

SOME NEW ESTIMATES FOR DISTANCES IN ANALYTIC  
FUNCTION SPACES OF SEVERAL COMPLEX VARIABLES  
AND DOUBLE BERGMAN REPRESENTATION FORMULA

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ABSTRACT. Using double Bergman representation formula we provide new sharp estimates for distances from fixed analytic functions to some subspaces of holomorphic functions in unit polydisk and unit ball. We will enlarge the list of previously known assertions of this type obtained recently by R. Zhao and W. Xu.

1. INTRODUCTION AND NOTATIONS

Let  $\mathbb{D}$  be, as usual, the unit disk on the complex plane,  $dm_2(z)$  the normalized Lebesgue measure on  $\mathbb{D}$  so that  $m_2(\mathbb{D}) = 1$  and  $d\xi$  be the Lebesgue measure on the circle  $T = \{\xi : |\xi| = 1\}$ . Let further  $H(\mathbb{D})$  be the space of all holomorphic functions on the unit disk  $\mathbb{D}$ .

For  $f \in H(\mathbb{D})$  and  $f(z) = \sum_k a_k z^k$ , we define the fractional derivative of the function  $f$  as usual in the following manner

$$D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha a_k z^k, \alpha \in \mathbb{R}.$$

We will write  $Df(z)$  if  $\alpha = 1$ . Obviously, for all  $\alpha \in \mathbb{R}$ ,  $D^\alpha f \in H(\mathbb{D})$ .

For  $a \in \mathbb{D}$ , let  $g(z, a) = \log \frac{1}{|\phi_a(z)|}$  be the Green's function for  $\mathbb{D}$  with pole at  $a$ , where  $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ . For  $0 < p < \infty$ ,  $-2 < q < \infty$ ,  $0 < s < \infty$ ,  $-1 < q + s < \infty$ , we say that

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$f \in F(p, q, s)$ , if  $f \in H(\mathbb{D})$  and

$$\|F\|_{F(p,q,s)}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |Df(z)|^p (1 - |z|^2)^q g^s(z, a) dm_2(z) < \infty.$$

As we know [14], if  $0 < p < \infty$ ,  $-2 < q < \infty$ ,  $0 < s < \infty$ ,  $-1 < q + s < \infty$ ,  $f \in F(p, q, s)$ , if and only if

$$\int_{\mathbb{D}} |Df(z)|^p (1 - |z|^2)^q (1 - |\phi_a(z)|)^s dm_2(z) < \infty.$$

It is known also that  $F(2, 0, 1) = BMOA$ .

We recall that weighted Bloch class  $\mathbf{B}^\alpha(\mathbb{D})$ ,  $\alpha > 0$ , is the collection of the analytic functions on the unit disk satisfying

$$\|f\|_{\mathbf{B}^\alpha} = \sup_{z \in \mathbb{D}} |Df(z)|(1 - |z|^2)^\alpha < \infty.$$

$\mathbf{B}^\alpha(\mathbb{D})$  is a Banach space with the norm  $\|f\|_{\mathbf{B}^\alpha}$ . Note  $\mathbf{B}^1(\mathbb{D}) = B(D)$  is a classical Bloch class (see [2] and the references there).

The well-known so called "duality" approach to extremal problems in theory of analytic functions leads to the following general formula

$$dist_Y(g, X) = \sup_{l \in X^\perp, \|l\| \leq 1} |l(g)| = \inf_{\varphi \in X} \|g - \varphi\|_Y,$$

where  $g \in Y$ ,  $X$  is subspace of a normed space  $Y$ ,  $Y \in H(D)$  and  $X^\perp$  it is orthogonal complement in  $Y^*$ , the dual space of  $Y$  and  $l$  is a linear functional on  $Y$  (see [6]). Various extremal problems in Hp Hardy classes in  $\mathbb{D}$  based on duality approach we mentioned were discussed in [3, Chapter 8]. In particular for a function  $K \in L^q(T)$  the following equality holds (see [3]),  $1 < p < \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$dist_{L^q}(K, H^q) = \inf_{g \in H^q, K \in L^q} \|K - g\|_{H^q} = \sup_{f \in H^p, \|f\|_{H^p} \leq 1} \frac{1}{2\pi} \left| \int_{|\xi|=1} f(\xi) K(\xi) d\xi \right|.$$

It is well known if  $p > 1$  then inf-dual extremal problem in analytic  $H^p$  Hardy classes has a solution, it is unique if an extremal function exists (see [3]). Note also that extremal problems for  $H^p$  spaces in multiply connected domains were studied before in [1], [7] and [8]. Various new results on extremal problems in  $A^p$  Bergman class and its subspaces were obtained recently by many authors (see [5] and the references there). In this paper, we will provide direct proofs for estimation of

$$dist_Y(f, X) = \inf_{g \in X} \|f - g\|_Y, X \subset Y, f \in Y, X, Y \subset H(\mathbb{D})$$

in higher dimension.

Let further  $\Omega_{\alpha, \varepsilon}^k = \{z \in \mathbb{D} : |D^k f(z)|(1 - |z|^2)^\alpha \geq \varepsilon\}$ ,  $\Omega_{\alpha, \varepsilon}^0 = \Omega_{\alpha, \varepsilon}$ .

Applying famous Fefferman duality theorem, P. Jones proved the following.

**Theorem A.** ([4], [14]) Let  $f \in B$ . Then the following are equivalent:

- (1)  $d_1 = dist_{\mathbf{B}}(f, BMOA)$ ;
- (2)  $d_2 = \inf\{\varepsilon : \chi_{\Omega_{1, \varepsilon}^1}(z) \frac{dm_2(z)}{1 - |z|^2} \text{ is a Carleson measure}\}$ ,

where  $\chi$  denotes characteristic function of the mentioned set.

Recently, R. Zhao (see [14]) and W. Xu (see [13]) obtained results on distances from Bloch functions to some Möbius invariant function spaces in a relatively direct way. The goal of this paper is to develop further their ideas and present new theorems in analytic function spaces in higher dimension.

In next sections various assertions for distance function will be given. We will indicate proofs of some assertions in details, short sketches in some cases will be also provided.

Throughout the paper, we write  $C$  ( sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

We will write for two expressions  $A \lesssim B$  if there is a positive constant  $C$  such that  $A < CB$ .

## 2. MAIN RESULTS

In this section applying rather unusual "double" Bergman representation formula, we indicate a precise formula for "dist" function that allows to estimate distances from any function from certain analytic BMOA type class to certain weighted Bergman space. The simple nature of our proof allows to consider more general weighted Bergman classes, but we restrict ourselves to standart weights. We assume in advance that inf of any set below and the set itself can not be empty. We denote below everywhere by  $\mathbb{D}^n$  as usual the unit polydisk (see for example [2] and references there) and use below all standart notations of function theory in polydisk that can be found for example in [2].

Let  $\alpha > r, r > 1, \alpha > 1$ . Then obviously

$$\begin{aligned} \|f\|_{Q_{r,\alpha}} &= \sup_{a_j \in \mathbb{D}} \int_{\mathbb{D}} \dots \int_{\mathbb{D}} \frac{|f(z_1, \dots, z_n)| \times \prod_{k=1}^n (1 - |z_k|)^{\alpha-2}}{\prod_{k=1}^n (|1 - \overline{a_k} z_k|)^{\alpha}} dm_{2n}(z) \times \prod_{k=1}^n (1 - |a_k|)^r \\ &\leq C \|f\|_{A_{r-2}^1} = C \int_{\mathbb{D}^n} |f(z_1, \dots, z_n)| \times \prod_{k=1}^n (1 - |z_k|)^{r-2} dm_{2n}(z) \end{aligned}$$

where  $\mathbb{D}^n$  is a unit polydisk,  $f \in H(\mathbb{D}^n)$ ,  $H(\mathbb{D}^n)$  is a space of holomorphic functions in  $\mathbb{D}^n$  and  $dm_{2n}(z)$  is a Lebesgue measure on  $\mathbb{D}^n$ . So we have a natural extremal problem in polydisk to find

$$dist_{Q_{r,\alpha}}(f, A_{r-2}^1) = \inf_{g \in A_{r-2}^1} \|f - g\|_{Q_{r,\alpha}}, f \in Q_{r,\alpha}.$$

For  $f \in H(\mathbb{D}^n)$ , we define a set

$$N_f = N_{f,\varepsilon}^{\alpha,r} = \{a \in \mathbb{D}^n : \int_{\mathbb{D}^n} \frac{|f(z_1, \dots, z_n)| \prod_{k=1}^n (1 - |z_k|)^{\alpha-2}}{\prod_{k=1}^n (|1 - \overline{a_k} z_k|)^{\alpha}} dm_{2n}(z) \times \prod_{k=1}^n (1 - |a_k|)^r \geq \varepsilon\},$$

$\varepsilon > 0, \alpha > 1, r > 0$ .

**Remark.** Note  $A_{r-2}^1$  is a standard Bergman class in polydisk  $\mathbb{D}^n$  studied before by various authors (see for example [2]). Classes  $Q_{r,\alpha}$  in disk  $\mathbb{D}$  are so called BMOA type spaces were also under investigation by many authors recently (see [11], [12]).

Applying classical Bergman representation for the unit disk  $n$  times by each variable we get Bergman representation formula in polydisk.

$$f(z_1, \dots, z_n) = C_\gamma \int_{\mathbb{D}^n} \frac{f(w_1, \dots, w_n) \prod_{k=1}^n (1 - |w_k|)^\gamma}{\prod_{k=1}^n (1 - \overline{w_k} z_k)^{\gamma+2}} dm_{2n}(w),$$

where  $C_\gamma$  is constant,  $z_j \in \mathbb{D}$ ,  $j = 1, \dots, n$ ,  $\gamma > -1$ ,  $f \in H(\mathbb{D}^n)$ . Choose  $\gamma_1 > -1$ , then applying Bergman representation formula twice we have a double Bergman representation for analytic  $f$  function in polydisk

$$f(z_1, \dots, z_n) = C_\gamma C_{\gamma_1} \int_{\mathbb{D}^n} \frac{\prod_{k=1}^n (1 - |w_k|)^\gamma}{\prod_{k=1}^n (1 - \overline{w_k} z_k)^{\gamma+2}} \times \int_{\mathbb{D}^n} \frac{\prod_{k=1}^n (1 - |\widetilde{w}_k|)^{\gamma_1} f(\widetilde{w}_1, \dots, \widetilde{w}_n)}{\prod_{k=1}^n (1 - \overline{\widetilde{w}_k} w_k)^{\gamma_1+2}} dm_{2n}(\widetilde{w}) dm_{2n}(w) = f_1(z_1, \dots, z_n) + f_2(z_1, \dots, z_n),$$

and

$$f_1(z_1, \dots, z_n) = C \int_{\mathbb{D}^n} \int_{\mathbb{D}^n \setminus N_f} \frac{\prod_{k=1}^n (1 - |w_k|)^\gamma}{\prod_{k=1}^n (1 - \overline{w_k} z_k)^{\gamma+2}} \times \frac{\prod_{k=1}^n (1 - |\widetilde{w}_k|)^{\gamma_1} f(\widetilde{w}_1, \dots, \widetilde{w}_n)}{\prod_{k=1}^n (1 - \overline{\widetilde{w}_k} w_k)^{\gamma_1+2}} dm_{2n}(\widetilde{w}) dm_{2n}(w),$$

$$f_2(z_1, \dots, z_n) = C \int_{\mathbb{D}^n} \int_{N_f} \frac{\prod_{k=1}^n (1 - |w_k|)^\gamma}{\prod_{k=1}^n (1 - \overline{w_k} z_k)^{\gamma+2}} \times \frac{\prod_{k=1}^n (1 - |\widetilde{w}_k|)^{\gamma_1} f(\widetilde{w}_1, \dots, \widetilde{w}_n)}{\prod_{k=1}^n (1 - \overline{\widetilde{w}_k} w_k)^{\gamma_1+2}} dm_{2n}(\widetilde{w}) dm_{2n}(w),$$

where  $dm_{2n}(w) = dm_2(w_1) \dots dm_2(w_n)$ ,  $C = C_{\gamma, \gamma_1}$ .

Fix some  $\varepsilon > 0$ . Our task to show  $f_1 \in Q_{r, \alpha}$ ,  $\|f_1\|_{Q_{r, \alpha}} \leq C\varepsilon$ ,  $f_2 \in A_{r-2}^1$ . Then we will have

$$\text{dist}_{Q_{r, \alpha}}(f, A_{r-2}^1) \leq C\|f - f_2\|_{Q_{r, \alpha}} = C\|f_1\|_{Q_{r, \alpha}} \leq C\varepsilon, f \in Q_{r, \alpha}.$$

Moreover it turns out the reverse estimate is also true and we have the following sharp theorem. We assume also that all inf everywhere in paper are taken from not empty sets.

**Theorem 2.1.** *Let  $r < \alpha - 1$ ,  $r > 1$ ,  $\alpha > 1$ ,  $f \in Q_{r, \alpha}$ , then the following are equivalent:*

- (1)  $[\text{dist}_{Q_{r, \alpha}}(f, A_{r-2}^1)]$ ;
- (2)  $\inf\{\varepsilon > 0, \int_{\mathbb{D}^n} (\chi_{N_f})(a_1, \dots, a_n) \prod_{k=1}^n (1 - |a_k|)^{-2} dm_{2n}(a) < \infty\}$ , where  $\chi_M$  is characteristic function of a set  $M$ ,  $M \subset \mathbb{D}^n$ ;

*Proof.* Let us show first the implication (2)  $\rightarrow$  (1) using arguments we provided above. We have for  $\gamma > r - 2$

$$\|f_2\|_{A_{r-2}^1} = \int_{\mathbb{D}^n} \prod_{k=1}^n (1 - |z_k|)^{r-2} |f_2(z_1, \dots, z_n)| dm_{2n}(z) \leq C \int_{\mathbb{D}^n} \prod_{k=1}^n (1 - |z_k|)^{r-2} \int_{\mathbb{D}^n} \int_{N_f} \frac{\prod_{k=1}^n (1 - |w_k|)^\gamma}{|\prod_{k=1}^n (1 - w_k z_k)^{\gamma+2}|} \times$$

$$\int_{\mathbb{D}^n} \frac{\prod_{k=1}^n (1 - |\widetilde{w}_k|)^{\gamma_1} |f(\widetilde{w}_1, \dots, \widetilde{w}_n)|}{|\prod_{k=1}^n (1 - \widetilde{w}_k w_k)^{\gamma_1+2}|} dm_{2n}(\widetilde{w}) dm_{2n}(w) dm_{2n}(z),$$

Hence using Fubini's theorem and putting  $\gamma_1 = \alpha - 2$ , we will have

$$\begin{aligned} \|f_2\|_{A_{r-2}^1} &= C \int_{\mathbb{D}^n} \int_{N_f} \frac{|f(\widetilde{w}_1, \dots, \widetilde{w}_n)| \prod_{k=1}^n (1 - |\widetilde{w}_k|)^{\gamma_1}}{|\prod_{k=1}^n (1 - \widetilde{w}_k w_k)^{\gamma_1+2}|} \times \prod_{k=1}^n (1 - |w_k|)^\gamma \\ &\times \prod_{k=1}^n (1 - |w_k|)^{r-\gamma-2} dm_{2n}(\widetilde{w}) dm_{2n}(w) \leq \sup_{w_k} C \int_{\mathbb{D}^n} \frac{|f(\widetilde{w}_1, \dots, \widetilde{w}_n)| \prod_{k=1}^n (1 - |\widetilde{w}_k|)^{\alpha-2}}{\prod_{k=1}^n (|1 - \widetilde{w}_k w_k|)^\alpha} \\ &\prod_{k=1}^n (1 - |w_k|)^r \left( \int_{\mathbb{D}^n} (\chi_{N_f})(w_1, \dots, w_n) \prod_{k=1}^n (1 - |w_k|)^{-2} dm_{2n}(w) \right) \end{aligned}$$

We used that

$$(3) \quad \int_{\mathbb{D}^n} \frac{\prod_{k=1}^n (1 - |w_k|)^{t_1} dm_{2n}(w)}{\prod_{k=1}^n |(1 - \widetilde{w}_k z_k)|^{t_2}} \asymp C \prod_{k=1}^n (1 - |z_k|)^{t_1+2-t_2}, \quad t_1 > -1, t_2 > t_1 + 2, z \in \mathbb{D}^n.$$

(See [2]). Now we show  $\|f_1\|_{Q_{r,\alpha}} \leq C\varepsilon$ . We have using (3) and Fubini's theorem for  $\gamma > r - 1$

$$\begin{aligned} \|f_1\|_{Q_{r,\alpha}} &= \left( \sup_{a_j \in \mathbb{D}} \int_{\mathbb{D}^n} \frac{|f_1(z_1, \dots, z_n)| \prod_{k=1}^n (1 - |z_k|)^{\alpha-2}}{\prod_{k=1}^n |(1 - \overline{a}_k z_k)|^\alpha} dm_{2n}(z) \times \prod_{k=1}^n (1 - |a_k|)^r \right. \\ &\leq (C \sup_{a_j \in \mathbb{D}} \int_{\mathbb{D}^n} \int_{\mathbb{D}^n} \int_{\mathbb{D}^n \setminus N_f} \frac{\prod_{k=1}^n (1 - |w_k|)^\gamma \prod_{k=1}^n (1 - |z_k|)^{\alpha-2}}{\prod_{k=1}^n |(1 - \overline{w}_k z_k)|^{\gamma+2}} \times \prod_{k=1}^n (1 - |a_k|)^r \\ &\quad \times \frac{\prod_{k=1}^n (1 - |\widetilde{w}_k|)^{\alpha-2} |f(\widetilde{w}_1, \dots, \widetilde{w}_n)|}{\prod_{k=1}^n |(1 - \widetilde{w}_k w_k)|^\alpha} dm_{2n}(z) dm_{2n}(\widetilde{w}) dm_{2n}(w) \leq \\ &\left. (C\varepsilon) \sup_{a_j \in \mathbb{D}} \int_{\mathbb{D}^n} \frac{\prod_{k=1}^n (1 - |a_k|)^r \prod_{k=1}^n (1 - |w_k|)^{\alpha-2-r}}{\prod_{k=1}^n |(1 - \overline{a}_k w_k)|^\alpha} dm_{2n}(w) \leq C_1 \varepsilon. \right. \end{aligned}$$

So we showed one part of our theorem. To show the reverse we assume the reverse to that assertion is true. Hence by assumption there are  $\varepsilon, \varepsilon_1, \varepsilon > 0, \varepsilon_1 > 0$  and there is  $f_{\varepsilon_1} \in A_{r-2}^1$ ,  $\varepsilon > \varepsilon_1$ ,  $\|f - f_{\varepsilon_1}\|_{Q_{r,\alpha}} \leq \varepsilon_1$  and  $K = \int_{\mathbb{D}^n} (\chi_{N_f}(a))(a_1, \dots, a_n) \prod_{k=1}^n (1 - |a_k|)^{-2} dm_{2n}(a) = \infty$ . Using this we arrive easily to contradiction. Indeed we have  $\widetilde{f} = f - f_{\varepsilon_1}$ ,  $\tau > \rho$ ,  $\beta - \tau + \rho = r - 2$ ,  $\rho > 1$ ,  $\beta > -1$ ,  $\tau > 0$ ,  $a \in \mathbb{D}^n$

$$\begin{aligned} \mathcal{M}(a) &= \int_{\mathbb{D}^n} \frac{|f_{\varepsilon_1}(z_1, \dots, z_n)| \prod_{k=1}^n (1 - |z_k|)^\beta}{\prod_{k=1}^n (|1 - \overline{a}_k z_k|)^\tau} dm_{2n}(z) \prod_{k=1}^n (1 - |a_k|)^\rho \geq \\ &\prod_{k=1}^n (1 - |a_k|)^\rho \int_{\mathbb{D}^n} \frac{|f(z_1, \dots, z_n)| \prod_{k=1}^n (1 - |z_k|)^\beta}{|\prod_{k=1}^n (1 - \overline{a}_k z_k)|^\tau} dm_{2n}(z) \\ &- \left( \sup_{a \in \mathbb{D}^n} \int_{\mathbb{D}^n} \frac{|f_{\varepsilon_1}(z) - f(z)| \prod_{k=1}^n (1 - |z_k|)^\beta}{|\prod_{k=1}^n (1 - \overline{a}_k z_k)|^\tau} dm_{2n}(z) \prod_{k=1}^n (1 - |a_k|)^\rho \right) \end{aligned}$$

Hence

$$\mathcal{M}(a) \geq ((\varepsilon - \varepsilon_1)(\chi_{N_f}(a)))$$

and by (3) after choosing appropriate  $\rho > 1$  and  $\beta > -1$ , we will have

$$\int_{\mathbb{D}^n} |f_{\varepsilon_1}(z)|(1 - |z_1|)^{r-2}(1 - |z_n|)^{r-2} dm_{2n}(z) \geq (\varepsilon - \varepsilon_1)K.$$

So we have an obvious contradiction. Our theorem 2.1 is proved.  $\square$

Let us note that similar results holds with similar proofs in unit ball  $B$  of  $\mathbb{C}^n$  and classes on subframes  $\tilde{\mathbb{D}}^n$  and expanded disk  $\mathbb{D}_*^n$ .

$$\mathbb{D}_*^n = \{z \in \mathbb{D}^n, z = (z_1, \dots, z_n), z_j = |z_j|\xi, |z_j| \in (0, 1), \xi \in T = \{|z| = 1\}.$$

$$\tilde{\mathbb{D}}^n = \{z \in \mathbb{D}^n, z = (z_1, \dots, z_n), z_j = |z|\xi_j, \xi_j \in T, j = 1, \dots, n, |z| \in (0, 1)\}.$$

Namely we consider Bergman classes  $A_\alpha^1(\mathbb{D}_*^n)$  and  $A_\alpha^1(\tilde{\mathbb{D}}^n)$  defined us

$$A_\alpha^1(\mathbb{D}_*^n) = \{f \in H(\mathbb{D}^n) : \int_0^1 \int_0^1 \int_T |f(r_1\xi, \dots, r_n\xi)| \prod_{k=1}^n (1 - r_k)^\alpha dr_1 \dots dr_n d\xi < \infty\}.$$

$$A_\alpha^1(\tilde{\mathbb{D}}^n) = \{f \in H(\mathbb{D}^n) : \int_{T^n} \int_0^1 |f(r\xi_1, \dots, r\xi_n)|(1 - r)^\alpha dr dm_n(\xi) < \infty\}.$$

Where  $dm_n(\xi)$  is standard Lebesgue measure in  $T^n$ ,  $\alpha > -1$  and  $H(\mathbb{D}^n)$  as above is a space of holomorphic function in  $\mathbb{D}^n$  and  $Q_{r,\alpha}(\mathbb{D}_*^n)$  and  $Q_{r,\alpha}(\tilde{\mathbb{D}}^n)$  are defined by replacing polydisk  $\mathbb{D}^n$  by  $\mathbb{D}_*^n$  or  $\tilde{\mathbb{D}}^n$  accordingly Note analytic classes on subframes and expanded disk were studied before in [9], [10]. We also note that proofs of mentioned assertions can be obtained by small modification of proofs of our main theorem 2.1.

Let  $\tilde{m}_{2n}(w)$  and  $m_{2n}^*(w)$  be Lebesgue measure in  $\tilde{\mathbb{D}}^n$  and  $\mathbb{D}_*^n$ , obviously for  $\alpha > \frac{1}{n}$ ,  $r > \frac{1}{n}$ ,  $\alpha > r$

$$\|f\|_{Q_{r,\alpha}^1(\tilde{\mathbb{D}}^n)} = (\sup_{a_j \in \mathbb{D}}) \int_0^1 \int_{T^n} \frac{|f(w)|(1 - |w|)^{n\alpha-2} d\tilde{m}_{2n}(w)}{\prod_{k=1}^n (|1 - \bar{a}_k w_k|)^\alpha} \times \prod_{k=1}^n (1 - |a_k|)^r \leq$$

$$C \int_0^1 \int_{T^n} |f(w)|(1 - |w|)^{n\alpha-2-(\alpha-r)n} d\tilde{m}_{2n}(w) \leq C \int_0^1 \int_{T^n} |f(w)|(1 - |w|)^{rn-2} d\tilde{m}_{2n}(w).$$

$$\|f\|_{Q_{r,\alpha}^2(\mathbb{D}_*^n)} = (\sup_{a_j \in \mathbb{D}}) \int_0^1 \dots \int_0^1 \int_T \frac{|f(w)| \prod_{k=1}^n (1 - |w_k|)^{\alpha-2}}{\prod_{k=1}^n (|1 - \bar{a}_k w_k|)^\alpha} dm_{2n}^*(w) \left( \prod_{k=1}^n (1 - |a_k|)^r \right)$$

$$\leq C(\|f\|_{A_{r-2}^1}) = C \int_0^1 \dots \int_0^1 \int_T |f(w)|(1 - |w_1|)^{r-2} \dots (1 - |w_n|)^{r-2} dm_{2n}^*(w),$$

$\alpha > r$ ,  $r > 1$ ,  $\alpha > 1$ .

So the problem of finding estimate for  $(dist_{Q_{r,\alpha}^2}(f, A_{r-2}^1(\mathbb{D}_*^n)), f \in (Q_{r,\alpha}^2)(\mathbb{D}_*^n))$ ; or  $(dist_{Q_{r,\alpha}^1}(f, A_{r-2}^1(\tilde{\mathbb{D}}^n)), f \in (Q_{r,\alpha}^1)(\tilde{\mathbb{D}}^n))$ ; appears naturally in subframe  $\tilde{\mathbb{D}}^n$  and expanded disk  $\mathbb{D}_*^n$  also.

We formulate a version of theorem 2.1 for unit ball, we use standard notations from [15].

Let  $dv(z)$  volume measure on ball  $\mathbb{B}$ ,  $d\sigma(\xi)$  standard Lebesgue measure on  $\mathbb{S}$  sphere. We have for  $r > n$ ,  $\alpha > r$ ,  $\alpha > n$

$$\begin{aligned} \|f\|_{Q_{r,\alpha}(\mathbb{B})} &= (\sup_{a \in \mathbb{B}}) \int_{\mathbb{B}} \frac{|f(w)| \prod_{k=1}^n (1 - |w|)^{\alpha-n-1}}{(|1 - \langle w, a \rangle|)^\alpha} dv(z) \times (1 - |a|)^r \leq \\ &C \int_{\mathbb{B}} |f(w)| (1 - |w|)^{r-n-1} dv(w) = \|f\|_{A_{r-n-1}^1(\mathbb{B})} \end{aligned}$$

Let

$$\tilde{N}_f = \{a \in \mathbb{B} : (\int_{\mathbb{B}} \frac{|f(w)|(1 - |w|)^{\alpha-n-1} dv(w)}{(|1 - \langle a, w \rangle|)^\alpha}) (1 - |a|)^r \geq \varepsilon\}.$$

**Theorem 2.2.** *Let  $r > n$ ,  $\alpha > n$ ,  $r < \alpha - n$ ,  $f \in Q_{r,\alpha}(\mathbb{B})$ , then the following are equivalent:*

- (1)  $[dist_{Q_{r,\alpha}(\mathbb{B})}(f, A_{r-n-1}^1(\mathbb{B}))]$ ;
- (2)  $\inf\{\varepsilon > 0, \int_{\mathbb{B}} [\chi_{\tilde{N}_f}(a)] (1 - |a|)^{-n-1} d\sigma(a) < \infty\}$ ;

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