nu(x^*, Tx^*) \quad \forall \nu \in A.
\]

Which is only possible if \(d_\nu(x^*, Tx^*) = 0\). Since the structure \(\{d_\nu : \nu \in A\}\) on \(X\) is separating, we have \(x^* \in Tx^*\).

In case of single valued mapping \(T: X \to X\) we have the following result:

**Theorem 2.10.** Let \(T: X \to X\) be a mapping such that for each \(\nu \in A\) we have

\[
d_\nu(Tx, Ty) \leq a_\nu d_\nu(x, y) + b_\nu d_\nu(x, Tx) + c_\nu d_\nu(y, Ty) + e_\nu d_\nu(x, Ty) + L_\nu d_\nu(y, Tx),
\]

\[\text{(2.6)}\]

for all \((x, y) \in E\), where, \(a_\nu, b_\nu, c_\nu, e_\nu, L_\nu \geq 0\), and \(sa_\nu + sb_\nu + sc_\nu + 2s^2e_\nu < 1\).

Assume that following conditions hold:

(i) there exists \(x_0 \in X\) such that \((x_0, Tx_0) \in E\);

(ii) for \((x, y) \in E\), we have \((Tx, Ty) \in E\), provided \(d_\nu(Tx, Ty) \leq d_\nu(x, y)\) for each \(\nu \in A\);

(iii) if \(\{x_n\}\) is a sequence in \(X\) such that \((x_n, x_{n+1}) \in E\) for each \(n \in \mathbb{N}\) and \(x_n \to x\) as \(n \to \infty\), then \((x_n, x) \in E\) for each \(n \in \mathbb{N}\);

Then \(T\) has a fixed point.
Example 2.11. Let $X = C([0, 10), [0, \infty))$ endowed with the $d_n(x(t), y(t)) = \max_{t \in [0, n]}(x(t) - y(t))^2$ for each $n \in \{1, 2, 3, \ldots, 10\}$ and the graph $G = (V, E)$ as $V = X$ and

$$E = \{(x(t), y(t)) : x(t) \leq y(t)\} \cup \{(x(t), x(t)) : x \in X\}.$$

Define $T: X \to X$ by $T x(t) = \frac{x(t) + 1}{5}$, for each $x \in X$. It is easy to see that (2.9) holds with $a_n = 1/5$ and $b_n = c_n = e_n = L_n = 0$ for each $n \in \{1, 2, 3, \ldots, 10\}$. For $x_0 = 0$ and $x_1 = T x_0 = 1/5$, we have $(x_0, T x_0) \in E$. Since $T$ is nondecreasing, for each $(x, y) \in E$, we have $(T x, Ty) \in E(G)$. For each sequence $(x_m)$ in $X$ such that $(x_m, x_{m+1}) \in E$ for each $m \in \mathbb{N}$ and $x_m \to x$ as $m \to \infty$, then $(x_m, x) \in E$ for each $m \in \mathbb{N}$. Therefore, all conditions of Theorem 2.10 are satisfied and $T$ has a fixed point.

Before going towards our next theorem, we have to define $\Psi_{x, \nu}$ family of mappings. Let $\psi: [0, \infty) \to [0, \infty)$ be a nondecreasing mappings such that it satisfies following conditions:

1. $\psi(0) = 0$;
2. $\psi(\rho t) < \rho t$ for each $\rho, t > 0$;
3. $\sum_{i=1}^{\infty} s^{2i} \psi^i(t) < \infty$;

where $s \geq 1$.

Theorem 2.12. Let $T: X \to CL(X)$ be a mapping such that for each $\nu \in A$ we have

$$H_\nu(T x, Ty) \leq \psi_\nu(d_\nu(x, y)), \quad \forall (x, y) \in E, \tag{2.7}$$

where $\psi_\nu \in \Psi_{x, \nu}$. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) if $(x, y) \in E$, for $u \in Tx$ and $v \in Ty$ such that $\frac{1}{2} d_\nu(u, v) < d_\nu(x, y)$ for each $\nu \in A$, then $(u, v) \in E$;

(iii) if $\{x_n\}$ is a sequence in $X$ such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$;

(iv) for each $x \in X$, we have $y \in Tx$ such that

$$d_\nu(x, y) \leq s d_\nu(x, Tx) \quad \forall \nu \in A.$$

Then $T$ has a fixed point.

Proof. By hypothesis we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. From (2.7), we get

$$d_\nu(x_1, Tx_1) \leq H_\nu(Tx_0, Tx_1) \leq \psi_\nu(d_\nu(x_0, x_1)) \quad \forall \nu \in A. \tag{2.8}$$

By hypothesis (iv), for $x_1 \in X$, we have $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \leq s d_\nu(x_1, Tx_1) \leq s \psi_\nu(d_\nu(x_0, x_1)) \quad \forall \nu \in A. \tag{2.9}$$

Applying $\psi_\nu$, we have

$$\psi_\nu(d_\nu(x_1, x_2)) \leq \psi_\nu(s \psi_\nu(d_\nu(x_0, x_1))) = s \psi_\nu^2(d_\nu(x_0, x_1)) \quad \forall \nu \in A.$$

From (2.9), it is clear that $(x_1, x_2) \in E$. Again from (2.7), we have

$$d_\nu(x_2, Tx_2) \leq H_\nu(Tx_1, Tx_2) \leq \psi_\nu(d_\nu(x_1, x_2)) \quad \forall \nu \in A. \tag{2.10}$$
By hypothesis (iv), for \( x_2 \in X \), we have \( x_3 \in Tx_2 \) such that
\[
d_\nu(x_2, x_3) \leq s_\nu(x_2, Tx_2) \leq s_\psi_\nu(d_\nu(x_1, x_2)) \leq s^2_\psi_\nu^2(d_\nu(x_0, x_1)) \quad \forall \nu \in \mathcal{A}.
\] (2.11)
Clearly, \((x_2, x_3) \in E\). Continuing in the same way, we get a sequence \( \{x_m\} \) in \( X \) such that \((x_m, x_{m+1}) \in E\) and
\[
d_\nu(x_m, x_{m+1}) \leq s^m_\psi_\nu^m(d_\nu(x_0, x_1)) \quad \forall \nu \in \mathcal{A}.
\]
Now, we show that \( \{x_m\} \) is a Cauchy sequence. For \( m, p \in \mathbb{N} \), we have
\[
d_\nu(x_m, x_{m+p}) \leq \sum_{i=m}^{m+p-1} s^i_\psi_\nu(d_\nu(x_i, x_{i+1}) \leq \sum_{i=m}^{m+p-1} s^{2i}_\psi_\nu^i(d_\nu(x_0, x_1)) < \infty
\]
This implies that \( \{x_m\} \) is a Cauchy sequence in \( X \). By completeness of \( X \), we have \( x^* \in X \) such that \( x_m \to x^* \) as \( m \to \infty \). Using hypothesis (iv), triangular inequality and (2.7), we have
\[
d_\nu(x^*, Tx^*) \leq s_\nu(x^*, x_{m-1}) + s_\nu(x_{m-1}, Tx^*)
\leq s_\nu(x^*, x_{m-1}) + sH_\nu(Tx_m, Tx^*)
\leq s_\nu(x^*, x_{m-1}) + s_\nu(d_n(x_m, x^*)) \quad \forall \nu \in \mathcal{A}.
\]
Letting \( m \to \infty \), we get \( d_\nu(x^*, Tx^*) = 0 \) for each \( \nu \in \mathcal{A} \). Since the structure \( \{d_\nu : \nu \in \mathcal{A}\} \) on \( X \) is separating, we have \( x^* \in Tx^* \).

By considering \( T : X \to X \) in above theorem we get the following one.

**Theorem 2.13.** Let \( T : X \to X \) be a mapping such that for each \( \nu \in \mathcal{A} \) we have
\[
d_\nu(Tx, Ty) \leq \psi_\nu(d_\nu(x, y)), \quad \forall (x, y) \in E,
\] (2.12)
where \( \psi_\nu \in \Psi_2 \). Assume that the following conditions hold:

(i) there exist \( x_0 \in X \) and \( x_1 \in Tx_0 \) such that \((x_0, x_1) \in E\);

(ii) for \((x, y) \in E\), we have \((Tx, Ty) \in E\) provided \( \frac{1}{2}d_\nu(Tx, Ty) < d_\nu(x, y) \) for each \( \nu \in \mathcal{A} \), then \((u, v) \in E\);

(iii) if \( \{x_n\} \) is a sequence in \( X \) such that \((x_n, x_{n+1}) \in E\) for each \( n \in \mathbb{N} \) and \( x_n \to x \) as \( n \to \infty \), then \((x_n, x) \in E\) for each \( n \in \mathbb{N} \).

Then \( T \) has a fixed point.

3. **Application**

Consider the Volterra integral equation of the form:
\[
x(t) = f(t) + \int_0^1 K(t, s, x(s))ds, \quad t \in I
\] (3.1)
where \( f : I \to \mathbb{R} \) is a continuous function, and \( K : I \times I \times \mathbb{R} \to \mathbb{R} \) is continuous and nondecreasing function.

Let \( X = (C[0, \infty), \mathbb{R}) \). Define the family of \( b_2 \)-pseudo norms by \( \|x\|_n = \max_{t \in [0,n]} |x(t)|^2, \ n \in \mathbb{N} \). By using this family of \( b_2 \)-pseudonorms we get a family of \( b_2 \)-pseudo metrics as \( d_n(x, y) = \|x - y\|_n \). Clearly, \( \mathcal{F} = \{d_n : n \in \mathbb{N}\} \) defines \( b_2 \)-gauge structure on \( X \), which is complete and separating. Define graph \( G = (V, E) \) such that \( V = X \) and \( E = \{(x, y) : x(t) \leq y(t), \forall t \geq 0\} \).
Theorem 3.1. Let $X = (C[0, \infty), \mathbb{R})$ and let the operator

$$T: X \to X \quad Tx(t) = f(t) + \int_0^t K(t, s, x(s))ds, \quad t \in I = [0, \infty),$$

where $f: I \to \mathbb{R}$ is a continuous function, and $K: I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function. Assume that the following conditions hold:

(i) for each $t, s \in [0, n]$ and $x, y \in X$ with $(x, y) \in E(G)$, there exists a continuous mapping $p: I \times I \to I$ such that

$$|K(t, s, x(s)) - K(t, s, y(s))| \leq \sqrt{p(t, s)d_n(x, y)} \text{ for each } n \in \mathbb{N};$$

(ii) $\sup_{t \geq 0} \int_0^t \sqrt{p(t, s)}ds = a < \frac{1}{\sqrt{2}}$;

(iii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.

Then the integral equation (3.1) has at least one solution.

Proof. First we show that for each $(x, y) \in E(G)$, the inequality (2.1) holds. For any $(x, y) \in E(G)$ and $t \in [0, n]$ for each $n \geq 1$, we have

$$(Tx(t) - Ty(t))^2 \leq \left( \int_0^t |K(t, s, x(s)) - K(t, s, y(s))|ds \right)^2 \leq \left( \int_0^t \sqrt{p(t, s)d_n(x, y)}ds \right)^2 \leq \left( \int_0^t \sqrt{p(t, s)}ds \right)^2 d_n(x, y) = a^2 d_n(x, y).$$

Thus, we get $d_n(Tx, Ty) \leq a^2 d_n(x, y)$ for each $(x, y) \in E$ and $n \in \mathbb{N}$ with $a^2 < 1/2$. This implies that (2.1) holds with $a_n = a^2$, and $b_n = c_n = e_n = L_n = 0$ for each $n \in \mathbb{N}$. As $K$ is nondecreasing, for each $(x, y) \in E(G)$, we have $(Tx, Ty) \in E(G)$. Therefore, by Theorem 2.10, there exists a fixed point of the operator $T$, that is, integral equation (3.1) has at least one solution. \hfill $\square$

References


