PIEZOELECTRIC VISCOELASTIC KELVIN-VOIGT CUSPED PRISMATIC SHELLS
LECTURE NOTES OF TICMI

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Preface

The present work is intended to provide some materials for participants of the forthcoming TICMI Advance Courses (September 22-25, 2019) on "Mathematical Models of Piezoelectric Solids and Related Problems". This work is oriented mainly on the lecture course of the same name "Piezoelectric Viscoelastic Kelvin-Voigt Cusped Prismatic Shells", foreseen in the prospective programme of the above-mentioned Advance Courses. It mainly contains unpublished results of the author concerning piezoelectrics. Some auxiliary materials, which make the work self-contained, are provided as well.

The aim of the present work is also to draw the attention of scientists, particularly of young researchers, to problems to be solved, connected with cusped shell-like elastic and viscoelastic piezoelectric bodies with voids and with related nonclassical BVPs and IBVPs for partial differential equations with order and type degeneracy. The development of the corresponding numerical methods and numerical calculations on computers are especially challenging.

George Jaiani
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Abstract. The present work is devoted to construction of hierarchical models for piezoelectric nonhomogeneous porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories. Using I. Vekua’s dimension reduction method, governing systems are derived and in the Nth approximation of hierarchical models boundary value problems (BVPs) and initial boundary value problems (IBVPs) are set. In the N = 0 approximation, considering, e.g., elastic, plates of a constant thickness, governing systems mathematically coincide with the governing systems of the plane strain corresponding to the basic three-dimensional (3D) linear theory up to a separate equation for the out of plane component of the displacement vector.

The ways of investigation of BVPs and IBVPs, including the case of cusped prismatic shells, are indicated and some preliminary results are presented. Antiplane deformation of piezoelectric nonhomogeneous materials in the three-dimensional formulation and in N = 0 approximation is analysed.

Well-posedness of Dirichlet and Keldysh type problems (BVP) are studied in the N = 0 order approximation of hierarchical models for cusped prismatic shells. Some BVPs are solved in explicit forms in concrete cases.

Key words and phrases: Hierarchical models, Piezoelectrics, Viscoelastic prismatic shells, Porous elastic prismatic shells, Materials with voids, Partial differential equations and Systems with order degeneracy.

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1 Introduction

In 1955 Ilia Vekua [1] published his models of elastic prismatic shells. In 1965 he offered analogous models for standard shells [2]. In both papers he considered a very important investigation of well-posedness of boundary value problems (BVPs) of peculiar types which could arise in the case of cusped shells.

Cusped prismatic shells considered as 3D bodies may have non-Lipschitz surfaces as the boundaries and their thicknesses may vanish at the edge. Using I. Vekua’s dimension reduction method, complexity of the 3D domain, occupied by the body will be transformed into the degeneracy of the order of the 2D governing equations of the constructed hierarchy of 2D models on the boundary of the 2D projection of the 3D bodies under consideration.

Consideration of BVPs and initial boundary value problems (IBVPs) within the framework of hierarchical models for elastic cusped prismatic shells leads to investigation of nonclassical BVPs and IBVPs for the governing elliptic and hyperbolic systems of equations of the second order with order degeneracy on the boundary of the domain under consideration in the case of two spatial variables [1]-[4]. We easily reduce this case to the case of systems of singular equations, i.e., to systems of equations with unbounded coefficients. Initial conditions (IC) for the so called weighted mathematical moments of displacements remain classical, while the boundary conditions (BC) for them are nonclassical, in general. It means that in certain cases the Dirichlet BCs should be replaced by the Keldysh BCs (i.e. some parts of the boundary, where the order of the equations degenerates, should be freed from the BCs) and in certain cases weighted BCs should be set (see [4]).

The present work is devoted to construction of hierarchical models for piezoelectric nonhomogeneous porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories [5]-[10]. Using I. Vekua’s [1] (see also [2]) dimension reduction method, governing systems are derived and in the Nth approximation of hierarchical models BVPs and IBVPs are set. In the $N=0$ approximation, considering, e.g., elastic plates of a constant thickness, governing systems mathematically coincide with the governing systems of the plane strain corresponding to the basic three-dimensional (3D) linear theory [1]-[4] up to a separate equation for the out of plane component of the displacement vector.

The ways of investigation of BVPs and IBVPs, including the case of cusped prismatic shells [4], are indicated and some preliminary results are presented. Antiplane deformation of piezoelectric nonhomogeneous materials in the three-dimensional formulation and in $N=0$ approximation is analysed. Some BVPs are solved in explicit forms in concrete cases.

The aim of the present work is also to draw the attention of scientists, particularly of young researchers, to problems to be solved connected with cusped shell-like elastic and viscoelastic piezoelectric bodies with voids and with re-
lated nonclassical BVPs and IBVPs for partial differential equations with order and type degeneracy. The development of the corresponding numerical methods and numerical calculations on computers are especially challenging.

The work is organized as follows. Introduction is devoted to motivations of our research and the main targets of the work are indicated as well. Section 2 contains 3D field equations for nonhomogeneous piezoelectric Kelvin-Voight materials with voids in the case of the general anisotropy. In Section 3 the hierarchical models are constructed and in the $N$th approximation BVPs and IBVPs are set in the case of noncusped prismatic shells. Section 4 deals with the analysis of Dirichlet and Keldysh type problems for the general governing system of the $N = 0$ approximation for nonhomogeneous piezoelectric Kelvin-Voight materials with voids and general anisotropy. To this end results of Section 8 are exploited. In Section 5 we consider transversely isotropic elastic piezoelectric nonhomogeneous bodies in the case when the poling axis coincides with one of the material symmetry axises. Namely, time-harmonic motion under conditions of anti-plane piezoelectric state is discussed. In Section 6 we study an antiplane deformation of piezoelectrics in $N = 0$ approximation of hierarchical models for prismatic shells, in particular, with cusped edges. In Section 7 we treat BVPs for porous isotropic elastic cusped prismatic shells. In Section 8 we examine well-posedness of BVPs for systems of elliptic equations of the second order with an order degeneracy, covering systems of elliptic equations arising in previous sections. In Section 9 for the convenience of the reader we repeat the relevant material, concerning $H$-weak solutions of BVPs for a single second order equation with an order degeneracy, from [15] with proofs in a slightly changed form, thus making our exposition of the present work self-contained. Section 10 provides some useful formulas for constructing the hierarchical models. Section 11 is devoted to conclusions, concerning mainly mechanical meaning.

2 Field Equations for Piezoelectric Kelvin-Voigt Materials with Voids

Let a piezoelectric solid occupy a reference configuration $\Omega \in \mathbb{R}^3$. Under the quasi-static conditions, when the rate of change of the magnetic field is small and there is no electric current, i.e., the electric field $\mathbf{E}$ and magnetic field $\mathbf{M}$ are curl free, the governing equations have the following form.

**Motion Equations**

\[
X_{ji,j} + \Phi_i = \rho \ddot{u}_i(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \quad \rho > 0, \quad i = 1, 3;
\]  

(2.1)

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\( H_{j,j} + H_0 + \mathcal{F} = \rho k \ddot{\varphi}, \)  

(2.2)

\[ D_{j,j} = f_e, \quad B_{j,j} = 0, \quad \Omega \times ]0, T[ , \]  

(2.3)

where \( X_{ij} \in C^1(\Omega) \) is the stress tensor; \( \Phi_i \) are the volume force components; \( k \) is equilibrated inertia, \( \rho \) is the mass density; \( \varphi := \nu - \nu_0 \in C^2(\Omega) \) is the change of the volume fraction from the matrix reference volume fraction \( \nu_0 \) (clearly, the bulk density \( \rho = \nu \gamma, \) 0 < \( \nu \) \leq 1, here \( \gamma \) is the matrix density); \( u_i \in C^2(\Omega) \) are the displacements; \( H_j \in C^1(\Omega) \) is the component of the equilibrated stress vector, \( H_0 \) and \( \mathcal{F} \) are the intrinsic and extrinsic equilibrated volume forces; Einstein’s summation convention is used; indices after comma mean differentiation with respect to the corresponding variables of the Cartesian frame \( O \hat{x}_1 \hat{x}_2 \hat{x}_3 \) (throughout the work we assume existence of the indicated (continuous) derivatives unless otherwise stated); dots as superscripts of the symbols mean derivatives with respect to time \( t; \) \( \chi : \Omega \times ]0, T[ \to \mathbb{R}^4 \) and \( \eta : \Omega \times ]0, T[ \to \mathbb{R}^4 \) are electric and magnetic potentials, respectively, i.e., \( \mathbf{E} = -\text{grad} \chi, \mathbf{M} = -\text{grad} \eta, \) \( f_e : \Omega \times ]0, T[ \to \mathbb{R}^4 \) is electric charge density, \( p_{lijk} \) are the piezoelectric coefficients, \( q_{lijk} \) are the piezomagnetic coefficients, \( \varsigma_{jl} \) and \( \xi_{jl} \) are the dielectric (permittivity) and magnetic permeability coefficients, respectively, \( \tilde{a}_{jl} \) are the coupling coefficients connecting electric and magnetic fields. \( \mathbf{D} := (D_1, D_2, D_3) : \Omega \times ]0, T[ \to \mathbb{R}^3 \) is the electrical displacement vector, \( \mathbf{B} := (B_1, B_2, B_3) : \Omega \times ]0, T[ \to \mathbb{R}^3 \) is the magnetic induction vector.

**Kinematic Relations**

\[ e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad i, j = \overline{1, 3}. \]  

(2.4)

**Constitutive Equations**

\[ X_{ji} = X_{ij} = E_{ijkl} \epsilon_{kl} + E_{ijkl}^* \epsilon_{kl} + \tilde{b}_{ijkl} \varphi + b_{ijkl}^* \varphi + d_{ijkl} \varphi_{.k} + d_{ijkl}^* \varphi_{.k} + p_{ijkl} \chi_{.k} + p_{ijkl}^* \chi_{.k} + q_{ijkl} \eta_{.k} + q_{ijkl}^* \eta_{.k}, \quad i, j = \overline{1, 3}, \]  

(2.5)

\[ H_j = d_{klj} \epsilon_{kl} + d_{klj}^* \epsilon_{kl} + d_{j} \varphi + d_{j}^* \varphi + \tilde{\alpha}_{jl} \varphi_{,i} + \alpha_{jl}^* \varphi_{,i}, \quad j = \overline{1, 3}, \]  

(2.6)

\[ H_0 = -\tilde{b}_{ij} \epsilon_{ij} - \tilde{\xi}_{ij} \varphi - d_{ij} \varphi_{,i} - b_{ij}^* \epsilon_{ij} - \xi_{ij}^* \varphi - d_{ij}^* \varphi_{,i}, \]  

(2.7)

\[ D_j = p_{jkl} \epsilon_{kl} + p_{jkl}^* \epsilon_{kl} - \varsigma_{jl} \chi_{,l} - \tilde{\alpha}_{jl} \eta_{,l}, \quad j = \overline{1, 3}, \]  

(2.8)

\[ B_j = q_{jkl} \epsilon_{kl} + q_{jkl}^* \epsilon_{kl} - \tilde{\alpha}_{jl} \chi_{,l} - \xi_{jl} \eta_{,l}, \quad j = \overline{1, 3}, \]  

(2.9)

where \( e_{ij} \in C^1(\Omega) \) is the strain tensor; the constitutive coefficients \( E_{ijkl}, E_{ijkl}^*, \tilde{b}_{ijkl}, b_{ijkl}^*, d_{ijkl}, d_{ijkl}^*, \alpha_{ijkl}, \alpha_{ijkl}^*, \xi_{ijkl}, \xi_{ijkl}^*, p_{ijkl}, p_{ijkl}^*, q_{ijkl}, q_{ijkl}^*, \varsigma_{jl}, \tilde{\alpha}_{jl}, \xi_{jl} \) satisfy the following relations

\[ E_{ijkl} = E_{jikl} = E_{jilk} = E_{klij}; \quad E_{ijkl}^* = E_{jikl}^* = E_{jilk}^* = E_{klij}^*. \]
\[
\tilde{b}_{ij} = \tilde{b}_{ji}, \quad d_{ijk} = d_{jik}, \quad \tilde{\alpha}_{ij} = \tilde{\alpha}_{ji};
\]

\[
\tilde{b}'_{ij} = \tilde{b}'_{ji}, \quad d'_{ijk} = d'_{jik}, \quad \alpha'_{ij} = \alpha'_{ji}; \quad p_{jkl} = p_{jlk}, \quad q_{jkl} = q_{jlk}, \quad \varsigma_{jl} = \varsigma_{lj};
\]

\[
\tilde{\alpha}_{jl} = \tilde{\alpha}_{lj}, \quad \xi_{jl} = \xi_{lj}, \quad p^*_{jkl} = p^*_{jlk}, \quad q^*_{jkl} = q^*_{jlk}.
\]

The constitutive equations also meet some other conditions, following from physical considerations (see [5], [8], and the references given there). With a view to apply I. Vekua’s dimension reduction method, we require the constitutive coefficients to be independent of \( x_3 \).

Let us consider the general BVPs and IBVPs with the following mixed BCs

\[
\begin{align*}
    u_i &= f_i \quad \text{on} \quad \Gamma_0, \quad X_{ij}n_j = g_i \quad \text{on} \quad \Gamma_1 = \partial \Omega \setminus \Gamma_0, \quad i = 1, 3, \\
    \varphi &= f^\varphi \quad \text{on} \quad \Gamma_0^\varphi, \quad H_{ij}n_j = g^\varphi \quad \text{on} \quad \Gamma_1^\varphi = \partial \Omega \setminus \Gamma_0^\varphi, \quad i = 1, 3, \\
    \chi &= f^\chi \quad \text{on} \quad \Gamma_0^\chi, \quad D_{ij}n_j = g^\chi \quad \text{on} \quad \Gamma_1^\chi = \partial \Omega \setminus \Gamma_0^\chi, \quad i = 1, 3, \\
    \eta &= f^n \quad \text{on} \quad \Gamma_0^n, \quad B_{ij}n_j = g^n \quad \text{on} \quad \Gamma_1^n = \partial \Omega \setminus \Gamma_0^n, \quad i = 1, 3,
\end{align*}
\]

and the standard ICs in the case of dynamical problems

\[
\begin{align*}
    u(x, 0) &= u^0(x), \quad \dot{u}(x, 0) = u^1(x), \quad \varphi(x, 0) = \varphi^0(x), \quad \dot{\varphi}(x, 0) = \varphi^1(x), \\
    \chi(x, 0) &= \chi^0(x), \quad \dot{\chi}(x, 0) = \chi^1(x), \quad \eta(x, 0) = \eta^0(x), \quad \dot{\eta}(x, 0) = \eta^1(x),
\end{align*}
\]

where \( n := (n_1, n_2, n_3) \) is the outward unit normal vector to \( \partial \Omega \), \((f_1, f_2, f_3), \ f^\varphi, \ f^\chi, \ f^n \) are the given displacement vector, volume fraction, electric and magnetic potentials, respectively, \((g_1, g_2, g_3), \ g^\varphi, \ g^\chi \) and \( g^n \) are the given stress vector, normal components of the equilibrated stress, electric displacement and magnetic induction vectors, respectively, while \( \mathbf{u}^0 \) and \( \mathbf{u}^1 \) are the initial mechanical displacement and velocity vectors, whereas \( \varphi^0 \) and \( \varphi^1 \) are the initial volume fraction distribution and its rate. Note that the sub-manifolds \( \Gamma_0, \ \Gamma_0^\varphi, \ \Gamma_0^\chi, \) and \( \Gamma_0^n, \) of the boundary \( \partial \Omega \) in boundary conditions (2.10)-(2.13) are different, in general, from each other and depending on the physical problem, some of them may be empty.

3 Construction of Hierarchical Models.

Nth Approximation

Now, we construct hierarchical models for piezoelectric Kelvin-Voigt prismatic shells. First a few words about prismatic shells.

Let us consider prismatic shells (see, e.g., Figure 3.1 and [4], [11]), occupying 3D domain \( \Omega \) with the projection \( \omega \) (on the plane \( x_3 = 0 \)) and the face surfaces

\[
x_3 = h(x_1, x_2) \in C^2(\omega) \quad \text{and} \quad x_3 = h(x_1, x_2) \in C^2(\omega), \quad (x_1, x_2) \in \omega.
\]
Figure 3.1: A prismatic shell of constant thickness. $\partial \Omega$ is a Lipschitz boundary

Figure 3.2: A sharp cusped prismatic shell with a semicircle projection. $\partial \Omega$ is a Lipschitz boundary

Figure 3.3: A cusped plate with sharp $\gamma_1$ and blunt $\gamma_2$ edges, $\gamma^0 := \gamma_1 \cup \gamma_2$. $\partial \Omega$ is a non-Lipschitz boundary

$$2h(x_1, x_2) := (+) h(x_1, x_2) - (-) h(x_1, x_2) > 0, \quad (x_1, x_2) \in \omega,$$

is the thickness of the prismatic shell. A part of $\partial \omega$, where the thickness vanishes, i.e., $2h = 0$, is said to be a cusped edge. If $\partial \Omega$ contains it smoothly, we shall call it a blunt edge, otherwise, i.e., the points of the cusped edge are points of nonsmoothness of $\partial \Omega$, we shall call it a sharp edge (see Figures 3.2, 3.3).

Let

$$2\tilde{h}(x_1, x_2) := (+) h(x_1, x_2) + (-) h(x_1, x_2), \quad (x_1, x_2) \in \omega.$$

Figure 3.4: Comparison of cross-sections of prismatic and standard shells

Figure 3.5: Cross-sections of a prismatic (left) and a standard shell with the same mid-surface
In the case of the symmetric prismatic shell, i.e., when
\[
(\pm) h(x_1, x_2) = - (\mp) h(x_1, x_2),
\]
evidently
\[
2\tilde{h}(x_1, x_2) \equiv 0, \quad (x_1, x_2) \in \omega.
\]

Distinctions between the prismatic shell of a constant thickness and the standard shell of a constant thickness are shown in the Figures 3.4, 3.5, where cross-sections of the prismatic shell of a constant thickness with its projection and of the standard shell of a constant thickness with its middle surface are given in red and green colors, respectively, with common parts in blue. In other words, the lateral boundary of the standard shell is orthogonal to the “middle surface” of the shell, while the lateral boundary of the prismatic shell is orthogonal to the prismatic shell’s projection on \(x_3 = 0\) (see [4]).

In particular, let \(\omega\) be a domain bounded by a sufficiently smooth arc \((\partial \omega \setminus \gamma^0)\) lying in the half-plane \(x_2 > 0\) and a segment \(\gamma^0\) of the \(x_1\)-axis \((x_2 = 0)\). Let the thickness look like (see Figures 3.2, 3.3)
\[
2h(x_1, x_2) = 2h_0x_2^\kappa, \quad h_0, \kappa = const > 0, \tag{3.1}
\]
which corresponds to the case
\[
\begin{align*}
(\pm) h(x_1, x_2) &= h_0x_2^\kappa, \quad h_0 = const, \quad h_0 > h_0, \quad 2h_0 := h_0 - h_0. \\
\end{align*}
\]
In this case we have to do with a blunt edge for \(\kappa < 1\) and with a sharp edge for \(\kappa \geq 1\), respectively.

In Figures 3.6-3.20 (\(\hat{\phi}\) is the angle at the cusp between tangents \(T^+\) and \(T^-\), \(\nu\) is an inward normal at \(O\) to \(\partial \omega\)) we show some characteristic (typical) profiles (cross-sections) of cusped prismatic shells (see also figures in [1], [11], [4]).

First we consider the general case of \(\omega\) and of positive thickness. In such a case the prismatic shell under consideration has not cusped edges.

Figure 3.6: A cross-section of a blunt cusped prismatic shell (\(\hat{\phi} = \pi/2\)). It has a Lipschitz boundary

Figure 3.7: A cross-section of a blunt cusped prismatic shell (\(\hat{\phi} \in [0, \pi/2]\)). It has a Lipschitz boundary
Figure 3.8: A cross-section of a blunt cusped prismatic shell \(\hat{\phi} = 0\). It has a non-Lipschitz boundary.

Figure 3.9: A cross-section of a blunt cusped plate \(\hat{\phi} = \pi\). It has a Lipschitz boundary.

Figure 3.10: A cross-section of a blunt cusped prismatic shell \(\hat{\phi} = \pi/2\). It has a Lipschitz boundary.

Figure 3.11: A cross-section of a blunt cusped prismatic shell \(\hat{\phi} \in ]\pi/2, \pi[\). It has a Lipschitz boundary.

Figure 3.12: \(\hat{\phi} = \pi\)

Figure 3.13: Wedge, \(\hat{\phi} \in ]0, \pi[\)

Figure 3.14: \(\hat{\phi} = 0\)
$r$th order moments of the following quantities are defined as the integrals

$$
\left(u_{ir}, X_{ijr}, e_{ijr}, \Phi_{jr}, H_{ir}, H_{0r}, \varphi_r, F_r, D_{jr}, B_{jr}, \chi_r, \eta_r, f_{er}\right)(x_1, x_2, t)
$$

$$
\begin{align*}
&:= \int \left(u_i, X_{ij}, e_{ij}, \Phi_j, H_i, H_0, \varphi, F, D_j, B_j, \chi, \eta, f_e\right)(x_1, x_2, x_3, t) \\
&\times P_r(ax_3 - b) \, dx_3, \quad i, j = 1, 3,
\end{align*}
$$

where

$$
P_r(ax_3 - b) \left(a(x_1, x_2) := \frac{2}{h - h} = \frac{1}{h}, \quad b(x_1, x_2) := \frac{(-) + (-)}{h + h} = \frac{\tilde{h}}{h}, \right)
$$

$$
\begin{align*}
&= \frac{1}{h}
\end{align*}
$$

are the $r$th order Legendre polynomials.

Under the well-know restrictions (see, e.g., [1]) the following Fourier-Legendre series

$$
\left(u_{ir}, X_{ijr}, e_{ijr}, \Phi_{jr}, H_{ir}, H_{0r}, \varphi_r, F_r, D_{jr}, B_{jr}, \chi_r, \eta_r, f_{er}\right)(x_1, x_2, x_3, t) = \sum_{r=0}^{\infty} a \left(r + \frac{1}{2}\right)
$$

$$
\times \left(u_{ir}, X_{ijr}, e_{ijr}, \Phi_{jr}, H_{ir}, H_{0r}, \varphi_r, F_r, D_{jr}, B_{jr}, \chi_r, \eta_r, f_{er}\right)(x_1, x_2, t)
$$

$$
\times P_r(ax_3 - b)
$$
are convergent.

Therefore on the upper and lower face surfaces of the prismatic shell under consideration

\[
(\pm) \ u_i := u_i(x_1, x_2, h(x_1, x_2), t) = \sum_{s=0}^{\infty} a(s + \frac{1}{2}) u_{is}(\pm 1)^s
\]

\[
= \sum_{s=0}^{\infty} \frac{(\pm 1)^s(2s + 1)}{2h} u_{is},
\]

whence

\[
(\pm) u_i - (-1)^r(\mp) u_i = - \sum_{s=0}^{\infty} a_{3s} u_{is}, \ i = 1, 3,
\]

\[
(\pm)(\pm) u_i h_{\cdot \alpha} - (-1)^r(\mp) u_i h_{\cdot \alpha} = \sum_{s=0}^{\infty} a_{\alpha s} u_{is}, \ i = 1, 3, \ \alpha = 1, 2,
\]

where

\[
a_{\alpha s}^* = a_{\alpha s}, \ s \neq r, \ a_{\alpha s}^* = (2r + 1) \frac{h_{\cdot \alpha}}{h},
\]

\[
a_{\alpha s} := (2s + 1) \frac{(-1)^r h_{\cdot \alpha}^{r+s}}{2h} a_{\alpha s}, \ s \neq r, \ a_{\alpha s}^* := r \frac{h_{\cdot \alpha}}{h},
\]

\[
a_{3s} := -(2s + 1) \frac{1 - (-1)^{s+r}}{2h}.
\]

\[
(b_{js} := -a_{js}, \ s > r; \ b_{js} = 0, \ s < r;
\]

\[
r_{\alpha s} := a_{\alpha s}^* - a_{\alpha s}^* = -r \frac{h_{\cdot \alpha} - h_{\cdot \alpha}}{2h}, \ r_{3s} = 0.
\]

Using (see formulas (10.4), (10.5) of Section 10)

\[
\int_{(\pm) h(x_1, x_2)} (a x_3 - b) f_{\cdot \alpha} \ dx_3 = f_{r, \alpha} + \sum_{s=0}^{r} a_{\alpha s} f_s - f_{h_{\cdot \alpha}}(-1)^r f_{h_{\cdot \alpha}},
\]

\[
\alpha = 1, 2,
\]

\[
\int_{(\pm) h(x_1, x_2)} (a x_3 - b) f_{\cdot \alpha} \ dx_3 = \sum_{s=0}^{r} a_{3s} f_s + f_{r} - (-1)^r f_{r},
\]

\[
\alpha = 1, 2,
\]
from (2.1)-(2.3), after multiplying them by \( P_r(ax_3 - b) \) for \( r = 0, 1, \ldots \), and then integrating within the limits \( h^- (x_1, x_2) \) and \( h^+ (x_1, x_2) \) with respect to the thickness variable \( x_3 \), we obtain the following formulas in \( \omega \):

\[
X_{\alpha r, \alpha} + \sum_{s=0}^{r} \alpha_isX_{js} + \frac{r}{X_i} = \frac{\partial^2 u_{ir}}{\partial t^2}, \quad i = 1, 3, \quad r = 0, 1, \ldots, \quad (3.4)
\]

\[
H_{\alpha r, \alpha} + \sum_{s=0}^{r} \alpha_isH_{is} + H_{0r} + \frac{r}{H} = \rho \frac{\partial^2 \varphi_r}{\partial t^2}, \quad r = 0, 1, \ldots, \quad (3.5)
\]

\[
D_{\alpha r, \alpha} + \sum_{s=0}^{r} \alpha_isD_{is} + \frac{r}{D} = f_{er}, \quad r = 0, 1, \ldots, \quad (3.6)
\]

\[
B_{\alpha r, \alpha} + \sum_{s=0}^{r} \alpha_isB_{is} + B = 0, \quad r = 0, 1, \ldots, \quad (3.7)
\]

where

\[
\begin{align*}
X_j := X_{3j} - X_{ajh_{\alpha}} + (-1)^r \left[ -X_{3j} + X_{ajh_{\alpha}} \right] + \Phi_jr \\
&= X_{n,j}^{+} \left[ 1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2 \right] + \Phi_jr, \\
&= \frac{X_{n,j}^{+}}{1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2}, \\
&= \frac{H_{1}n_{i}}{1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2}, \\
&= \frac{D_{1}n_{i}}{1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2},
\end{align*}
\]

\( X_{n,j}^{+} \) and \( X_{n,j}^- \) are components of the stress vectors acting on the upper and lower face surfaces with normals \( \alpha_n \) and \( \alpha_n^- \), respectively;

\[
H_j := H_3 - H_ah_{\alpha} + (-1)^r \left[ -H_3 + H_ah_{\alpha} \right] + \Phi_j \\
= H_{1}n_{i} \left[ 1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2 \right] + \Phi_j, \\
= \frac{H_{1}n_{i}}{1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2}, \\
= \frac{H_{1}n_{i}}{1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2}, \\
= \frac{H_{1}n_{i}}{1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2},
\]

\( H_j^+ \) and \( H_j^- \) are components of the equilibrated stress vectors on the upper and lower face surfaces with normals \( \alpha_n^+ \) and \( \alpha_n^- \), respectively;

\[
\begin{align*}
D := D_3 - D_{\gamma}h_{\gamma} + (-1)^r \left[ -D_3 + D_{\gamma}h_{\gamma} \right] \\
&= D_{1}n_{i} \left[ 1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2 \right] + D_{1}n_{i}, \\
&= \frac{D_{1}n_{i}}{1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2}, \\
&= \frac{D_{1}n_{i}}{1 + \left( h_{1} \right)^2 + \left( h_{2} \right)^2},
\end{align*}
\]
\[ B := (\pm) B_3 - B_\gamma h_\gamma + (-1)^r \left[ - (\pm) B_3 + B_\gamma h_\gamma \right] \]
\[
= (\pm) \left( B_i u_i \sqrt{1 + (h_{1i})^2 + (h_{2i})^2} \right) \left( B_i u_i \sqrt{1 + (h_{1i})^2 + (h_{2i})^2} \right) \left( - (\pm) B_i u_i \sqrt{1 + (h_{1i})^2 + (h_{2i})^2} \right)
\]

(in the above calculations we have used formulas (10.13) and (10.14) of Section 10 corresponding for \( X_{\alpha i} \) and \( X_{3i} \), \( H_{\alpha} \) and \( H_{3} \), \( D_{\alpha} \) and \( D_{3} \), \( B_{\alpha} \) and \( B_{3} \) instead of \( f \)).

Using (10.5), (10.6) of Section 10 for \( u_i \) instead of \( f \) and the Fourier-Legendre expansions of \( u_i \) on the upper and lower face surfaces

\[
(\pm) u_i = \sum_{s=0}^{\infty} \left( \frac{-(1)^s(2s+1)}{2h} \right) u_{is}, \quad i = 1, 3,
\]
from (2.4), similarly to (10.10), (10.11), we obtain

\[
e_{ijr} = \frac{1}{2} (u_{ir,j} + u_{jr,i}) + \frac{1}{2} \sum_{s=r}^{\infty} b_{is} u_{js} + \frac{1}{2} \sum_{s=r}^{\infty} b_{js} u_{is}, \quad i, j = 1, 3, \quad r = 0, 1, \ldots
\]

(3.8)

\[
\theta_r := e_{iir} = u_{\gamma r, \gamma} + \sum_{s=r}^{\infty} b_{ks} u_{ks}, \quad r = 0, 1, \ldots
\]

In view of

\[
b_{3r} = 0, \quad h^{r+1}(h^{-r-1}_{-\alpha})_{-\alpha} = b_{r\alpha}, \quad \alpha = 1, 2,
\]
we can rewrite (3.8) for

\[
v_{ir} := h^{-r-1} u_{ir}
\]
as follows

\[
e_{ijr} = \frac{1}{2} h^{r+1} (v_{ir,j} + v_{jr,i}) + \frac{1}{2} \sum_{s=r+1}^{\infty} h^{s+1} \left( b_{is} v_{js} + b_{js} v_{is} \right), \quad i, j = 1, 3, \quad r = 0, 1, \ldots
\]

(3.9)

\[
\theta_r := e_{iir} = h^{r+1} v_{\gamma r, \gamma} + \sum_{s=r+1}^{\infty} h^{s+1} b_{ks} v_{ks}, \quad r = 0, 1, \ldots
\]

(3.10)

If we apply formulas (10.11), (10.12) of Section 10, using the Fourier-Legendre expansions of \( \chi \) and \( \eta \) on the upper and lower face surfaces

\[
(\pm) \chi = \sum_{s=0}^{\infty} \left( \frac{-(1)^s(2s+1)}{2h} \right) \chi_s,
\]

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from (2.5) we obtain

\[
X_{ijr} = E_{ijkl}e_{klr} + E^*_{ijkl}\dot{e}_{klr} + \ddot{b}_{ij}\phi_r + b^*_ij\dot{\phi}_r + d_{ij}\gamma_r + \sum_{s=r}^{\infty} b^r_{\gamma s}\phi_s
\]

\[
- \sum_{s=r+1}^{\infty} a^r_{\gamma s}\phi_s + d^r_{ij}\gamma_r + \sum_{s=r}^{\infty} b^r_{\gamma s}\phi_s
\]

\[
+ p_{\gamma ij}(\chi_{r,\gamma} + \sum_{s=r}^{\infty} b^r_{\gamma s}\chi_s) - p^{*}_{3ij}(\dot{\chi}_{r,\gamma} + \sum_{s=r}^{\infty} b^r_{\gamma s}\dot{\chi}_s)
\]

\[
- p^{*}_{3ij}(\sum_{s=r+1}^{\infty} a^r_{3s}\dot{\chi}_s + q_{\gamma ij}(\eta_{r,\gamma} + \sum_{s=r}^{\infty} b^r_{\gamma s}\eta_s) - q^{*}_{3ij}(\sum_{s=r+1}^{\infty} a^r_{3s}\eta_s)
\]

\[
+ q^{*}_{\gamma ij}(\eta_{r,\gamma} + \sum_{s=r}^{\infty} b^r_{\gamma s}\eta_s) - q^{*}_{3ij}(\sum_{s=r+1}^{\infty} a^r_{3s}\eta_s)
\]

Therefore, by virtue of (3.8),

\[
X_{ijr} = \frac{1}{2} E_{ijkl}(u_{kr,l} + u_{lr,k}) + \frac{1}{2} E_{ijkl} \sum_{s=r}^{\infty} (b^r_{ks}u_{ls} + b^r_{ls}u_{ks})
\]

\[
+ \frac{1}{2} E^*_{ijkl}(\dot{u}_{kr,l} + \dot{u}_{lr,k}) + \frac{1}{2} E^*_{ijkl} \sum_{s=r}^{\infty} (b^r_{ks}\dot{u}_{ls} + b^r_{ls}\dot{u}_{ks})
\]

\[
+ \ddot{b}_{ij}\phi_r + b^*_ij\dot{\phi}_r + d_{ij}\gamma_r + \sum_{s=r}^{\infty} b^r_{\gamma s}\phi_s
\]

\[
- \sum_{s=r+1}^{\infty} a^r_{\gamma s}\phi_s + d^r_{ij}\gamma_r + \sum_{s=r}^{\infty} b^r_{\gamma s}\phi_s
\]

\[
+ p_{\gamma ij}(\chi_{r,\gamma} + \sum_{s=r}^{\infty} b^r_{\gamma s}\chi_s) - p^{*}_{3ij}(\dot{\chi}_{r,\gamma} + \sum_{s=r}^{\infty} b^r_{\gamma s}\dot{\chi}_s)
\]

\[
- p^{*}_{3ij}(\sum_{s=r+1}^{\infty} a^r_{3s}\dot{\chi}_s + q_{\gamma ij}(\eta_{r,\gamma} + \sum_{s=r}^{\infty} b^r_{\gamma s}\eta_s) - q^{*}_{3ij}(\sum_{s=r+1}^{\infty} a^r_{3s}\eta_s)
\]

\[
+ q^{*}_{\gamma ij}(\eta_{r,\gamma} + \sum_{s=r}^{\infty} b^r_{\gamma s}\eta_s) - q^{*}_{3ij}(\sum_{s=r+1}^{\infty} a^r_{3s}\eta_s)
\]

\[
, ~ i,j = 1,3, ~ r = 0, 1, \cdots. \quad (3.11)
\]

Let

\[
v_r := \frac{u_r}{h^{r+1}}, \quad \psi_r := \frac{\varphi_r}{h^{r+1}}, \quad \tilde{\chi}_r := \frac{\chi_r}{h^{r+1}}, \quad \tilde{\eta}_r := \frac{\eta_r}{h^{r+1}}. \quad (3.13)
\]
Substituting (3.9) into (3.11), and taking into account (3.13), it follows that

\[
X_{ijr} = \frac{1}{2} E_{ijkl} h^{r+1} \left( v_{kr,l} + v_{lr,k} \right) + \frac{1}{2} E_{ijkl} \sum_{s=r+1}^{\infty} h^{s+1} \left( b_{ks} v_{sl} + b_{ls} v_{ks} \right) \\
+ \frac{1}{2} E_{ijkl} h^{r+1} \left( \psi_{kr,l} + \dot{\psi}_{lr,k} \right) + \frac{1}{2} E_{ijkl} \sum_{s=r+1}^{\infty} h^{s+1} \left( b_{ks} \dot{v}_{ls} + b_{ls} \dot{v}_{ks} \right) \\
+ b_{ij} h^{r+1} \psi_r + b_{ij} h^{r+1} \psi_r + d_{ij \gamma} h^{r+1} \psi_{r,\gamma} + d_{ijk} \sum_{s=r+1}^{\infty} h^{s+1} b_{ks} \phi_s + d_{ij \gamma} h^{r+1} \dot{\psi}_{r,\gamma} \\
+ d_{ijk} \sum_{s=r+1}^{\infty} h^{s+1} b_{ks} \dot{\psi}_s + p_{\gamma ij} h^{r+1} \dot{\chi}_{r,\gamma} + p_{klj} \sum_{s=r+1}^{\infty} h^{s+1} b_{ks} \dot{\chi}_s \\
P_{\gamma ij} h^{r+1} \dot{\chi}_{r,\gamma} + p_{klj} \sum_{s=r+1}^{\infty} h^{s+1} b_{ks} \dot{\chi}_s + q_{\gamma ij} h^{r+1} \eta_{r,\gamma} + q_{klj} \sum_{s=r+1}^{\infty} h^{s+1} b_{ks} \dot{\eta}_s \\
+ q_{\gamma ij} h^{r+1} \dot{\eta}_{r,\gamma} + q_{klj} \sum_{s=r+1}^{\infty} h^{s+1} b_{ks} \dot{\eta}_s, \quad i, j = \overline{1, 3}, \quad r = 0, 1, \ldots 
\]

(because of

\[
(h^{r+1} \dot{\chi}_r)_{\gamma} - h^{r+1} (r + 1) \frac{\chi_{r,\alpha}}{h} \dot{\chi}_r = h^{r+1} \dot{\chi}_{r,\gamma},
\]

and the similar formulas for \( \psi \) and \( \dot{\eta} \).

Analogously, from (2.6) we have (compare with (10.11), (10.12))

\[
H_{jr} = \frac{1}{2} d_{klj}(u_{kr,l} + u_{lr,k}) + \frac{1}{2} d_{klj} \sum_{s=r}^{\infty} (b_{ks} u_{ls} + b_{ls} u_{ks}) \\
+ \frac{1}{2} d_{klj}(u_{kr,l} + u_{lr,k}) + \frac{1}{2} d_{klj} \sum_{s=r}^{\infty} (b_{ks} u_{ls} + b_{ls} u_{ks}) + d_{jr} \psi_r + d_{jr} \dot{\varphi}_r \\
+ \tilde{\alpha}_{j3} \left[ \psi_{r,\beta} + \sum_{s=0}^{r} a_{\beta s} \psi_s - (\psi \phi)_s h_{\beta} + (-1)^r \phi h_{\beta} \right] \\
+ \tilde{\alpha}_{j3} \left[ \sum_{s=0}^{r} a_{3s} \phi_s + (\phi \phi)_s - (-1)^r \phi \phi \right] \\
+ \tilde{\alpha}_{j3} \left[ \phi_{r,\beta} + \sum_{s=0}^{r} a_{\beta s} \phi_s - \phi h_{\alpha} - (-1)^r \phi h_{\alpha} \right] \\
+ \tilde{\alpha}_{j3} \left[ \sum_{s=0}^{r} a_{3s} \dot{\phi}_s + \dot{\phi} - (-1)^r \dot{\phi} \right], \quad j = \overline{1, 3},
\]

and substituting here the corresponding Fourier-Legendre expansions of \( \varphi \) on
the upper and lower face surfaces

$$\varphi = \sum_{s=0}^{\infty} \frac{(\pm 1)^{s}(2s + 1)}{2h} \varphi_s,$$

we get

$$H_{jr} = \frac{1}{2}d_{klj}(u_{kr,l} + u_{lr,k}) + \frac{1}{2}d_{klj} \sum_{s=r}^{\infty} (b_{ks} u_{ls} + b_{ls} u_{ks})$$

$$\frac{1}{2}d_{klj}^*(u_{kr,l} + u_{lr,k}) + \frac{1}{2}d_{klj}^* \sum_{s=r}^{\infty} (b_{ks} u_{ls} + b_{ls} u_{ks}) + d_{j} \varphi_r + d_{j}^* \dot{\varphi}_r$$

$$+ \tilde{a}_{jk} \left( \varphi_{r,k} + \sum_{s=r}^{\infty} b_{ks} \varphi_s \right) + \alpha^*_{jk} \left( \dot{\varphi}_{r,k} + \sum_{s=r}^{\infty} b_{ks} \dot{\varphi}_s \right), \quad j = 1, 3, \quad r = 0, 1, \ldots, \quad (3.15)$$

i.e. (see (3.13))

$$H_{jr} = \frac{1}{2}d_{klj} h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2}d_{klj} \sum_{s=r+1}^{\infty} h^{s+1} (b_{ks} v_{hs} + b_{ls} v_{ks})$$

$$+ \frac{1}{2}d_{klj}^* h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2}d_{klj}^* \sum_{s=r+1}^{\infty} h^{s+1} (b_{ks} \dot{v}_{hs} + b_{ls} \dot{v}_{ks})$$

$$+ d_{j} h^{r+1} \dot{\psi}_r + d_{j}^* h^{r+1} \dot{\psi}_r$$

$$+ \tilde{a}_{ji} \left( h^{r+1} \psi_{r,i} + \sum_{s=r+1}^{\infty} h^{s+1} b_{is} \psi_s \right) + \alpha^*_{ji} \left( h^{r+1} \dot{\psi}_{r,i} + \sum_{s=r+1}^{\infty} h^{s+1} b_{is} \dot{\psi}_s \right), \quad (3.16)$$

$$j = 1, 3, \quad r = 0, 1, \ldots$$

From (2.7), on account of combined (10.11), (10.12), evidently,

$$H_{0r} = -d_{i}(\varphi_{r,i} + \sum_{s=r}^{\infty} b_{is} \varphi_s) - \tilde{b}_{ij} e_{ijr} - \tilde{\xi} \varphi_r$$

$$- d^* (\dot{\varphi}_{r,i} + \sum_{s=r}^{\infty} b_{is} \dot{\varphi}_s) - \tilde{b}_{ij} \dot{e}_{ijr} - \tilde{\xi}^* \dot{\varphi}_r, \quad r = 0, 1, \ldots$$

and, in view of (3.8),

$$H_{0r} = -d_{i}(\varphi_{r,i} + \sum_{s=r}^{\infty} b_{is} \varphi_s)$$

$$- \tilde{b}_{ij} \left[ \frac{1}{2} (u_{ir,j} + u_{jr,i}) + \frac{1}{2} \sum_{s=r}^{\infty} b_{is} u_{js} + \frac{1}{2} \sum_{s=r}^{\infty} b_{js} u_{is} \right]$$

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\[-\ddot{\xi}\varphi_r - d^*(\dot{\varphi}_{r,i} + \sum_{s=r} b_{is} \dot{\varphi}_s)\]

\[-b_{ij} \left[ \frac{1}{2} \left( u_{ir,j} + u_{jr,i} \right) + \frac{1}{2} \sum_{s=r} b_{is} \dot{u}_{js} + \frac{1}{2} \sum_{s=r} b_{js} \dot{u}_{is} \right] - \xi^*\dot{\varphi}_r, \quad (3.17)\]

while, by virtue of (3.9) and (3.13),

\[H_{0r} = -\ddot{b}_{ij} \left[ \frac{1}{2} h^{r+1} \left( v_{ir,j} + v_{jr,i} \right) + \frac{1}{2} \sum_{s=r+1} h^{s+1} \left( b_{is} \dot{v}_{js} + b_{js} \dot{v}_{is} \right) \right] \]

\[-\ddot{\xi} h^{r+1} \psi_r - d_i \left( h^{r+1} \dot{\psi}_{r,i} + \sum_{s=r+1} b_{is} \dot{\psi}_s \right) \]

\[-b_{ij} \left[ \frac{1}{2} h^{r+1} \left( \dot{v}_{ir,j} + \dot{v}_{jr,i} \right) + \frac{1}{2} \sum_{s=r+1} h^{s+1} \left( b_{is} \dot{v}_{js} + b_{js} \dot{v}_{is} \right) \right] \]

\[-\xi^* h^{r+1} \dot{\psi}_r - d_i^* \left( h^{r+1} \dot{\psi}_{r,i} + \sum_{s=r+1} b_{is} \dot{\psi}_s \right), \quad r = 0, 1, \cdots. \quad (3.18)\]

Similarly, from (2.8) it follows

\[D_{jr} = p_{jkl} e_{klr} + p^*_{jkl} \dot{e}_{klr} - \varsigma_j \gamma \left( x_{r,\gamma} + \sum_{s=r} b_{\gamma s} x_s \right) + \varsigma_j \sum_{s=r+1} a_{\gamma s} x_s \]

\[-\tilde{a}_{j} \gamma \left( \eta_{r,\gamma} + \sum_{s=r} b_{\gamma s} \eta_s \right) + \tilde{a}_{j} \sum_{s=r+1} a_{\gamma s} \eta_s, \quad j = 1, 3, \quad r = 0, 1, \cdots, \]

i.e., in view of (3.8),

\[D_{jr} = \frac{1}{2} p_{jkl} (u_{kr,l} + u_{lr,k}) + \frac{1}{2} p_{jkl} \sum_{s=r} (b_{ks} u_{ls} + b_{ls} u_{ks}) \]

\[+ \frac{1}{2} p^*_{jkl} (\dot{u}_{kr,l} + \dot{u}_{lr,k}) + \frac{1}{2} p^*_{jkl} \sum_{s=r} (b_{ks} \dot{u}_{ls} + b_{ls} \dot{u}_{ks}) \]

\[-\varsigma_j \gamma \left( x_{r,\gamma} + \sum_{s=r} b_{\gamma s} x_s \right) + \varsigma_j \sum_{s=r+1} a_{\gamma s} x_s \]

\[-\tilde{a}_{j} \gamma \left( \eta_{r,\gamma} + \sum_{s=r} b_{\gamma s} \eta_s \right) + \tilde{a}_{j} \sum_{s=r+1} a_{\gamma s} \eta_s, \quad j = 1, 3, \quad r = 0, 1, \cdots. \quad (3.19)\]
while by virtue of (3.9) and (3.13), we have

\[
D_{jr} = \frac{1}{2} p_{jkl} h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2} p_{jkl} \sum_{s=r+1}^{\infty} h^{s+1}(b_{ks} v_{ls} + b_{ls} v_{ks}) \\
+ \frac{1}{2} p_{jkl} h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} p_{jkl} \sum_{s=r+1}^{\infty} h^{s+1}(b_{ks} \dot{v}_{ls} + b_{ls} \dot{v}_{ks}) \\
- \xi_{j7} \left( h^{r+1} \chi_{r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1} b_{\gamma s} \tilde{x}_{s} \right) + \xi_{j3} \sum_{s=r+1}^{\infty} h^{s+1} a_{3s} \tilde{x}_{s} \\
- \alpha_{j7} \left( h^{r+1} \eta_{r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1} b_{\gamma s} \tilde{\eta}_{s} \right) + \alpha_{j3} \sum_{s=r+1}^{\infty} h^{s+1} a_{3s} \tilde{\eta}_{s},
\]

(3.20)

\[j = 1, 3, \ r = 0, 1, \cdots.\]

In the same way from (2.9) we get

\[
B_{jr} = \frac{1}{2} q_{jkl}(u_{kr,l} + u_{lr,k}) + \frac{1}{2} q_{jkl} \sum_{s=r}^{\infty} (b_{ks} u_{ls} + b_{ls} u_{ks}) \\
+ \frac{1}{2} q_{jkl} (\dot{u}_{kr,l} + \dot{u}_{lr,k}) + \frac{1}{2} q_{jkl} \sum_{s=r}^{\infty} (b_{ks} \dot{u}_{ls} + b_{ls} \dot{u}_{ks}) \\
- \dot{\alpha}_{j7} \left( \chi_{r,\gamma} + \sum_{s=r}^{\infty} b_{\gamma s} \chi_{s} \right) + \dot{\alpha}_{j3} \sum_{s=r}^{\infty} a_{3s} \chi_{s} \\
- \dot{\xi}_{j7} \left( \eta_{r,\gamma} + \sum_{s=r}^{\infty} b_{\gamma s} \eta_{s} \right) + \dot{\xi}_{j3} \sum_{s=r}^{\infty} a_{3s} \eta_{s},
\]

(3.21)

i.e.

\[
B_{jr} = \frac{1}{2} q_{jkl} h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2} q_{jkl} \sum_{s=r+1}^{\infty} h^{s+1}(b_{ks} v_{ls} + b_{ls} v_{ks}) \\
+ \frac{1}{2} q_{jkl} h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} q_{jkl} \sum_{s=r+1}^{\infty} h^{s+1}(b_{ks} \dot{v}_{ls} + b_{ls} \dot{v}_{ks}) \\
- \dot{\alpha}_{j7} \left( h^{r+1} \chi_{r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1} b_{\gamma s} \tilde{x}_{s} \right) + \dot{\alpha}_{j3} \sum_{s=r+1}^{\infty} h^{s+1} a_{3s} \tilde{x}_{s} \\
- \dot{\xi}_{j7} \left( h^{r+1} \eta_{r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1} b_{\gamma s} \tilde{\eta}_{s} \right) + \dot{\xi}_{j3} \sum_{s=r+1}^{\infty} h^{s+1} a_{3s} \tilde{\eta}_{s},
\]

(3.22)

\[j = 1, 3, \ r = 0, 1, \cdots;\]

We remind that

\[ r \ b_{\alpha r} := -(r+1) \frac{h_{\alpha r}}{h}, \ r \ b_{3r} = 0, \]

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with respect to the into (3.6) and (3.7), respectively, we construct an equivalent infinite system in the following sense: if will satisfy the infinite relations (3.4)-(3.8), (3.12), (3.15), (3.17), (3.19), (3.21),

\[
\begin{align*}
\tilde{a}_{\alpha} &= -2s + 1 \frac{(-1)^{s+r}}{2h}, & s \leq r, \\
\tilde{a}_{3s} &= 2s + 1 \frac{(-1)^{s+r}}{2h}, & s > r, \\
\end{align*}
\]

\[\alpha = 1, 2, \quad j = 1, 3, \quad r, s = 0, 1, 2, \ldots .\]

So, we get the equivalent to (2.1)-(2.9), infinite system (3.4)-(3.8), (3.12), (3.15), (3.17), (3.19), (3.21), (3.8) with respect to the so called r-th order moments \(X_{ijr}, e_{ijr}, u_{ir}, H_{jr}, H_{0r}, \varphi, \chi, \eta_r\). Then, substituting (3.12) into (3.4), expressions (3.15) and (3.17) into (3.5); expressions (3.19) and (3.21) into (3.6) and (3.7), respectively, we construct an equivalent infinite system with respect to the r-th order moments \(u_{ir}, \varphi, \chi, \eta_r\). Namely,

\[
\begin{align*}
\frac{1}{2} \left( E_{\alpha i k l} u_{kr, \delta} \right)_{,\alpha} &= \frac{1}{2} \left( E_{\alpha i l} u_{rl, \gamma} \right)_{,\alpha} + \frac{1}{2} \left( E_{\alpha i k l} \sum_{s=r}^{\infty} b_{ks} u_{ls} \right)_{,\alpha} \\
+ \frac{1}{2} \left( E_{\alpha i k l} \sum_{s=r}^{\infty} b_{ls} u_{ks} \right)_{,\alpha} &= \frac{1}{2} \left( E_{\alpha i l} \sum_{s=r}^{\infty} b_{ks} u_{ls} \right)_{,\alpha} + \frac{1}{2} \left( E_{\alpha i l} \sum_{s=r}^{\infty} b_{ls} u_{ks} \right)_{,\alpha} + (b_{\alpha l} \varphi_r)_{,\alpha} + (d_{\alpha i} \varphi_{r, \gamma})_{,\alpha} + (d_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{ls} \varphi_{s} \gamma)_{,\alpha} - (d_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{s} \varphi_{s} \delta)_{,\alpha} \\
+ \left( d_{\alpha i} \varphi_{r, \gamma} \right)_{,\alpha} + \left( d_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{ls} \varphi_{s} \gamma \right)_{,\alpha} - \left( d_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{s} \varphi_{s} \delta \right)_{,\alpha} \\
+ \left( p_{\alpha i} \chi_{r, \gamma} \right)_{,\alpha} + \left( p_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{ls} \chi_{s} \gamma \right)_{,\alpha} - \left( p_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{s} \chi_{s} \delta \right)_{,\alpha} \\
+ \left( q_{\alpha i} \tilde{\chi}_{r, \gamma} \right)_{,\alpha} + \left( q_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{ls} \tilde{\chi}_{s} \gamma \right)_{,\alpha} - \left( q_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{s} \tilde{\chi}_{s} \delta \right)_{,\alpha} \\
+ \left( q_{\alpha i} \tilde{\eta}_{r, \gamma} \right)_{,\alpha} + \left( q_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{ls} \tilde{\eta}_{s} \gamma \right)_{,\alpha} - \left( q_{\alpha i} \sum_{s=r}^{\infty} b_{r} b_{s} \tilde{\eta}_{s} \delta \right)_{,\alpha}
\end{align*}
\]

\[\text{in the following sense: if } X_{ij}, e_{ij}, u_{i}, H_{i}, H_{0}, D_{j}, B_{j}, \chi, \eta, \text{ and } \varphi \text{ satisfy the relations (2.1)-(2.9), then constructed by (3.2) functions } X_{ijr}, e_{ijr}, u_{ir}, H_{jr}, H_{0r}, D_{jr}, B_{jr}, \chi, \eta, \text{ and } \varphi \text{ will satisfy the infinite relations (3.4)-(3.8), (3.12), (3.15), (3.17), (3.19), (3.21), (3.8) and, vice versa, if } X_{ijr}, e_{ijr}, u_{ir}, H_{jr}, H_{0r}, D_{jr}, B_{jr}, \chi, \eta, \text{ and } \varphi \text{ satisfy the infinite relations (3.4)-(3.8), (3.12), (3.15), (3.17), (3.19), (3.21), (3.8), then constructed by means of (3.3) functions } X_{ij}, e_{ij}, u_{i}, H_{i}, H_{0}, D_{j}, B_{j}, \chi, \eta, \text{ and } \varphi \text{ will satisfy the relations (2.1)-(2.9).} \]
\[ + \sum_{s=0}^{r} a_{js} \left[ \frac{1}{2} E_{jkl} (u_{ks,l} + u_{ls,k}) + \frac{1}{2} E_{jkl} \sum_{s'=s}^{\infty} (b_{ks'}u_{ls'} + b_{ls'}u_{ks'}) \right] \\
+ \frac{1}{2} E_{jkl} (u_{ks,l} + u_{ls,k}) + \frac{1}{2} E_{jkl} \sum_{s'=s}^{\infty} (b_{ks'}u_{ls'} + b_{ls'}u_{ks'}) \\
+ b_{ij} \varphi + b_{ij} \varphi + d_{ij} \left( \varphi_{\gamma} + \sum_{s'=s}^{\infty} b_{ijs'} \varphi_{s'} \right) - d_{ij} \sum_{s'=s+1}^{\infty} a_{3s'} \varphi_{s'} \\
+ d_{ij} \left( \varphi_{\gamma} + \sum_{s'=s}^{\infty} b_{ijs'} \varphi_{s'} \right) - d_{ij} \sum_{s'=s+1}^{\infty} a_{3s'} \varphi_{s'} \\
+ p_{\gamma} \left( \chi_{s,\gamma} + \sum_{s'=s}^{\infty} b_{s'\chi} \chi_{s'} \right) - p_{\gamma} \sum_{s'=s+1}^{\infty} a_{3s'} \chi_{s'} \\
+ p_{\gamma} \left( \chi_{s,\gamma} + \sum_{s'=s}^{\infty} b_{s'\chi} \chi_{s'} \right) - p_{\gamma} \sum_{s'=s+1}^{\infty} a_{3s'} \chi_{s'} \\
+ q_{\gamma} \left( \eta_{s,\gamma} + \sum_{s'=s}^{\infty} b_{s'\eta} \eta_{s'} \right) - q_{\gamma} \sum_{s'=s+1}^{\infty} a_{3s'} \eta_{s'} \\
+ q_{\gamma} \left( \eta_{s,\gamma} + \sum_{s'=s}^{\infty} b_{s'\eta} \eta_{s'} \right) - q_{\gamma} \sum_{s'=s+1}^{\infty} a_{3s'} \eta_{s'} \\
+ \rho \frac{\partial^{2} u_{ir}}{\partial t^{2}} + X_{i} \]
- \hat{b}_{ij} \left[ \frac{1}{2} (u_{ir,j} + u_{jr,i}) + \frac{1}{2} \sum_{s=r}^{\infty} b_{is} u_{js} + \frac{1}{2} \sum_{s=r}^{\infty} b_{js} u_{is} \right] - \hat{\xi} \varphi_r

- d_i (\varphi_{r,i} + \sum_{s=r}^{\infty} b_{is} \varphi_s) - b_{ij} \sum_{s=r}^{\infty} (u_{ir,j} + u_{jr,i}) + \frac{1}{2} \sum_{s=r}^{\infty} b_{is} u_{js} + \frac{1}{2} \sum_{s=r}^{\infty} b_{js} u_{is} \right]

- \xi^* \varphi_r - d_i^* (\varphi_{r,i} + \sum_{s=r}^{\infty} b_{is} \varphi_s) + \hat{H} = \rho \hbar \frac{\partial^2 \varphi_r}{\partial t^2}, \quad r = 0, 1, \ldots ,

\frac{1}{2} (p_{akl} u_{kr,l})_{\alpha} + \frac{1}{2} (p_{akl} u_{lr,k})_{\alpha} + \frac{1}{2} (p_{akl} \sum_{s=r}^{\infty} b_{ks} u_{ls})_{\alpha} + \frac{1}{2} (p_{akl} \sum_{s=r}^{\infty} b_{ls} u_{ks})_{\alpha}

+ \frac{1}{2} (p_{akl} \hat{u}_{kr,l})_{\alpha} + \frac{1}{2} (p_{akl} \hat{u}_{lr,k})_{\alpha} + \frac{1}{2} (p_{akl} \sum_{s=r}^{\infty} b_{ks} \hat{u}_{ls})_{\alpha} + \frac{1}{2} (p_{akl} \sum_{s=r}^{\infty} b_{ls} \hat{u}_{ks})_{\alpha}

- \left( s_{r\gamma} \chi_{r,\gamma} \right)_{\alpha} - \left( s_{r\gamma} \sum_{s=r}^{\infty} b_{\gamma\alpha} \chi_{s} \right)_{\alpha} + \left( s_{r\gamma} \sum_{s=r+1}^{\infty} a_{3\gamma} \chi_{s} \right)_{\alpha}

- \left( \dot{a}_{r\gamma} \chi_{r,\gamma} \right)_{\alpha} - \left( \dot{a}_{r\gamma} \sum_{s=r}^{\infty} b_{\gamma\alpha} \chi_{s} \right)_{\alpha} + \left( \dot{a}_{r\gamma} \sum_{s=r+1}^{\infty} a_{3\gamma} \chi_{s} \right)_{\alpha}

\left( \dot{\alpha}_{r\gamma} \chi_{r,\gamma} + \sum_{s=s}^{\infty} b_{\gamma\alpha} \chi_{s} \right) + \eta_{r\gamma} \sum_{s=s+1}^{\infty} a_{3\gamma} \chi_{s} + D = f_{cr}, \quad r = 0, 1, \ldots ,

\frac{1}{2} (q_{akl} u_{kr,l})_{\alpha} + \frac{1}{2} (q_{akl} u_{lr,k})_{\alpha} + \frac{1}{2} (q_{akl} \sum_{s=r}^{\infty} b_{ks} u_{ls})_{\alpha} + \frac{1}{2} (q_{akl} \sum_{s=r}^{\infty} b_{ls} u_{ks})_{\alpha}

+ \frac{1}{2} (q_{akl} \hat{u}_{kr,l})_{\alpha} + \frac{1}{2} (q_{akl} \hat{u}_{lr,k})_{\alpha} + \frac{1}{2} (q_{akl} \sum_{s=r}^{\infty} b_{ks} \hat{u}_{ls})_{\alpha} + \frac{1}{2} (q_{akl} \sum_{s=r}^{\infty} b_{ls} \hat{u}_{ks})_{\alpha}

- \left( \dot{\alpha}_{r\gamma} \chi_{r,\gamma} \right)_{\alpha} - \left( \dot{\alpha}_{r\gamma} \sum_{s=r}^{\infty} b_{\gamma\alpha} \chi_{s} \right)_{\alpha} + \left( \dot{\alpha}_{r\gamma} \sum_{s=r+1}^{\infty} a_{3\gamma} \chi_{s} \right)_{\alpha}

- \left( \dot{\alpha}_{r\gamma} \eta_{r,\gamma} \right)_{\alpha} - \left( \dot{\alpha}_{r\gamma} \sum_{s=r}^{\infty} b_{\gamma\alpha} \eta_{s} \right)_{\alpha} + \left( \dot{\alpha}_{r\gamma} \sum_{s=r+1}^{\infty} a_{3\gamma} \eta_{s} \right)_{\alpha}
+ \sum_{s=0}^{r} a_{is} \left[ \frac{1}{2} q_{ikl}(u_{ks,l} + u_{ls,k}) + \frac{1}{2} q_{ikl} \sum_{s'=s}^{\infty} (b_{ks'}u_{ls'} + b_{ls'}u_{ks'}) \right]
+ \frac{1}{2} q_{ikl}(u_{ks,l} + u_{ls,k}) + \frac{1}{2} q_{ikl} \sum_{s'=s}^{\infty} (b_{ks'}u_{ls'} + b_{ls'}u_{ks'})
- \hat{a}_{i\gamma} \left( \chi_{s,\gamma} + \frac{1}{2} \tilde{b}_{\gamma s'} \chi_{s'} \right) + \tilde{a}_{i3} \sum_{s'=s+1}^{s} \tilde{a}_{3s'} \chi_{s'}
- \xi_{i\gamma} \left( \eta_{s,\gamma} + \sum_{s'=s}^{s} \tilde{b}_{\gamma s'} \eta_{s'} \right) + \xi_{i3} \sum_{s'=s+1}^{s} \tilde{a}_{3s'} \eta_{s'} \right] + \frac{r}{B} = 0, \quad r = 0, 1, \cdots.

If we suppose that the moments, whose subscripts, indicating moments’ order, are greater than \(N\), equal to zero and consider only the first \(N + 1\) equations \((r = 0, N)\) for each \(i = 1, 2, 3, \varphi_r, \chi_r, \eta_r\), from the obtained infinite system of equations with respect to

\[ u_{ir}, \quad i = \overline{1, 3}, \ \varphi_r, \ \chi_r, \ \eta_r \quad r = 0, 1, \cdots, \]

we obtain the \(N\)th order approximation (hierarchical model) governing system consisting of \(6N + 6\) equations with respect to \(6N + 6\) unknown functions

\[ u_{ir}, \ \varphi_r, \ \chi_r, \ \eta_r, \quad i = \overline{1, 3}, \ \ r = 0, N \]

(roughly speaking \(u_{ir}, \varphi_r, \chi_r, \eta_r\) are “approximate values” of \(u_{ir}, \varphi_r, \chi, \eta\), since \(u_{ir}, \varphi_r, \chi, \eta\) are solutions of the derived finite system; below superscript \(N\) is omitted in order to avoid overloading the symbols):

\[
\frac{1}{2} (E_{aikl}u_{kr,l})_{\alpha} + \frac{1}{2} (E_{aikl}u_{lr,k})_{\alpha} + \frac{1}{2} (E_{aikl} \sum_{s=r}^{N} b_{ks}u_{ls})_{\alpha}
+ \frac{1}{2} (E_{aikl} \sum_{s=r}^{N} b_{ks}u_{ks})_{\alpha} + \frac{1}{2} (E_{aikl} \sum_{s=r}^{N} b_{ls}u_{ks})_{\alpha}
+ \frac{1}{2} (E_{aikl} \sum_{s=r}^{N} b_{ls}u_{ks})_{\alpha} + \frac{1}{2} (E_{aikl} \sum_{s=r}^{N} b_{ls}u_{ks})_{\alpha}
+ (b_{a1\varphi_r})_{\alpha} + (d_{a1\gamma} \varphi_{r,\gamma})_{\alpha} + (d_{a1r} \sum_{s=r}^{N} b_{\gamma s} \varphi_{s})_{\alpha} - (d_{a1r} \sum_{s=r+1}^{N} b_{3s} \varphi_{s})_{\alpha}
+ (d_{a1r} \varphi_{r,\gamma})_{\alpha} + (d_{a1r} \varphi_{r,\gamma})_{\alpha} - (d_{a1r} \sum_{s=r+1}^{N} b_{3s} \varphi_{s})_{\alpha}
\]
\[(p_{\gamma ai} x_{r,\gamma}) \cdot \alpha + (p_{\gamma ai} \sum_{s=r}^{N} b_{\gamma s} x_{s}) \cdot \alpha - (p_{3\alpha i} \sum_{s=r+1}^{N} a_{3s} x_{s}) \cdot \alpha + (p_{\gamma ai}^{*} x_{r,\gamma}) \cdot \alpha + (p_{\gamma ai}^{*} \sum_{s=r}^{N} b_{\gamma s} x_{s}) \cdot \alpha - (p_{3\alpha i}^{*} \sum_{s=r+1}^{N} a_{3s} x_{s}) \cdot \alpha + (q_{\gamma ai} \eta_{r,\gamma}) \cdot \alpha + (q_{\gamma ai} \sum_{s=r}^{N} b_{\gamma s} \eta_{s}) \cdot \alpha - (q_{3\alpha i} \sum_{s=r+1}^{N} a_{3s} \eta_{s}) \cdot \alpha + (q_{\gamma ai}^{*} \eta_{r,\gamma}) \cdot \alpha + (q_{\gamma ai}^{*} \sum_{s=r}^{N} b_{\gamma s} \eta_{s}) \cdot \alpha - (q_{3\alpha i}^{*} \sum_{s=r+1}^{N} a_{3s} \eta_{s}) \cdot \alpha + \sum_{s=0}^{r} \frac{1}{2} E_{ijkl} (u_{ks,l} + u_{ls,k}) + \sum_{s'=s}^{N} \frac{1}{2} E_{ijkl} (b_{ks,s'} u_{ls'} + b_{ls'} u_{ks'}) + \sum_{s'=s}^{N} \frac{1}{2} E_{ijkl} (\dot{u}_{ks,l} + \dot{u}_{ls,k}) + \sum_{s'=s}^{N} \frac{1}{2} E_{ijkl} (b_{ks,s'} u_{ls'} + b_{ls'} u_{ks'}) + b_{ij} \varphi_{s} + b_{ij}^{*} \dot{\varphi}_{s} + d_{ij} \gamma (\varphi_{s,\gamma} + \sum_{s'=s}^{N} b_{s,s'} \varphi_{s'}) - d_{ij3} \sum_{s'=s+1}^{N} a_{3s'} \varphi_{s'} + d_{ij}^{*} \gamma (\varphi_{s,\gamma} + \sum_{s'=s}^{N} b_{s,s'} \varphi_{s'}) - d_{ij3}^{*} \sum_{s'=s+1}^{N} a_{3s'} \dot{\varphi}_{s'} + p_{\gamma ji} (\chi_{s,\gamma} + \sum_{s'=s}^{N} b_{s,s'} \chi_{s'}) - p_{3ij} \sum_{s'=s+1}^{N} a_{3s'} \chi_{s'} + p_{\gamma ji}^{*} (\dot{\chi}_{s,\gamma} + \sum_{s'=s}^{N} b_{s,s'} \dot{\chi}_{s'}) - p_{3ij}^{*} \sum_{s'=s+1}^{N} a_{3s'} \dot{\chi}_{s'} + q_{\gamma ji} (\eta_{s,\gamma} + \sum_{s'=s}^{N} b_{s,s'} \eta_{s'}) - q_{3ji} \sum_{s'=s+1}^{N} a_{3s'} \eta_{s'} + q_{\gamma ji}^{*} (\dot{\eta}_{s,\gamma} + \sum_{s'=s}^{N} b_{s,s'} \dot{\eta}_{s'}) - q_{3ji}^{*} \sum_{s'=s+1}^{N} a_{3s'} \dot{\eta}_{s'} + X_{i} = \rho \frac{\partial^{2} u_{ir}}{\partial t^{2}}, i = 1, 3, r = 0, N,
\]

\[
\frac{1}{2} (d_{kla} u_{kr,l}) \cdot \alpha + \frac{1}{2} (d_{kla} u_{lr,k}) \cdot \alpha + \frac{1}{2} (d_{kla} \sum_{s=r}^{N} b_{ks} u_{ls}) \cdot \alpha + \frac{1}{2} (d_{kla}^{*} u_{kr,l}) \cdot \alpha + \frac{1}{2} (d_{kla}^{*} u_{lr,k}) \cdot \alpha
\]
\[ +\frac{1}{2} \left( d_{k\alpha}^{s} \sum_{s=r}^{N} b_{ks} \dot{u}_{ls} \right) , \alpha + \frac{1}{2} \left( d_{k\alpha}^{s} \sum_{s=r}^{N} b_{ks} \dot{u}_{ks} \right) , \alpha + \left( d_{\alpha r}^{s} \dot{\varphi}_{r} \right) , \alpha \]

\[ + \left( \tilde{a}_{\alpha k} \dot{\varphi}_{r,k} \right) , \alpha + \left( \tilde{a}_{\alpha k} \sum_{s=r}^{N} b_{ks} \varphi_{s} \right) , \alpha + \left( \alpha_{\alpha k}^{s} \dot{\varphi}_{r,k} \right) , \alpha + \left( \alpha_{\alpha k}^{s} \sum_{s=r}^{N} b_{ks} \dot{\varphi}_{s} \right) , \alpha \]

\[ + \sum_{s=0}^{r} a_{is} \left[ \frac{1}{2} d_{kli} \left( u_{ks,l} + u_{ls,k} \right) + \frac{1}{2} d_{kli} \sum_{s'=s}^{N} b_{ks'} \dot{u}_{ls'} + b_{ls'} \dot{u}_{ks'} \right] \]

\[ + \frac{1}{2} d_{kli}^{s} \left( u_{ks,l} + u_{ls,k} \right) + \frac{1}{2} d_{kli}^{s} \sum_{s'=s}^{N} \left( b_{ks'} \dot{u}_{ls'} + b_{ls'} \dot{u}_{ks'} \right) + d_{i} \dot{\varphi}_{s} + d_{i} \dot{\varphi}_{s} \]

\[ + \tilde{a}_{ik} \left( \varphi_{s,k} + \sum_{s'=s}^{N} b_{ks'} \dot{\varphi}_{s'} \right) + \alpha_{ik}^{s} \left( \dot{\varphi}_{s,k} + \sum_{s'=s}^{N} b_{ks'} \dot{\varphi}_{s'} \right) \]

\[ - \hat{b}_{ij} \left[ \frac{1}{2} \left( u_{ir,j} + u_{jr,i} \right) + \frac{1}{2} \sum_{s=r}^{N} b_{is} \dot{u}_{js} + \frac{1}{2} \sum_{s=r}^{N} b_{js} \dot{u}_{is} \right] - \frac{1}{2} \sum_{s=r}^{N} d_{i} \left( \varphi_{r,i} + \sum_{s=r}^{N} b_{is} \dot{\varphi}_{s} \right) + \frac{1}{2} \sum_{s=r}^{N} \left( \dot{u}_{ir,j} + \dot{u}_{jr,i} \right) + \frac{1}{2} \sum_{s=r}^{N} b_{is} \dot{u}_{js} + \frac{1}{2} \sum_{s=r}^{N} b_{js} \dot{u}_{is} \]

\[ H = \rho k \frac{\partial^{2} \dot{\varphi}_{s}}{\partial t^{2}} \quad r = 0, N, \]

\[ \frac{1}{2} \left( p_{a kl} u_{kr,l} \right) , \alpha + \frac{1}{2} \left( p_{a kl} \dot{u}_{lr,k} \right) , \alpha + \frac{1}{2} \left( p_{a kl} \sum_{s=r}^{N} b_{ks} \dot{u}_{ls} \right) , \alpha + \frac{1}{2} \left( p_{a kl} \sum_{s=r}^{N} b_{ls} \dot{u}_{ks} \right) , \alpha \]

\[ + \frac{1}{2} \left( p_{a kl}^{s} \dot{u}_{kr,l} \right) , \alpha + \frac{1}{2} \left( p_{a kl}^{s} \dot{u}_{lr,k} \right) , \alpha + \frac{1}{2} \left( p_{a kl}^{s} \sum_{s=r}^{N} b_{ks} \dot{u}_{ls} \right) , \alpha + \frac{1}{2} \left( p_{a kl}^{s} \sum_{s=r}^{N} b_{ls} \dot{u}_{ks} \right) , \alpha \]

\[ - \left( s_{a \gamma} \chi_{r,\gamma} \right) , \alpha - \left( s_{a \gamma} \sum_{s=r}^{N} b_{gs} \chi_{s} \right) , \alpha + \left( s_{a \alpha} \sum_{s=r}^{N} b_{gs} \chi_{s} \right) , \alpha \]

\[ - \left( \tilde{a}_{a \gamma} \eta_{r,\gamma} \right) , \alpha - \left( \tilde{a}_{a \gamma} \sum_{s=r}^{N} b_{gs} \eta_{s} \right) , \alpha + \left( \tilde{a}_{a \alpha} \sum_{s=r}^{N} b_{gs} \eta_{s} \right) , \alpha \]

\[ + \sum_{s=0}^{r} \tilde{a}_{is} \left[ \frac{1}{2} p_{i kl} \left( u_{ks,l} + u_{ls,k} \right) + \frac{1}{2} p_{i kl} \sum_{s'=s}^{N} \left( b_{ks'} \dot{u}_{ls'} + b_{ls'} \dot{u}_{ks'} \right) \right] \]

\[ + \frac{1}{2} \left( p_{i kl}^{s} \dot{u}_{ks,l} + \dot{u}_{ls,k} \right) + \frac{1}{2} p_{i kl}^{s} \sum_{s'=s}^{N} \left( b_{ks'} \dot{u}_{ls'} + b_{ls'} \dot{u}_{ks'} \right) \]

\[ - \xi_{r} \left( \chi_{r,\gamma} + \sum_{s'=s}^{N} b_{gs} \chi_{s} \right) + \xi_{s} \sum_{s'=s+1}^{N} b_{gs} \chi_{s} \]

\[ 28 \]
\[-\ddot{a}_{i\gamma} \left( \eta_{s,\gamma} + \sum_{s'=s}^{N} b_{s's'} \eta_{s'} \right) + \ddot{a}_{i3} \sum_{s'=s+1}^{N} a_{3s'} \eta_{s'} \right] + \dot{D} = f_{er}, \quad r = 0, N, \]

\[\frac{1}{2} \left( q_{\alpha kl} \dot{u}_{kr,l} \right)_{\alpha} + \frac{1}{2} \left( q_{\alpha kl} \dot{u}_{lr,k} \right)_{\alpha} + \left( q_{\alpha kl} \sum_{s=r}^{N} b_{ks} \dot{u}_{ls} \right)_{\alpha} + \frac{1}{2} \left( q_{\alpha kl} \sum_{s=r}^{N} b_{ls} \dot{u}_{ks} \right)_{\alpha} + \frac{1}{2} \left( q_{\alpha kl} \sum_{s=r}^{N} b_{ls} \dot{u}_{ks} \right)_{\alpha} + \frac{1}{2} \left( q_{\alpha kl} \sum_{s=r}^{N} b_{ls} \dot{u}_{ks} \right)_{\alpha} \]

In the $N$th approximation (hierarchical model)

\[\left( u_{r}, \varphi, x, \eta \right)_{r} \left( x_{1}, x_{2}, x_{3}, t \right) \]

\[\cong \sum_{r=0}^{N} a \left( r + \frac{1}{2} \right) \left( \dot{u}_{ir}, \dot{\varphi}_{ir}, \dot{x}_{ir}, \dot{\eta}_{ir} \right)_{r} \left( x_{1}, x_{2}, t \right) P_{r} \left( a x_{3} - b \right), \]

where \( \left( \dot{u}_{ir}, \dot{\varphi}_{ir}, \dot{x}_{ir}, \dot{\eta}_{ir} \right)_{r} \) is a solution of the above system.

**Remark 3.1.** Note that solutions

\[\sum_{i=1}^{3} , \quad \dot{x}_{ir}, \dot{\eta}_{ir} \]

of the governing system of the $N$th approximation are not $r$th order moments in sense of (3.2) any more, in general.
It should be also mentioned that by deriving the governing system of the $N$th approximation we use the following expressions prescribed on the face surfaces of the prismatic shell:

\[
X_{ni} = X_{ji} n_j, \quad i = 1, 3; \\
H_n = (H, n) = H_{jn} n_j; \\
D_n = (D, n) = D_{jn} n_j; \\
B_n = (B, n) = B_{jn} n_j.
\]

In the $N$th order approximation BCs (2.10)-(2.13) on the lateral boundary of the prismatic shell and ICs (2.14) should be written in terms of $r$-the order, $r = 0, N$, moments for functions participating in BCs.

BCs (on the lateral surface of the prismatic shell) and ICs we easily get from BCs (2.10)-(2.13) and ICs (2.14) after multiplying them by $P_r(ax_3 - b)$ and integrating the obtained within the limits $h(x_1, x_2)$ and $h(x_1, x_2)$ with respect to $x_3$, provided the constitutive coefficients and the thickness do not vanish on $\bar{\omega}$:

\[
u_{ir} = f_{ir} \quad \text{on} \quad \gamma_0, \quad X_{jir} n_j = g_{jir} \quad \text{on} \quad \gamma_1 = \partial \omega \backslash \gamma_0, \quad i = 1, 3, \quad (3.23)
\]

\[
\varphi_r = f^{\varphi}_r \quad \text{on} \quad \gamma_0^\varphi, \quad H_{jir} n_j = g^{\varphi}_r \quad \text{on} \quad \gamma_1^\varphi = \partial \omega \backslash \gamma_0^\varphi, \quad i = 1, 3, \quad (3.24)
\]

\[
\chi_r = f^{\chi}_r \quad \text{on} \quad \gamma_0^\chi, \quad D_{jir} n_j = g^{\chi}_r \quad \text{on} \quad \gamma_1^\chi = \partial \omega \backslash \gamma_0^\chi, \quad i = 1, 3, \quad (3.25)
\]

\[
\eta_r = f^{\eta}_r \quad \text{on} \quad \gamma_0^{\eta}, \quad B_{jir} n_j = g^{\eta}_r \quad \text{on} \quad \gamma_1^{\eta} = \partial \omega \backslash \gamma_0^{\eta}, \quad i = 1, 3, \quad (3.26)
\]

and the standard ICs in dynamical problems

\[
\mathbf{u}_r(x, 0) = \mathbf{u}^0_r(x), \quad \dot{\mathbf{u}}_r(x, 0) = \mathbf{u}^1_r(x), \\
\varphi_r(x, 0) = \varphi^0_r(x), \quad \dot{\varphi}_r(x, 0) = \varphi^1_r(x), \quad x \in \Omega;
\]

here $\mathbf{n} := (n_1, n_2)$ is the outward unit normal vector to $\partial \omega$, $(f_{1r}, f_{2r}, f_{3r})$, $f^{\varphi}_r$, $f^{\chi}_r$, $f^{\eta}_r$ are the $r$th order moments of the given displacement vector, volume fraction, electric and magnetic potentials, respectively, $(g_{1r}, g_{2r}, g_{3r})$, $g^{\varphi}_r$, $g^{\chi}_r$ and $g^{\eta}_r$ are the $r$th order moments of the given stress vector, normal components of the equilibrated stress, electric displacement and magnetic induction vectors, respectively, while $\mathbf{u}_r^0$ and $\mathbf{u}_r^1$ are the $r$th order moments of the initial mechanical displacement and velocity vectors, whereas $\varphi_r^0$ and $\varphi_r^1$ are the initial volume fraction distribution and its rate of change. Note that the curves $\gamma_0$, $\gamma_0^\varphi$, $\gamma_0^\chi$, and $\gamma_0^{\eta}$, which represent projections on plane $x_3 = 0$ of the corresponding parts of the lateral boundary of the prismatic shell, in BCs (3.23)-(3.26) are different, in general, from each other and depending on the physical problem some of them may be empty.
Remark 3.2. In case of cusped prismatic shells the setting of BCs is not classical, in general, and is depending on the character of tapering: either the Dirichlet problem should be replaced by the Keldysh problem or BCs should be weighted ones, in general. On the parts of $\partial \omega$, where the thickness is positive (i.e., it does not vanish) the BCs have the form (3.23)-(3.26).

Multiplying equality (3.4) by $h^r$ and, taking into account that $\sum_{s=0}^{r-1} (\cdots) \equiv 0$, $i = 1, 3, \quad r = 0, 1, \cdots$.

(3.27)

$r_{ar} = rh^{-1}h_{\alpha},$

we get

\[
(h^r X_{ar})_{\alpha} + h^r \sum_{s=0}^{r-1} a_{js} X_{jis} + h^r r_{ir} X_i = \rho h^r \frac{\partial^2 h^{r+1} v_{ir}}{\partial t^2},
\]

\[
\sum_{s=0}^{r-1} (\cdots) \equiv 0, \quad i = 1, 3, \quad r = 0, 1, \cdots
\]

(3.27)

Multiplying (3.5) by $h^r$ we obtain

\[
(h^r H_{ar})_{\alpha} + h^r \sum_{s=0}^{r-1} a_{is} H_{is} + h^r H_0 h^r + h^r H = \rho kh^r \frac{\partial^2 h^{r+1} \psi_r}{\partial t^2}.
\]

(3.28)

Similarly, from (3.6) and (3.7) we have

\[
(h^r D_{ar})_{\alpha} + h^r \sum_{s=0}^{r-1} a_{is} D_{is} + h^r \psi_r = h^r f_{er}, \quad r = 0, 1, \cdots
\]

(3.29)

and

\[
(h^r B_{ar})_{\alpha} + h^r \sum_{s=0}^{r-1} a_{is} B_{is} + h^r \psi_r = 0, \quad r = 0, 1, \cdots
\]

(3.30)

Substituting (3.14) into (3.27); (3.16) and (3.18) into (3.28); (3.20) into (3.29); (3.22) into (3.30), respectively, we construct an equivalent infinite system with respect $v_{ir}, \psi_r, \hat{X}_r,$ and $\hat{\eta}_r$:
\begin{align}
&+ \left( b_{\alpha i}^* h^{2r+1} \psi_r \right)_{\alpha} + \left( d_{\alpha i}^* h^{2r+1} \psi_{r,\gamma} \right)_{\alpha} + \left( d_{\alpha i k} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \psi_s \right)_{\alpha} \\
&+ \left( d_{\alpha i r}^* h^{2r+1} \psi_{r,\gamma} \right)_{\alpha} + \left( d_{\alpha i k}^* \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \psi_s \right)_{\alpha} \\
&+ \left( p_{\gamma \alpha i} h^{2r+1} \tilde{\chi}_{r,\gamma} \right)_{\alpha} + \left( p_{k \alpha i} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \tilde{\chi}_s \right)_{\alpha} \\
&+ \left( p_{\gamma \alpha i}^* h^{2r+1} \tilde{\chi}_{r,\gamma} \right)_{\alpha} + \left( p_{k \alpha i}^* \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \tilde{\chi}_s \right)_{\alpha} \\
&+ \left( q_{\gamma \alpha i} h^{2r+1} \tilde{\eta}_{r,\gamma} \right)_{\alpha} + \left( q_{k \alpha i} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \tilde{\eta}_s \right)_{\alpha} \\
&+ \left( q_{\gamma \alpha i}^* h^{2r+1} \tilde{\eta}_{r,\gamma} \right)_{\alpha} + \left( q_{k \alpha i}^* \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \tilde{\eta}_s \right)_{\alpha} \\
&+ \sum_{s=0}^{r} a_{js} \left[ \frac{1}{2} E_{jikl} h^{r+s+1} \left( v_{ks,l} + v_{ts,k} \right) + \frac{1}{2} E_{jikl} \sum_{s'=s+1}^{\infty} h^{r+s'+1} \left( b_{ks'} v_{ls'} + b_{lts'} v_{ks'} \right) \right] \\
&+ \frac{1}{2} E_{jikl} h^{r+s+1} \left( \dot{v}_{ks,l} + \dot{v}_{ts,k} \right) + \frac{1}{2} E_{jikl} \sum_{s'=s+1}^{\infty} h^{r+s'+1} \left( b_{ks'} \dot{v}_{ls'} + b_{lts'} \dot{v}_{ks'} \right) \\
&+ \dot{b}_{ij} h^{r+s+1} \dot{\psi}_s + b_{ij}^* h^{r+s+1} \dot{\psi}_s + d_{ji} h^{r+s+1} \dot{\psi}_{s,\gamma} + d_{jik} \sum_{s'=s+1}^{\infty} b_{ks'} h^{r+s'+1} \dot{\psi}_{s'} \right) \\
&+ \frac{1}{2} d_{ji}^* h^{r+s+1} \dot{\psi}_{s,\gamma} + d_{jik}^* \sum_{s'=s+1}^{\infty} b_{ks'} h^{r+s'+1} \dot{\psi}_{s'} + p_{\gamma ji} h^{r+s+1} \dot{\chi}_{s,\gamma} \right) \\
&+ p_{kji} \sum_{s'=s+1}^{\infty} b_{ks'}^* h^{r+s'+1} \dot{\chi}_{s'} + p_{\gamma ji}^* h^{r+s+1} \dot{\chi}_{s,\gamma} + p_{kji}^* \sum_{s'=s+1}^{\infty} b_{ks'}^* h^{r+s'+1} \dot{\chi}_{s'} \right) \\
&+ q_{\gamma ji} h^{r+s+1} \dot{\eta}_{s,\gamma} + q_{kji} \sum_{s'=s+1}^{\infty} b_{ks'}^* h^{r+s'+1} \dot{\eta}_{s'} + q_{\gamma ji}^* h^{r+s+1} \dot{\eta}_{s,\gamma} \right) \\
&+ q_{kji}^* \sum_{s'=s+1}^{\infty} b_{ks'}^* h^{r+s'+1} \dot{\eta}_{s'} \right] + h^r X_i = \rho h^r \frac{\partial^2 h^{r+1} \psi_i}{\partial t^2}, \quad i = 1, 3, \quad r = 0, 1, \cdots ,
\end{align}

\begin{align}
\frac{1}{2} \left( d_{k \alpha i} h^{2r+1} v_{kr,l} \right)_{\alpha} + \frac{1}{2} \left( d_{k \alpha i} h^{2r+1} v_{l r,k} \right)_{\alpha} + \frac{1}{2} \left( d_{k \alpha i} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} v_{ls} \right)_{\alpha} \\
+ \frac{1}{2} \left( d_{k \alpha i} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} v_{ks} \right)_{\alpha} + \frac{1}{2} \left( d_{k \alpha i}^* h^{2r+1} v_{kr,l} \right)_{\alpha} + \frac{1}{2} \left( d_{k \alpha i}^* h^{2r+1} v_{l r,k} \right)_{\alpha}
\end{align}
\[ + \frac{1}{2} \left( d_{knL} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \dot{v}_{ls} \right)_{,\alpha} + \frac{1}{2} \left( d_{knL} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \dot{v}_{ks} \right)_{,\alpha} \\
+ \left( d_{s} h^{2r+1} \psi_{r} \right)_{,\alpha} + \left( d_{s}^* h^{2r+1} \dot{\psi}_{r} \right)_{,\alpha} + \left( \tilde{\alpha}_{ok} h^{2r+1} \dot{\psi}_{r,k} \right)_{,\alpha} \\
+ \left( \tilde{\alpha}_{ok} \sum_{s=r+1}^{r} b_{ks} h^{r+s+1} \psi_{s} \right)_{,\alpha} + \left( \alpha_{ok} h^{2r+1} \dot{\psi}_{r,k} \right)_{,\alpha} + \left( \alpha_{ok} \sum_{s=r+1}^{r} b_{ks} h^{r+s+1} \dot{\psi}_{s} \right)_{,\alpha} \\
+ \sum_{s=0}^{r} d_{li} \left[ \frac{1}{2} d_{kli} h^{r+s+1} \left( \dot{v}_{ks,l} + \dot{v}_{ls,k} \right) + \frac{1}{2} d_{kli} \sum_{s'=s+1}^{\infty} h^{r+s'+1} \left( b_{ks'} v_{ls'} + b_{ls'} v_{ks'} \right) \right] \\
+ \frac{1}{2} d_{kli} h^{r+s+1} \left( \dot{v}_{ks,l} + \dot{v}_{ls,k} \right) + \frac{1}{2} d_{kli} \sum_{s'=s+1}^{\infty} h^{r+s'+1} \left( b_{ks'} v_{ls'} + b_{ls'} v_{ks'} \right) \\
+ d_{i} h^{r+s+1} \dot{\psi}_{s} + d_{i}^* h^{r+s+1} \dot{\psi}_{s} + \tilde{\alpha}_{ik} \left( h^{r+s+1} \dot{\psi}_{s,k} + \sum_{s'=s+1}^{\infty} b_{ks'} h^{r+s'+1} \dot{\psi}_{s'} \right) \\
+ \alpha_{ik} \left( h^{r+s+1} \dot{\psi}_{s,k} + \sum_{s'=s+1}^{\infty} b_{ks'} h^{r+s'+1} \dot{\psi}_{s'} \right) \\
\]
\[ - \tilde{b}_{lj} \left[ \frac{1}{2} h^{2r+1} \left( \dot{v}_{ir,j} + \dot{v}_{jr,i} \right) + \frac{1}{2} \sum_{s=r+1}^{\infty} b_{ls} h^{r+s+1} \dot{v}_{js} + \frac{1}{2} \sum_{s=r+1}^{\infty} b_{js} h^{r+s+1} \dot{v}_{ls} \right] \\
- \xi \dot{h}^{2r+1} \psi_{r} - d_{i} \left( h^{2r+1} \dot{\psi}_{r,i} + \sum_{s=r+1}^{\infty} b_{ls} h^{r+s+1} \dot{\psi}_{s} \right) \\
- \delta_{lj} \left[ \frac{1}{2} h^{2r+1} \left( \dot{v}_{ir,j} + \dot{v}_{jr,i} \right) + \frac{1}{2} \sum_{s=r+1}^{\infty} b_{ls} h^{r+s+1} \dot{v}_{js} + \frac{1}{2} \sum_{s=r+1}^{\infty} b_{js} h^{r+s+1} \dot{v}_{ls} \right] \\
- \xi^* h^{2r+1} \psi_{r} - d_{i}^* \left( h^{2r+1} \dot{\psi}_{r,i} + \sum_{s=r+1}^{\infty} b_{ls} h^{r+s+1} \dot{\psi}_{s} \right) + h^r H = \rho k h^r \frac{\partial^2 h^{r+1} \psi_r}{\partial t^2}, \]
\[ r = 0, 1, \ldots, \]
\[ \frac{1}{2} \left( p_{okl} h^{2r+1} v_{kr,l} \right)_{,\alpha} + \frac{1}{2} \left( p_{okl} h^{2r+1} v_{lr,k} \right)_{,\alpha} + \frac{1}{2} \left( p_{okl} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} v_{ls} \right)_{,\alpha} \\
+ \frac{1}{2} \left( p_{okl} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} v_{ks} \right)_{,\alpha} + \frac{1}{2} \left( p_{okl}^* h^{2r+1} v_{kr,l} \right)_{,\alpha} + \frac{1}{2} \left( p_{okl}^* h^{2r+1} v_{lr,k} \right)_{,\alpha} \\
+ \frac{1}{2} \left( p_{okl}^* \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} v_{ls} \right)_{,\alpha} + \frac{1}{2} \left( p_{okl} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} v_{ks} \right)_{,\alpha} \\
- \left( \xi_{\alpha} h^{2r+1} \tilde{\chi}_{r,\gamma} \right)_{,\alpha} - \left( \xi_{\alpha} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \tilde{\chi}_{s} \right)_{,\alpha} \]
\[
- \left( \tilde{a}_{\alpha r} \gamma_h^{2r+1} \tilde{\eta}_{r,\gamma} \right)_{\alpha} - \left( \tilde{a}_{\alpha k} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \tilde{\eta}_s \right)_{\alpha} \\
+ \sum_{s=0}^{r} a_{ls} \left[ \frac{1}{2} p_{ikl}(v_{kl} + v_{ls}) + \frac{1}{2} \sum_{s'=s+1}^{\infty} h^{r+s'} (b_{ks'} v_{ls'} + \tilde{b}_{ks'} v_{ks'}) \right] + \frac{1}{2} \sum_{s'=s+1}^{\infty} h^{r+s'} (b_{ks'} v_{ls'} + \tilde{b}_{ks'} v_{ks'}) \\
- \sum_{s'=s+1}^{\infty} b_{ks'} h^{r+s'+1} \tilde{\eta}_{s'} \\
- \tilde{a}_{r+1} - \tilde{a}_{ik} \sum_{s'=s+1}^{\infty} b_{ks'} h^{r+s'+1} \tilde{\eta}_{s'} + h^r D = h^r f_{er}, \quad r = 0, 1, \ldots
\]

\[
\frac{1}{2} \left( q_{abk} h^{2r+1} v_{kr,l} \right)_{\alpha} + \frac{1}{2} \left( q_{abk} h^{2r+1} v_{lr,k} \right)_{\alpha} + \frac{1}{2} \left( q_{abk} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} v_{ls} \right)_{\alpha} \\
+ \frac{1}{2} \left( q_{abk} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} v_{ks} \right)_{\alpha} \\
- \left( \tilde{a}_{\alpha r} \gamma_h^{2r+1} \tilde{\eta}_{r,\gamma} \right)_{\alpha} - \left( \tilde{a}_{\alpha k} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \tilde{\eta}_s \right)_{\alpha} \\
- \left( \tilde{a}_{\alpha r} \gamma_h^{2r+1} \tilde{\eta}_{r,\gamma} \right)_{\alpha} - \left( \tilde{a}_{\alpha k} \sum_{s=r+1}^{\infty} b_{ks} h^{r+s+1} \tilde{\eta}_s \right)_{\alpha} \\
+ \sum_{s=0}^{r} a_{ls} \left[ \frac{1}{2} q_{ikl}(v_{kl} + v_{ls}) + \frac{1}{2} \sum_{s'=s+1}^{\infty} h^{r+s'} (b_{ks'} v_{ls'} + \tilde{b}_{ks'} v_{ks'}) \right] + \frac{1}{2} \sum_{s'=s+1}^{\infty} h^{r+s'} (b_{ks'} v_{ls'} + \tilde{b}_{ks'} v_{ks'}) \\
- \tilde{a}_{r+1} - \tilde{a}_{ik} \sum_{s'=s+1}^{\infty} b_{ks'} h^{r+s'+1} \tilde{\eta}_{s'} \\
- \tilde{a}_{r+1} - \tilde{a}_{ik} \sum_{s'=s+1}^{\infty} b_{ks'} h^{r+s'+1} \tilde{\eta}_{s'} + h^r B = 0, \quad r = 0, 1, \ldots
\]

Assuming

\[v_{ir} = 0, \quad i = 1, 3; \quad \psi_r = 0, \quad \tilde{\chi}_r = 0, \quad \tilde{\eta}_r = 0 \text{ for } r > N, \quad (3.31)\]
we obtain governing system consisting of $6N + 6$ equations of the $N$th approximation with respect to (see Remark 3.1)

\[
\tilde{v}_{ir}, \quad i = 1, 3; \quad N \psi_r, \quad \tilde{X}_r, \quad \tilde{\eta}_r, \quad r = 0, N,
\]

provided we consider the first $N + 1$ equations for $r = 0, N$ for each $i = 1, 2, 3$, $\psi_r$, $\tilde{X}_r$, $\tilde{\eta}_r$.

Namely (below superscript $N$ on unknown functions is omitted),

\[
\frac{1}{2}(E_{aikd}h^{2r+1}v_{kr,\delta})_a + \frac{1}{2}(E_{airh}h^{2r+1}v_{ir,\gamma})_a + \frac{1}{2}(E_{aikl} \sum_{s=r+1}^N b_{ks} h^{r+s+1}v_{ls})_a
\]

\[
+ \frac{1}{2}(E_{aikl} \sum_{s=r+1}^N b_{ks} h^{r+s+1}v_{ks})_a + \frac{1}{2}(E_{aikl} \sum_{s=r+1}^N b_{ks} h^{r+s+1}v_{ls})_a + \frac{1}{2}(E_{airh}h^{2r+1}v_{ir,\gamma})_a
\]

\[
+ \frac{1}{2}(E_{aikl} \sum_{s=r+1}^N b_{ks} h^{r+s+1}v_{ls})_a + \frac{1}{2}(E_{aikl} \sum_{s=r+1}^N b_{ks} h^{r+s+1}v_{ls})_a + \frac{1}{2}(E_{airh}h^{2r+1}v_{ir,\gamma})_a
\]
\[\begin{align*}
&+d_{ij}\gamma h^{r+s+1}\psi_{s,\gamma} + d_{ijk}^s \sum_{s'=s+1}^N b_{ks} h^{r+s'+1}\psi_{s'} + p_{\gamma ji} h^{r+s+1}\tilde{\chi}_{s,\gamma} \\
&+p_{kji} \sum_{s'=s+1}^N b_{ks} h^{r+s'+1}\tilde{\chi}_{s'} + p_{\gamma ji} h^{r+s+1}\tilde{\chi}_{s,\gamma} + p_{kji}^s \sum_{s'=s+1}^N b_{ks} h^{r+s'+1}\tilde{\chi}_{s'} \\
&+q_{\gamma ji} h^{r+s+1}\eta_{s,\gamma} + q_{kji} \sum_{s'=s+1}^N b_{ks} h^{r+s'+1}\eta_{s'} + q_{\gamma ji} h^{r+s+1}\tilde{\psi}_{s,\gamma} \\
&+q_{kji}^s \sum_{s'=s+1}^N b_{ks} h^{r+s'+1}\tilde{\psi}_{s'} + h^r X_i = \rho h^r \frac{\partial^2 h^{r+1} v_{ir}}{\partial t^2}, \quad (3.32)
\end{align*}\]
\begin{align*}
&\xi h^{2r+1}\psi_r - d_i\left(h^{2r+1}\psi_{r,i} + \sum_{s=r+1}^{N} b_{is} h^{r+s+1}\psi_{s}\right) \\
&- b_{ij}^* \left[\frac{1}{2} h^{2r+1} \left(\dot{v}_{ir,j} + \dot{v}_{jr,i}\right) + \frac{1}{2} \sum_{s=r+1}^{N} b_{is} h^{r+s+1}\dot{v}_{js} + \frac{1}{2} \sum_{s=r+1}^{N} b_{js} h^{r+s+1}\dot{v}_{is}\right] \\
&- \xi^* h^{2r+1}\psi_r - d_i^* \left(h^{2r+1}\psi_{r,i} + \sum_{s=r+1}^{N} b_{is} h^{r+s+1}\psi_{s}\right) \\
&+ h^r H = \rho k h^{r+1} \psi_r - d_i, \quad r = 0, N, \quad (3.33)
\end{align*}

\begin{align*}
&\frac{1}{2} \left(p_{akl} h^{2r+1}\dot{v}_{kr,l}\right)_{\alpha} + \frac{1}{2} \left(p_{akl} h^{2r+1}\dot{v}_{lr,k}\right)_{\alpha} + \frac{1}{2} \left(p_{akl} \sum_{s=r+1}^{N} b_{ks} h^{r+s+1}\dot{v}_{ls}\right)_{\alpha} \\
&+ \frac{1}{2} \left(p_{akl} \sum_{s=r+1}^{N} b_{ks} h^{r+s+1}\dot{v}_{ks}\right)_{\alpha} + \frac{1}{2} \left(p_{akl} \sum_{s=r+1}^{N} b_{ls} h^{r+s+1}\dot{v}_{ks}\right)_{\alpha} \\
&- \left(\omega_{\gamma} h^{2r+1}\dot{\chi}_{r,\gamma}\right)_{\alpha} - \left(\omega_{\alpha} \sum_{s=r+1}^{N} b_{ks} h^{r+s+1}\dot{\chi}_{s}\right)_{\alpha} \\
&- \left(\tilde{\alpha}_{\alpha} h^{2r+1}\dot{\tilde{\eta}}_{r,\gamma}\right)_{\alpha} - \left(\tilde{\alpha}_{\alpha} \sum_{s=r+1}^{N} b_{ks} h^{r+s+1}\dot{\tilde{\eta}}_{s}\right)_{\alpha} \\
&+ \sum_{s=0}^{r} \frac{r}{2} h^{r+s+1} p_{ikl} (v_{ks,l} + v_{ls,k}) + \frac{1}{2} p_{ikl} \sum_{s'=s+1}^{N} h^{r+s'+1}(b_{ks'}\dot{v}_{ls'} + b_{ks'}\dot{v}_{ks'}) \\
&+ \frac{1}{2} p_{ikl} h^{r+s+1}(v_{ks,l} + v_{ls,k}) + \frac{1}{2} p_{ikl} \sum_{s'=s+1}^{N} h^{r+s'+1}(b_{ks'}\dot{v}_{ls'} + b_{ks'}\dot{v}_{ks'}) \\
&- \varsigma_{\gamma} h^{r+s+1}\dot{\chi}_{s,\gamma} - \varsigma_{ik} \sum_{s'=s+1}^{N} b_{ks'} h^{r+s'+1}\dot{\chi}_{s'} \\
&+ \tilde{\alpha}_{\gamma} h^{r+s+1}\dot{\tilde{\eta}}_{s,\gamma} + \tilde{\alpha}_{ik} \sum_{s'=s+1}^{N} b_{ks'} h^{r+s'+1}\dot{\tilde{\eta}}_{s'} + h^r D = h^r f_{er}, \quad (3.34)
\end{align*}
\[
\frac{1}{2}(q_{\alpha kl} h^{2r+1} v_{kr,l})_{\alpha} + \frac{1}{2}(q_{\alpha kl} h^{2r+1} v_{lr,k})_{\alpha} + \frac{1}{2}(q_{\alpha kl} \sum_{s=r+1}^{N} b_{ks} h^{r+s+1} v_{ls})_{\alpha} \\
+ \frac{1}{2}(q_{\alpha kl} \sum_{s=r+1}^{N} b_{ls} h^{r+s+1} v_{ks})_{\alpha} + \frac{1}{2}(q_{\alpha kl} h^{2r+1} v_{kr,l})_{\alpha} + \frac{1}{2}(q_{\alpha kl} h^{2r+1} v_{lr,k})_{\alpha} \\
+ \frac{1}{2}(q_{\alpha kl} \sum_{s=r+1}^{N} b_{ks} h^{r+s+1} \dot{v}_{ls})_{\alpha} + \frac{1}{2}(q_{\alpha kl} \sum_{s=r+1}^{N} b_{ls} h^{r+s+1} \dot{v}_{ks})_{\alpha} \\
- (\tilde{a}_{\alpha}, h^{2r+1} \tilde{x}_{r,\gamma})_{\alpha} - (\tilde{a}_{\alpha k} \sum_{s=r+1}^{N} b_{ks} h^{r+s+1} \tilde{x}_{s})_{\alpha} \\
- (\xi_{\alpha k} h^{2r+1} \tilde{\eta}_{r,\gamma})_{\alpha} - (\xi_{\alpha k} \sum_{s=r+1}^{N} b_{ks} h^{r+s+1} \tilde{\eta}_{s})_{\alpha} \\
+ \sum_{s=0}^{r} a_{ls} \left[ \frac{1}{2} q_{\alpha kl} h^{r+s+1} (v_{ks,l} + v_{ls,k}) + \frac{1}{2} q_{ikl} \sum_{s'=s+1}^{N} h^{r+s'+1} (b_{ks} v_{ls'} + b_{ls'} v_{ks'}) \right] \\
+ \sum_{s'=s+1}^{r} a_{ks} \left( \frac{1}{2} q_{\alpha kl} h^{r+s+1} (\dot{v}_{ks,l} + \dot{v}_{ls,k}) + \frac{1}{2} q_{ikl} \sum_{s'=s+1}^{N} h^{r+s'+1} (b_{ks} \dot{v}_{ls'} + b_{ls'} \dot{v}_{ks'}) \right) \\
- \tilde{a}_{lr} h^{r+s+1} \tilde{x}_{s,\gamma} - \tilde{a}_{ik} \sum_{s'=s+1}^{s} b_{ks} h^{r+s'+1} \tilde{x}_{s'} \\
- \xi_{\alpha k} h^{r+s+1} \tilde{\eta}_{s,\gamma} - \xi_{ik} \sum_{s'=s+1}^{s} b_{ks} h^{r+s'+1} \tilde{\eta}_{s'} + h^{r} B = 0, \quad (3.35)
\]

In the $N$th approximation (hierarchical model):

\[
(u_{t}, \varphi, \chi, \eta)(x_{1}, x_{2}, x_{3}, t) = \sum_{r=0}^{N} \left[ r + \frac{1}{2} \right] h^{r} (v_{lr}, \psi_{r}, \tilde{x}_{r}, \tilde{\eta}_{r}) (x_{1}, x_{2}, x_{3}, t) P_{r}(ax_{3} - b),
\]

where \( (v_{lr}, \psi_{r}, \tilde{x}_{r}, \tilde{\eta}_{r}) \) is a solution of the above system (3.32)-(3.35);
\[ X_{ijr} = \frac{1}{2} E_{ijkl} h^{r+1} (v_{kr,l} + v_{lr,k}) + \frac{1}{2} E_{ijkl} \sum_{s=r+1}^{N} h^{s+1} (b_{ks} v_{ls} + b_{ls} v_{ks}) \]

\[ + \frac{1}{2} E_{ijkl} h^{r+1} (\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} E_{ijkl} \sum_{s=r+1}^{N} h^{s+1} (\dot{b}_{ks} \dot{v}_{ls} + \dot{b}_{ls} \dot{v}_{ks}) \]

\[ + \bar{b}_{ij} h^{r+1} \psi_r + b_{ij} h^{r+1} \dot{\psi}_r + d_{ijr} h^{r+1} \psi_{r,\gamma} + d_{ijk} \sum_{s=r+1}^{N} h^{s+1} b_{ks} \psi_s + d_{ijr}^{*} h^{r+1} \dot{\psi}_{r,\gamma} \]

\[ + d_{ijk} \sum_{s=r+1}^{N} h^{s+1} b_{ks} \dot{\psi}_s + p_{\gamma ij} h^{r+1} \chi_{r,\gamma} + p_{ki} \sum_{s=r+1}^{N} h^{s+1} b_{ks} \tilde{\chi}_s \]

\[ + p_{\gamma ij} h^{r+1} \chi_{r,\gamma} + p_{ki} \sum_{s=r+1}^{N} h^{s+1} b_{ks} \tilde{\chi}_s, \quad \gamma, i, j = \Gamma, 3, \quad r = 0, N; \]

\[ H_{jr} = \frac{1}{2} d_{klj} h^{r+1} (v_{kr,l} + v_{lr,k}) + \frac{1}{2} d_{klj} \sum_{s=r+1}^{N} h^{s+1} (b_{ks} v_{hs} + b_{ls} v_{ks}) \]

\[ + \frac{1}{2} d_{klj} h^{r+1} (\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} d_{klj} \sum_{s=r+1}^{N} h^{s+1} (\dot{b}_{ks} \dot{v}_{hs} + \dot{b}_{ls} \dot{v}_{ks}) \]

\[ + d_{j} h^{r+1} \psi_r + d_{j}^{*} h^{r+1} \dot{\psi}_{r} \]

\[ + \tilde{\alpha}_{ji} (h^{r+1} \psi_{r,i} + \sum_{s=r+1}^{N} h^{s+1} b_{is} \psi_s) + \alpha_{ji} (h^{r+1} \dot{\psi}_{r,i} + \sum_{s=r+1}^{N} h^{s+1} b_{is} \dot{\psi}_s), \quad j = \Gamma, 3, \quad r = 0, N; \]

\[ H_{0r} = -\tilde{b}_{ij} \left[ \frac{1}{2} h^{r+1} (\psi_{r,i} + \psi_{r,j}) + \frac{1}{2} \sum_{s=r+1}^{N} h^{s+1} (b_{is} \psi_{js} + b_{js} \psi_{is}) \right] \]

\[ - \tilde{\xi} h^{r+1} \psi_r - d_{i} (h^{r+1} \psi_{r,i} + \sum_{s=r+1}^{N} \tilde{b}_{is} \psi_s) \]

\[ - b_{ij} \left[ \frac{1}{2} h^{r+1} (\dot{\psi}_{r,j} + \dot{\psi}_{r,i}) + \frac{1}{2} \sum_{s=r+1}^{N} h^{s+1} (\dot{b}_{is} \dot{\psi}_{js} + \dot{b}_{js} \dot{\psi}_{is}) \right] \]

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\[-\xi^* h^{r+1} \dot{\psi}_r - \xi^* \left( h^{r+1} \dot{\psi}_{r,i} + \sum_{s=r+1}^{N} b_{is} \dot{\psi}_s \right), \quad r = 0, N;\]

\[D_{jr} = \frac{1}{2} p_{jkl} h^{r+1} (v_{kr,l} + v_{lr,k}) + \frac{1}{2} p_{jkl} \sum_{s=r+1}^{N} h^{s+1} (b_{ks} v_{ls} + b_{ls} v_{ks})\]

\[+ \frac{1}{2} p^*_{jkl} h^{r+1} (\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} p^*_{jkl} \sum_{s=r+1}^{N} h^{s+1} (b_{ks} \dot{v}_{ls} + b_{ls} \dot{v}_{ks})\]

\[-\varsigma_{jr} h^{r+1} \chi_{r,\gamma} - \varsigma_{jk} \sum_{s=r+1}^{N} h^{s+1} b_{ks} \chi_s\]

\[-\tilde{\alpha}_{jr} h^{r+1} \eta_{r,\gamma} - \tilde{\alpha}_{jk} \sum_{s=r+1}^{N} h^{s+1} b_{ks} \eta_s,\]

\[j = 1, 3, \quad r = 0, N;\]

\[B_{jr} = \frac{1}{2} q_{jkl} h^{r+1} (v_{kr,l} + v_{lr,k}) + \frac{1}{2} q_{jkl} \sum_{s=r+1}^{N} h^{s+1} (b_{ks} v_{ls} + b_{ls} v_{ks})\]

\[+ \frac{1}{2} q^*_{jkl} h^{r+1} (\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} q^*_{jkl} \sum_{s=r+1}^{N} h^{s+1} (b_{ks} \dot{v}_{ls} + b_{ls} \dot{v}_{ks})\]

\[-\tilde{\alpha}_{jr} h^{r+1} \chi_{r,\gamma} - \tilde{\alpha}_{jk} \sum_{s=r+1}^{N} h^{s+1} b_{ks} \chi_s\]

\[-\xi_{jr} h^{r+1} \eta_{r,\gamma} - \xi_{jk} \sum_{s=r+1}^{N} h^{s+1} b_{ks} \eta_s,\]

\[j = 1, 3, \quad r = 0, N;\]

\[e_{ijr} = \frac{1}{2} h^{r+1} (v_{ir,j} + v_{jr,i}) + \frac{1}{2} \sum_{s=r+1}^{N} h^{s+1} (b_{is} v_{js} + b_{js} v_{is}),\]

\[i, j = 1, 3, \quad r = 0, N;\]

\[X_{njr} = X_{ijr} n_i\]

\[= \left\{ \begin{array}{l}
\frac{1}{2} E_{ijkl} h^{r+1} (v_{kr,l} + v_{lr,k}) + \frac{1}{2} E_{ijkl} \sum_{s=r+1}^{N} h^{s+1} (b_{ks} v_{ls} + b_{ls} v_{ks}) \\
+ \frac{1}{2} E^*_{ijkl} h^{r+1} (\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} E^*_{ijkl} \sum_{s=r+1}^{N} h^{s+1} (b_{ks} \dot{v}_{ls} + b_{ls} \dot{v}_{ks})
\end{array} \right\} \]
\[ +\dot{h}_{ij} h^{r+1} \psi_r + \dot{b}_{ij} h^{r+1} \psi_r + \dot{d}_{ij} h^{r+1} \psi_{r,\gamma} + \dot{d}_{ijk} \sum_{s=r+1}^N h^{s+1} b_{ks} \psi_s + d_{ijk}^* h^{r+1} \psi_{r,\gamma} \\
+ \dot{d}_{ijk}^* \sum_{s=r+1}^N h^{s+1} b_{ks} \dot{\psi}_s + p_{\gamma ij} h^{r+1} \dot{\chi}_{r,\gamma} + p_{kj} \sum_{s=r+1}^N h^{s+1} b_{ks} \dot{\chi}_s \\
+ p_{\gamma ij}^* h^{r+1} \dot{\tilde{\chi}}_{r,\gamma} + p_{kj}^* \sum_{s=r+1}^N h^{s+1} b_{ks} \dot{\tilde{\chi}}_s + q_{\gamma ij} h^{r+1} \dot{\eta}_{r,\gamma} + p_{kj} \sum_{s=r+1}^N h^{s+1} b_{ks} \dot{\eta}_s \\
+ q_{\gamma ij}^* h^{r+1} \dot{\tilde{\eta}}_{r,\gamma} + q_{kj}^* \sum_{s=r+1}^N h^{s+1} b_{ks} \dot{\tilde{\eta}}_s \right) n_j, \quad j = 1, 3, \quad r = 0, N; \\
\]

\[ H_{nr} := H_{jr} n_j = \left\{ \frac{1}{2} d_{klj} h^{r+1} (v_{kr,l} + v_{lr,k}) + \frac{1}{2} d_{ldj} \sum_{s=r+1}^N h^{s+1} (b_{ks} v_{hs} + b_{ls} v_{ks}) \\
+ \frac{1}{2} d_{klj}^* h^{r+1} (\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} d_{ldj}^* \sum_{s=r+1}^N h^{s+1} (b_{ks} \dot{v}_{hs} + b_{ls} \dot{v}_{ks}) \\
+ d_{lj} h^{r+1} \dot{\psi}_r + d_{lj}^* h^{r+1} \dot{\psi}_r \\
+ \tilde{\alpha}_{ji} \left( h^{r+1} \dot{\psi}_{r,i} + \sum_{s=r+1}^N h^{s+1} b_{is} \dot{\psi}_s \right) + \alpha_{ji}^* \left( h^{r+1} \dot{\psi}_{r,i} + \sum_{s=r+1}^N h^{s+1} b_{is} \dot{\psi}_s \right) \right\} n_j, \quad r = 0, N; \\
\]

\[ D_{nr} := D_{jr} n_j = \left\{ \frac{1}{2} p_{kjl} h^{r+1} (v_{kr,l} + v_{lr,k}) + \frac{1}{2} p_{ljk} \sum_{s=r+1}^N h^{s+1} (b_{ks} v_{ls} + b_{ls} v_{ks}) \\
+ \frac{1}{2} p_{kjl}^* h^{r+1} (\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} p_{ljk}^* \sum_{s=r+1}^N h^{s+1} (b_{ks} \dot{v}_{ls} + b_{ls} \dot{v}_{ks}) \\
- \varsigma_{ji} h^{r+1} \dot{\chi}_{r,\gamma} - \varsigma_{jk} \sum_{s=r+1}^N h^{s+1} b_{ks} \dot{\chi}_s \\
- \tilde{\alpha}_{ji} h^{r+1} \dot{\eta}_{r,\gamma} - \tilde{\alpha}_{jk} \sum_{s=r+1}^N h^{s+1} b_{ks} \dot{\eta}_s \right\} n_j, \quad r = 0, N; \\
\]
\[B_{nr} := B_{jr} n_j = \left\{ \frac{1}{2} q_{jkl} h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2} q^*_{jkl} N \sum_{s=r+1}^N h^{s+1}(b_{ks} v_{ls} + b_{ls} v_{ks}) \right. \\
+ \frac{1}{2} q^*_{jkl} h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} q^*_{jkl} N \sum_{s=r+1}^N h^{s+1}(b_{ks}\dot{v}_{ls} + b_{ls}\dot{v}_{ks}) \\
- \tilde{a}_{jr} h^{r+1}\tilde{\chi}_{r,\gamma} - \tilde{a}_{jk} \sum_{s=r+1}^N h^{s+1}b_{ks}\tilde{\chi}_s \\
- \tilde{\xi}_{jr} h^{r+1}\tilde{\eta}_{r,\gamma} - \xi_{jk} \sum_{s=r+1}^N h^{s+1}b_{ks}\tilde{\eta}_s \right\} n_j, \quad r = 0, N.
\]

From (3.23)-(3.26) and from the ICs following them we easily obtain BCs and ICs in terms of \(v_{ir}, \psi_r, \tilde{\chi}_r, \tilde{\eta}_r\), provided prismatic shells are not cusped ones (concerning BCs for cusped prismatic shells see Remark 3.2).

### 4 \(N = 0\) Approximation

For the sake of simplicity we rewrite the equations (3.27), (3.28), (3.29), (3.30); (3.14), (3.16), (3.18), (3.20), (3.22), (3.9) for the \(N = 0\) approximation and derive the governing equations of the \(N = 0\) approximation. We first have

\[X_{\alpha j0,\alpha} + X_j = \rho \frac{\partial^2 h v_{j0}}{\partial t^2}, \quad j = 1, 3, \quad (4.1)\]

\[H_{\alpha 0,\alpha} + H_{00} + H = \rho k \frac{\partial^2 h \psi_0}{\partial t^2}, \quad (4.2)\]

\[D_{\alpha 0,\alpha} + D = f_{e0}, \quad (4.3)\]

\[B_{\alpha 0,\alpha} + B = 0, \quad (4.4)\]

\[v_{j0} = \frac{u_{j0}}{h}, \quad \psi_0 = \frac{\varphi_0}{h}; \quad (4.5)\]

\[X_{ij0} = \frac{1}{2} E_{ijkl} h (v_{k0,l} + v_{l0,k}) + \frac{1}{2} E^*_{ijkl} h (\dot{v}_{k0,l} + \dot{v}_{l0,k}) + \tilde{b}_{ij} h \psi_0 + b^*_{ij} h \dot{\psi}_0 + d_{ij,\gamma} h \psi_{0,\gamma} + d^*_{ij,\gamma} h \dot{\psi}_{0,\gamma} + p_{\gamma ij} h \tilde{\chi}_{r,\gamma} + p^*_{\gamma ij} h \tilde{\psi}_{0,\gamma} + q_{\gamma ij} h \tilde{\eta}_{0,\gamma} + q^*_{\gamma ij} h \dot{\tilde{\eta}}_{0,\gamma}, \quad i, j = 1, 3, \quad (4.5)\]

\[\tilde{\chi}_0 := \frac{\chi_0}{h}, \quad \tilde{\eta}_0 := \frac{\eta_0}{h}; \quad (42)\]
\[
H_{j0} = \frac{1}{2} d_{ki} h(v_{k0,i} + v_{l0,k}) + \frac{1}{2} d_{ki}^* h(\dot{v}_{k0,i} + \dot{v}_{l0,k}) + d_j h\psi_0 + d_j^* h\dot{\psi}_0
\]
\[
+ \alpha_j h\psi_{0,i} + \alpha_j^* h\dot{\psi}_{0,i}, \quad j = \overline{1,3}, \quad (4.6)
\]
\[
H_{00} = -\frac{1}{2} b_{ij} h(v_{i0,j} + v_{j0,i}) - \xi h\psi_0 - d_i h\psi_{0,i} - \frac{1}{2} b_{ij}^* h(\dot{v}_{i0,j} + \dot{v}_{j0,i})
\]
\[
- \xi^* h\dot{\psi}_0 - d_i^* h\dot{\psi}_{0,i}; \quad (4.7)
\]
\[
D_{j0} = \frac{1}{2} p_{jk} h(v_{k0,l} + v_{l0,k}) + \frac{1}{2} p_{jk}^* h(\dot{v}_{k0,l} + \dot{v}_{l0,k}) + \zeta_j h\tilde{\chi}_{0,\gamma}
\]
\[
+ \tilde{a}_j h\tilde{\eta}_{0,\gamma}, \quad j = \overline{1,3}; \quad (4.8)
\]
\[
B_{j0} = \frac{1}{2} q_{jk} h(v_{k0,l} + v_{l0,k}) + \frac{1}{2} q_{jk}^* h(\dot{v}_{k0,l} + \dot{v}_{l0,k}) + \tilde{a}_j h\tilde{\chi}_{0,\gamma}
\]
\[
+ \xi_j h\tilde{\eta}_{0,\gamma}, \quad j = \overline{1,3}; \quad (4.9)
\]
\[
e_{ij0} = \frac{1}{2} h(v_{i0,j} + v_{j0,i}), \quad i, j = \overline{1,3}, \quad (4.10)
\]
respectively.

Now substituting (4.5)-(4.9) into (4.1)-(4.4), respectively, for the \( N = 0 \) approximation we obtain the following governing system of equations with respect to \( v_{i0}, \psi_0, \tilde{\chi}_0, \tilde{\eta}_0 \) in the following form:

\[
\frac{1}{2} \left( E_{\alpha i k} h v_{k0,\delta} \right)_{,\alpha} + \frac{1}{2} \left( E_{\alpha i \gamma} h v_{\gamma 0,\gamma} \right)_{,\alpha} + \frac{1}{2} \left( E_{\alpha k}^* h \dot{v}_{k0,\delta} \right)_{,\alpha} + \frac{1}{2} \left( E_{\alpha \gamma}^* h \dot{v}_{\gamma 0,\gamma} \right)_{,\alpha}
\]
\[
+ \left( b_{\alpha \gamma} h\psi_0 \right)_{,\alpha} + \left( h_{\alpha \gamma}^* h\dot{\psi}_0 \right)_{,\alpha} + \left( d_{\alpha i \gamma} h\psi_{0,\gamma} \right)_{,\alpha} + \left( d_{\alpha i}^* h\dot{\psi}_{0,\gamma} \right)_{,\alpha} + \left( p_{\gamma \alpha} h\tilde{\chi}_{0,\gamma} \right)_{,\alpha}
\]
\[
+ \left( p_{\gamma \alpha}^* h\tilde{\eta}_{0,\gamma} \right)_{,\alpha} + \left( q_{\gamma \alpha} h\tilde{\chi}_{0,\gamma} \right)_{,\alpha} + \left( q_{\gamma \alpha}^* h\tilde{\eta}_{0,\gamma} \right)_{,\alpha} + X_i = \rho \frac{\partial^2 h v_{i0}}{\partial t^2}, \quad i = \overline{1,3}, \quad (4.11)
\]
i.e.

\[
\frac{1}{2} \left( E_{\alpha i k} h v_{k0,\delta} \right)_{,\alpha} + \frac{1}{2} \left( E_{\alpha i \gamma} h v_{\gamma 0,\gamma} \right)_{,\alpha} + \frac{1}{2} \left( E_{\alpha k}^* h \dot{v}_{k0,\delta} \right)_{,\alpha} + \frac{1}{2} \left( E_{\alpha \gamma}^* h \dot{v}_{\gamma 0,\gamma} \right)_{,\alpha}
\]
\[
+ b_{\alpha} h\psi_{0,\alpha} + \left( h_{\alpha}^* h\dot{\psi}_0 \right)_{,\alpha} + b_{\alpha}^* h\psi_{0,\alpha} + b_{\alpha}^* h \psi_0
\]
\[
+ \left( d_{\alpha i} h\psi_{0,\gamma} \right)_{,\alpha} + \left( d_{\alpha i}^* h\dot{\psi}_{0,\gamma} \right)_{,\alpha} + \left( p_{\gamma \alpha}^* h\tilde{\chi}_{0,\gamma} \right)_{,\alpha}
\]
\[
+ \left( p_{\gamma \alpha}^* h\tilde{\eta}_{0,\gamma} \right)_{,\alpha} + \left( q_{\gamma \alpha} h\tilde{\chi}_{0,\gamma} \right)_{,\alpha} + \left( q_{\gamma \alpha}^* h\tilde{\eta}_{0,\gamma} \right)_{,\alpha} + X_i = \rho \frac{\partial^2 h v_{i0}}{\partial t^2}, \quad i = \overline{1,3}, \quad (4.11)
\]
1 \frac{1}{2}(d_{\alpha\delta}h_{\frac{\nu}{\kappa}}\delta_{\alpha\beta})_{\alpha} + \frac{1}{2}(d_{\gamma\lambda}h_{\frac{\nu}{\kappa}}\gamma_{\frac{\nu}{\kappa}})_{\alpha} + \frac{1}{2}(d_{\lambda\beta}h_{\frac{\nu}{\kappa}}\lambda_{\beta})_{\alpha} + \frac{1}{2}(d_{\gamma\lambda}h_{\frac{\nu}{\kappa}}\lambda_{\gamma})_{\alpha} \\
+ (d_{\alpha}h_{\psi_{0}})_{\alpha} + (d^{*}_{\alpha}h_{\psi_{0}})_{\alpha} + (\tilde{\alpha}_{\alpha\beta}h_{\psi_{0}})_{\alpha} + (\alpha^{*}_{\alpha\beta}h_{\psi_{0}})_{\alpha} \\
- \frac{1}{2}b_{i\beta}h_{\psi_{0\beta}} - \frac{1}{2}b_{\alpha\beta}h_{\psi_{0\alpha\beta}} - \tilde{\zeta}h_{\psi_{0\alpha}} - d_{\beta}h_{\psi_{0\alpha}} \\
- \frac{1}{2}b^{*}_{i\beta}h_{\psi_{0\beta}} - \frac{1}{2}b^{*}_{\alpha\beta}h_{\psi_{0\alpha\beta}} - \xi^{*}h_{\psi_{0\alpha}} - d^{*}_{\beta}h_{\psi_{0\alpha}} + \frac{0}{0} + H = \rho k \frac{\partial^{2}h_{\psi_{0}}}{\partial t^{2}}, \\
i.e., \because of \tilde{b}_{ij} = \tilde{b}_{ji}, \quad b^{*}_{ij} = b^{*}_{ji}, \\
\frac{1}{2}(d_{\alpha\delta}h_{\frac{\nu}{\kappa}}\delta_{\alpha\beta})_{\alpha} + \frac{1}{2}(d_{\gamma\lambda}h_{\frac{\nu}{\kappa}}\gamma_{\frac{\nu}{\kappa}})_{\alpha} + \frac{1}{2}(d_{\lambda\beta}h_{\frac{\nu}{\kappa}}\lambda_{\beta})_{\alpha} + \frac{1}{2}(d_{\gamma\lambda}h_{\frac{\nu}{\kappa}}\lambda_{\gamma})_{\alpha} \\
+ (d_{\alpha}h_{\psi_{0}})_{\alpha} + (d^{*}_{\alpha}h_{\psi_{0}})_{\alpha} + (\tilde{\alpha}_{\alpha\beta}h_{\psi_{0}})_{\alpha} + (\alpha^{*}_{\alpha\beta}h_{\psi_{0}})_{\alpha} - \tilde{b}_{i\alpha}h_{\psi_{0\alpha}} \\
- \xi h_{\psi_{0\alpha}} - b^{*}_{i\alpha}h_{\psi_{0\alpha}} - \xi^{*}h_{\psi_{0\alpha}} + \frac{0}{0} + H = \rho k \frac{\partial^{2}h_{\psi_{0}}}{\partial t^{2}}, \\
(4.12) \\
\frac{1}{2}(p_{\alpha\delta}h_{\frac{\nu}{\kappa}}\delta_{\alpha\beta})_{\alpha} + \frac{1}{2}(p_{\gamma\lambda}h_{\frac{\nu}{\kappa}}\gamma_{\frac{\nu}{\kappa}})_{\alpha} + \frac{1}{2}(p_{\lambda\beta}h_{\frac{\nu}{\kappa}}\lambda_{\beta})_{\alpha} + \frac{1}{2}(p_{\gamma\lambda}h_{\frac{\nu}{\kappa}}\lambda_{\gamma})_{\alpha} \\
+ (p_{\alpha}h_{\psi_{0}})_{\alpha} + (p^{*}_{\alpha}h_{\psi_{0}})_{\alpha} + (\tilde{\alpha}_{\alpha\beta}h_{\psi_{0}})_{\alpha} + (\alpha^{*}_{\alpha\beta}h_{\psi_{0}})_{\alpha} + 0 + D = f_{0}. \\
(4.13) \\
\frac{1}{2}(q_{\alpha\delta}h_{\frac{\nu}{\kappa}}\delta_{\alpha\beta})_{\alpha} + \frac{1}{2}(q_{\gamma\lambda}h_{\frac{\nu}{\kappa}}\gamma_{\frac{\nu}{\kappa}})_{\alpha} + \frac{1}{2}(q_{\lambda\beta}h_{\frac{\nu}{\kappa}}\lambda_{\beta})_{\alpha} + \frac{1}{2}(q_{\gamma\lambda}h_{\frac{\nu}{\kappa}}\lambda_{\gamma})_{\alpha} \\
+ (q^{*}_{\alpha})h_{\psi_{0\alpha}} + (\tilde{\alpha}_{\alpha\beta}h_{\psi_{0\alpha}})_{\alpha} + (\alpha^{*}_{\alpha\beta}h_{\psi_{0\alpha}})_{\alpha} + 0 + B = 0. \\
(4.14) \\
\text{For such equations and systems with order degeneracy see Section 8 of the present work and also [11]-[18].} \\
\text{Remark 4.1. System (4.11)-(4.14) we may also obtain from system (3.32)-(3.35) assuming there (3.31).} \\
\text{Evidently,} \\
X_{n_{j0}} = X_{ij0n_{i}} = \left\{ \frac{1}{2}E_{ijkl}(v_{0k0l} + v_{0k0l}) + \frac{1}{2}E_{ijkl}^{*}(v_{0k0l} + v_{0k0l}) \right. \\
+ \tilde{b}_{ij}h_{\psi_{0}} + b^{*}_{ij}h_{\psi_{0}} + d_{ij\gamma}h_{\psi_{0\gamma}} + d^{*}_{ij\gamma}h_{\psi_{0\gamma}} + p_{ij\gamma}h_{\tilde{\chi}_{0\gamma}} \\
\left. + p^{*}_{ij\gamma}h_{\tilde{\chi}_{0\gamma}} + q_{ij\gamma}h_{\tilde{\eta}_{0\gamma}} + q^{*}_{ij\gamma}h_{\tilde{\eta}_{0\gamma}} \right\} n_{i}, \quad j = 1,3, \\
H_{n_{0}} = H_{j0n_{j}} = \left\{ \frac{1}{2}d_{ij}h_{(v_{0k0l} + v_{0k0l})} + \frac{1}{2}d^{*}_{ij}h_{(v_{0k0l} + v_{0k0l})} \right. \\
+ d_{ij}h_{\psi_{0}} + d^{*}_{ij}h_{\psi_{0}} + \tilde{\alpha}_{ij}h_{\psi_{0i}} + \alpha^{*}_{ij}h_{\psi_{0i}} \right\} n_{j}, \quad j = 1,3.
Now we apply the results of Section 8. To this end we rewrite the system (4.11)-(4.14) for the static case in the following matrix form

\[ Lu := (A^{\alpha\beta} u_{\alpha}),_{\beta} + E^{\alpha} u_{\alpha} + C u = F, \quad x \in \omega, \]  

where

\[ x := (x_1, x_2), \quad u = (u_1, \ldots, u_6)^\top \equiv (v_{10}, v_{20}, v_{30}, \tilde{\phi}_0, \eta_0)^\top, \]  

\[ F := (F_1, \ldots, F_6)^\top \equiv (-0, -0, -0, -0, -H, 0, -0, D, -B)^\top, \]  

\[ A^{\alpha\beta} := \|a^{\alpha\beta}_{kl}\|, \quad E^{\alpha} := \|e^{\alpha}_{kl}\| \in C^1(\tilde{\omega}); \]  

\[ C := \|c_{kl}\| \in C(\tilde{\omega}), \quad \alpha, \beta = 1, 2, \quad k, l = 1, 6, \]
\[
\begin{align*}
\begin{bmatrix}
E_{2111} & \frac{1}{2}(E_{2121} + E_{2112}) & \frac{1}{2}(E_{2131} + E_{2113}) & d_{211} & p_{121} & q_{121} \\
E_{2211} & \frac{1}{2}(E_{2221} + E_{2212}) & \frac{1}{2}(E_{2231} + E_{2213}) & d_{221} & p_{122} & q_{122} \\
E_{2311} & \frac{1}{2}(E_{2321} + E_{2312}) & \frac{1}{2}(E_{2331} + E_{2313}) & d_{231} & p_{123} & q_{123} \\
d_{112} & \frac{1}{2}(d_{212} + d_{122}) & \frac{1}{2}(d_{312} + d_{321}) & \tilde{\alpha}_{21} & 0 & 0 \\
p_{211} & \frac{1}{2}(p_{221} + p_{212}) & \frac{1}{2}(p_{231} + p_{213}) & 0 & -\tilde{\gamma}_{21} & -\tilde{\alpha}_{21} \\
qu_{211} & \frac{1}{2}(q_{221} + q_{212}) & \frac{1}{2}(q_{231} + q_{213}) & 0 & -\tilde{\delta}_{21} & -\tilde{\xi}_{21}
\end{bmatrix} = h
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\frac{1}{2}(E_{1122} + E_{1112}) & E_{1122} & \frac{1}{2}(E_{1132} + E_{1123}) & d_{112} & p_{211} & q_{211} \\
\frac{1}{2}(E_{1212} + E_{1221}) & E_{1222} & \frac{1}{2}(E_{1232} + E_{1223}) & d_{122} & p_{212} & q_{212} \\
\frac{1}{2}(E_{1312} + E_{1321}) & E_{1322} & \frac{1}{2}(E_{1332} + E_{1323}) & d_{132} & p_{213} & q_{213} \\
\frac{1}{2}(d_{121} + d_{211}) & d_{221} & \frac{1}{2}(d_{321} + d_{312}) & \tilde{\alpha}_{12} & 0 & 0 \\
\frac{1}{2}(p_{112} + p_{121}) & p_{122} & \frac{1}{2}(p_{132} + p_{123}) & 0 & -\tilde{\gamma}_{12} & -\tilde{\alpha}_{12} \\
\frac{1}{2}(q_{112} + q_{121}) & q_{122} & \frac{1}{2}(q_{132} + q_{123}) & 0 & -\tilde{\delta}_{12} & -\tilde{\xi}_{12}
\end{bmatrix} = h
\end{align*}
\]

\[E^{\alpha} := h\]

\[
\begin{bmatrix}
0 & 0 & 0 & \tilde{b}_{a1} & 0 & 0 \\
0 & 0 & 0 & \tilde{b}_{a2} & 0 & 0 \\
0 & 0 & 0 & \tilde{b}_{a3} & 0 & 0 \\
-\tilde{b}_{1\alpha} & -\tilde{b}_{2\alpha} & -\tilde{b}_{3\alpha} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \alpha = 1, 2;
\]

\[C := \]

\[
\begin{bmatrix}
0 & 0 & 0 & (\tilde{b}_{a1} h)_{,\alpha} & 0 & 0 \\
0 & 0 & 0 & (\tilde{b}_{a2} h)_{,\alpha} & 0 & 0 \\
0 & 0 & 0 & (\tilde{b}_{a3} h)_{,\alpha} & 0 & 0 \\
0 & 0 & 0 & (d_{,a} h)_{,\alpha} - \tilde{\xi} h & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Keeping in mind some symmetry properties of the constitutive coefficients, from (4.20), (4.21) it follows that
\[
A^{12} := h
\begin{bmatrix}
E_{2111} & E_{2121} & E_{2131} & d_{211} & p_{121} & q_{121} \\
E_{2211} & E_{2221} & E_{2231} & d_{221} & p_{122} & q_{122} \\
E_{2311} & E_{2321} & E_{2331} & d_{231} & p_{123} & q_{123} \\
d_{112} & d_{212} & d_{312} & \tilde{\alpha}_{21} & 0 & 0 \\
p_{211} & p_{221} & p_{231} & 0 & -\zeta_{21} & -\tilde{\alpha}_{21} \\
q_{211} & q_{221} & q_{231} & 0 & -\tilde{\alpha}_{21} & -\xi_{21}
\end{bmatrix}, \quad (4.24)
\]

\[
A^{21} := h
\begin{bmatrix}
E_{1112} & E_{1122} & E_{1132} & d_{112} & p_{211} & q_{211} \\
E_{1212} & E_{1222} & E_{1232} & d_{122} & p_{212} & q_{212} \\
E_{1312} & E_{1322} & E_{1332} & d_{132} & p_{213} & q_{213} \\
d_{121} & d_{221} & d_{321} & \tilde{\alpha}_{12} & 0 & 0 \\
p_{112} & p_{122} & p_{132} & 0 & -\zeta_{12} & -\tilde{\alpha}_{12} \\
q_{112} & q_{122} & q_{132} & 0 & -\tilde{\alpha}_{12} & -\xi_{12}
\end{bmatrix}, \quad (4.25)
\]

Let for any vector
\[
\xi^{(\alpha)} = \left( \xi_1^{(\alpha)}, ... , \xi_6^{(\alpha)} \right), \quad \alpha = 1, 2,
\]
satisfying
\[
\sum_{\alpha=1}^{2} \sum_{k=1}^{6} \left( \xi_k^{(\alpha)} \right)^2 > 0,
\]
inequality
\[
\xi^{(\alpha)} A^{\alpha\beta} \xi^{(\beta)} > 0, \quad x \in \bar{\omega} \setminus \gamma^0
\]
be valid, where
\[
\gamma^0 := \{ x \in \gamma := \partial \omega : A^{\alpha\beta}(x) = 0, \ \alpha, \beta = 1, 2 \}.
\]

So, system (4.15) with (4.19)-(4.23) is strongly elliptic on \( \bar{\omega} \setminus \gamma^0 \) while on \( \gamma^0 \) the order of equations degenerates.

In the case under consideration all the elements of the matrices \( A^{\alpha\beta} \) contain as factors \( h(x) \). Therefore, \( \gamma^0 \subseteq \partial \omega \) coincides with the set, where \( h(x) = 0 \), provided all the constitutive coefficients do not vanish on \( \partial \omega \), otherwise they will participate in formation of \( \gamma^0 \).
Let a piecewise smooth curve $\gamma$ consist of smooth non-intersecting curves $\tilde{\gamma}(k)$ which may have only endpoints in common:

$$\gamma = \bigcup_{k=1}^{p} \tilde{\gamma}(k)$$

we assume that the Gauss-Ostrogradsky formula is applicable to the domain $\omega$.

At points of smoothness of $\gamma^0$ let us consider the matrix

$$\Phi := E^\alpha \nu_\alpha,$$

where $\nu := (\nu_1, \nu_2)$ is the inward normal at the above boundary points. Let further

$$\gamma_0 := \{x \in \gamma^0 : \Phi(x) = 0\}, \; \gamma_1 := \{x \in \gamma^0 : \Phi(x) > 0\},$$

$$\gamma_2 := \{x \in \gamma^0 : \Phi(x) < 0\}, \; \gamma_3 := \gamma \setminus \gamma^0.$$ 

Under matrix inequalities we mean inequalities for corresponding quadratic forms on vectors with nonzero length.

Let $x_0 \in \gamma^0$ be a common point of the neighbour pairs of the curves among $\gamma^{(1)}, ..., \gamma^{(p)}$. If there exists a neighbourhood of $x_0$ on $\gamma$ which $\subset (\gamma_2 \cup \gamma_3)$, then the above point will be added to $\gamma_2 \cup \gamma_3$, otherwise it will be added to $\gamma_0 \cup \gamma_1$.

Let further

$$\gamma^0 = \gamma_0 \cup \gamma_1 \cup \gamma_2,$$

and consider BC

$$u|_{\gamma_2 \cup \gamma_3} = 0, \quad (4.26)$$

**Definition 4.1.** Let $C_L$ be the class of bounded vectors $u$ such that

$$u \in C^2(\omega) \cap C(\omega \cup \gamma_2 \cup \gamma_3),$$

$$A^{\alpha\beta} u_{,\alpha}, \; A^{\alpha\beta} u E^\alpha u \in C^1(\bar{\omega}),$$

$$\left. (A^{\alpha\beta} u_{,\alpha}) \right|_{\gamma^0} = 0, \; Lu \; \text{be bounded}.$$ 

**Definition 4.2.** A vector $u \in C_L$ satisfying system (4.15) and BC (4.26) will be called a regular solution of the BVP (4.15), (4.26).

**Definition 4.3.** A vector $u \in L_2(\omega)$ will be called a weak solution of the BVP (4.15), (4.26) if the vector $F \in L_2(\omega)$ and $u$ satisfies

$$\int_\omega v F dx = \int_\omega L^* v \cdot u dx$$

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for any \( v \in C_L \) satisfying the condition
\[
v|_{\gamma_1 \cup \gamma_3} = 0, \quad v = (v_1, \ldots, v_6)
\]
(the space of such vectors \( v \) will be denoted by \( V \)), \( L^* \) is the adjoint operator to \( L \):
\[
L^*(v) := (v_\alpha A^{\alpha \beta})_{\beta} - (v E^\alpha)_{\alpha} + vC.
\]

**Theorem 4.1.** Let \( A^{12} = A^{21} \), let the matrices \( E^\alpha \), \( \alpha = 1, 2 \) be symmetric, and
\[
E^\alpha_{\alpha} - 2C \geq 0, \quad in \ \omega,
\]
then homogeneous BVP, corresponding to the BVP (4.15), (4.26) has only the trivial solution in \( C_L \) if
\[
u A^{\alpha \beta} u_\beta + \frac{1}{2} u E^\alpha u \in C^1(\bar{G}), \quad \gamma_1 \cup \gamma_2 \cup \gamma_3 \neq \emptyset.
\]

**Theorem 4.2.** Let \( A^{12} = A^{21} \), let the matrices \( E^\alpha \) be symmetric, and
\[
\xi((E^\alpha_{\alpha} - 2C)\xi \geq c_0 \sum_{k=1}^m \xi_k^2, \quad x \in \omega, \quad c_0 = const > 0, \quad \xi := (\xi_1, \ldots, \xi_6) \quad (4.28)
\]
then there exists a weak solution of the BVP (4.15), (4.26).

Comparing (4.24) and (4.25) we easily obtain additional restrictions on the constitutive coefficients for fulfillment of \( A^{12} = A^{21} \). Namely,
\[
E_{1212} = E_{2211}, \quad E_{1312} = E_{2311}, \quad E_{1322} = E_{2321};
\]
\[
d_{121} = d_{112}, \quad d_{221} = d_{321} = d_{312};
\]
\[
p_{112} = p_{211}, \quad p_{122} = p_{221}, \quad p_{132} = p_{321};
\]
\[
q_{112} = q_{211}, \quad q_{122} = q_{221}, \quad q_{132} = q_{321}.
\]

For symmetry of matrices \( E^\alpha \), \( \alpha = 1, 2 \), we have to assume \( \tilde{b}_{ij} = 0 \) (see (4.22)), which implies that \( E^\alpha \equiv 0, \alpha = 1, 2 \). Since \( \xi > 0, \) (4.27) holds because of \( E^\alpha \equiv 0, \alpha = 1, 2 \), and (4.23). Indeed,
\[
\xi(-2C)\xi = 2 \left[ \xi h - (d_\alpha h)_{\alpha} \right] (\xi_4)^2 \geq 0 \quad on \ \bar{\omega}.
\]
For the same reason
\[
\Phi|_{\gamma_0} = E^\alpha \nu^\alpha|_{\gamma_0} = 0,
\]
hence
\[
\gamma_1 = \emptyset, \quad \gamma_2 = \emptyset, \quad \gamma^0 = \gamma_0, \quad \gamma = \gamma_0 \cup \gamma_3.
\]
So, we have to do with the Keldysh type BVP.
Thus, according to Theorem 4.1, a regular solution of BVP (4.15), (4.26) (the last takes the form
\[ u_3|_{\gamma_3} = 0 \]
is unique, provided it exists.

In the case under consideration
\[ \xi(E^{\alpha}, -2C)\xi = -2\xi C \xi = 2\left[ \tilde{\xi}h - (d_\alpha h)_\alpha \right](\xi_\alpha)^2 > 0 \text{ on } \omega \cup \gamma_3 \]
and (4.28) will be fulfilled if
\[ \min_{\omega} \left[ \tilde{\xi}h - (d_\alpha h)_\alpha \right] > \frac{c_0}{2}. \]
But \( h|_{\gamma_0} = 0 \) and \( \tilde{\xi} \) should be unbounded in a neighborhood of \( \gamma_0 \). If the last reasoning has a physical sense, then (4.28) holds and, according to Theorem 4.2, a weak solution of problem (4.15), (4.26) exists. If this physical assumption is not reasonable, then we apply an approach mentioned in Remark 8.2 and choose the multiplier \( \tilde{\psi} \) suitably for ensuring the fulfillment of condition (4.28).

5 Transversely Isotropic Solids

Let us now consider the transversely isotropic elastic piezoelectric material in the case when the poling axis coincides with one of the material symmetry axes [19]. A material behavior is said to be transversely isotropic if it is invariant with respect to an arbitrary rotation about a given axis. This material behavior is of special importance in the modelling of fibre-reinforced composite materials with a coordinate axis in the fibre direction and assumed isotropic in cross-sections orthogonal to fibre direction [20] (in our case to poling axis as well, since in the case under consideration they coincide). The transverse isotropic model is also suitable for biological applications because it adequately describes the elastic properties of bundled fibers aligned in one direction [21] (see also [22]).

It is well-known [19] that the electric field that develops in piezoelectrics can be assumed to be quasi-static because the velocity of the elastic waves is much smaller than the velocity of electromagnetic waves. Therefore, the magnetic field due to the elastic waves is negligible \( B \approx 0 \). This fact implies that
\[ \frac{\partial B}{\partial t} \approx 0. \]
So one of Maxwell’s equations of electrodynamics becomes
\[ \text{rot } E = \frac{\partial B}{\partial t} \approx 0 \]
and, as it was already assumed,
\[ E = -\text{grad } \chi. \]
Consequently, considering transversely isotropic piezoelectric continuum, it will be based on the governing equations of elastodynamics in the case of small deformations and quasi-electrostatic fields. Note that piezoelectric materials show in most cases a crystal structure with a symmetry of hexagonal 6 mm class. In the case when the poling axis coincides with one of the material symmetry axes these materials become transversely isotropic. Restricting to the case of time-harmonic motion with frequency \( \omega \), i.e., all the sought quantities, s.c. free members of governing equations, and boundary data are represented as the products of \( e^{i\omega t} \) and of the same quantities (to avoid redundant indices and symbols we leave the same notation) depending only on the space variables, from the governing equations of dynamics (2.1), (2.3), (2.4), (2.5), (2.8), we get the following governing equations

\[
X_{ij,j} + \rho \omega^2 u_i = -\Phi_i, \quad i = 1, 3; \tag{5.1}
\]

\[
D_{j,j} = f_e; \tag{5.2}
\]

\[
e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 3; \tag{5.3}
\]

\[
E_i = -\chi_i, \quad i = 1, 3;
\]

where (see [19])

\[
\begin{pmatrix}
X_{11} \\
X_{22} \\
X_{33} \\
X_{23} \\
X_{31} \\
X_{12} \\
D_1 \\
D_2 \\
D_3
\end{pmatrix}
= 
\begin{pmatrix}
e_{11} \\
e_{22} \\
e_{33} \\
2\epsilon_{23} \\
2\epsilon_{31} \\
2\epsilon_{12} \\
E_1 \\
E_2 \\
E_3
\end{pmatrix}
\tag{5.3}
\]

\[
\begin{pmatrix}
E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 & 0 & 0 & p_{311} \\
E_{1122} & E_{1111} & E_{1133} & 0 & 0 & 0 & 0 & 0 & p_{311} \\
E_{1133} & E_{1133} & E_{3333} & 0 & 0 & 0 & 0 & 0 & p_{333} \\
0 & 0 & 0 & E_{2323} & 0 & 0 & 0 & p_{113} & 0 \\
0 & 0 & 0 & 0 & E_{2323} & 0 & 0 & p_{113} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(E_{1111} - E_{1122}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_{113} & 0 & -\varsigma_{11} & 0 & 0 \\
0 & 0 & 0 & p_{113} & 0 & 0 & 0 & -\varsigma_{11} & 0 \\
p_{311} & p_{311} & p_{333} & 0 & 0 & 0 & 0 & 0 & -\varsigma_{33}
\end{pmatrix}
\tag{5.4}
\]

From (5.3), (5.4) we have

\[
X_{11} = E_{1111}e_{11} + E_{1122}e_{22} + E_{1133}e_{33} - p_{311}E_3,
\]
\[ X_{22} = E_{1122}e_{11} + E_{1111}e_{22} + E_{1133}e_{33} - p_{311}E_3, \]
\[ X_{33} = E_{1133}e_{11} + E_{1133}e_{22} + E_{3333}e_{33} - p_{333}E_3, \]
\[ X_{23} = 2E_{2323}e_{23} - p_{113}E_2, \quad X_{31} = 2E_{2323}e_{31} - p_{113}E_1, \]
\[ X_{12} = (E_{1111} - E_{1122})e_{12}, \]
\[ D_1 = 2p_{113}e_{13} + \varsigma_{11}E_1, \quad D_2 = 2p_{113}e_{23} + \varsigma_{11}E_2, \]
\[ D_3 = p_{311}e_{11} + p_{311}e_{22} + p_{333}e_{33} + \varsigma_{33}E_3, \]

i.e.,
\[ X_{11} = E_{1111}u_{1,1} + E_{1122}u_{2,2} + E_{1133}u_{3,3} - p_{311}E_3, \]
\[ X_{22} = E_{1122}u_{1,1} + E_{1111}u_{2,2} + E_{1133}u_{3,3} - p_{311}E_3, \]
\[ X_{33} = E_{1133}u_{1,1} + E_{1133}u_{2,2} + E_{3333}u_{3,3} - p_{333}E_3, \]
\[ X_{23} = E_{2323}(u_{2,3} + u_{3,3}) - p_{113}E_2, \quad X_{31} = E_{2323}(u_{3,1} + u_{1,3}) - p_{113}E_1, \quad (5.5) \]
\[ X_{12} = \frac{1}{2}(E_{1111} - E_{1122})(u_{1,2} + u_{2,1}), \]
\[ D_1 = p_{113}(u_{3,1} + u_{1,3}) + \varsigma_{11}E_1, \quad D_2 = p_{113}(u_{2,3} + u_{3,2}) + \varsigma_{11}E_2, \]
\[ D_3 = p_{311}u_{1,1} + p_{311}u_{2,2} + p_{333}u_{3,3} + \varsigma_{33}E_3. \]

Conditions of Anti-plane Piezoelectric State [19] have the form

1. \( u_1 \equiv 0, \quad u_2 \equiv 0, \quad u_3 \not\equiv 0; \)
2. \( X_{13} \not\equiv 0, \quad X_{23} \not\equiv 0; \quad X_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1, 2; \quad X_{33} \equiv 0; \)
3. \( e_{13} \not\equiv 0, \quad e_{23} \not\equiv 0; \quad e_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1, 2; \quad e_{33} \equiv 0; \quad (5.6) \)
4. \( E_1 \not\equiv 0, \quad E_2 \not\equiv 0, \quad E_3 \equiv 0; \)
5. \( D_1 \not\equiv 0, \quad D_2 \not\equiv 0, \quad D_3 \equiv 0. \)

Taking into account (5.6), from the first three relations of (5.5) we have
\[ u_{3,3} \equiv 0, \quad u_3 = u_3(x_1, x_2); \]

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the fourth and fifth relations of (5.5) give

\[ X_{23} = E_{2323}u_{3,2} - p_{113}E_2, \]  
\[ X_{31} = E_{2323}u_{3,1} - p_{113}E_1, \]

respectively;
the sixth of (5.5) is identically fulfilled;
the seventh and eighth relations of (5.5) give

\[ D_1 = p_{113}u_{3,1} + \varsigma_{11}E_1, \] 
\[ D_2 = p_{113}u_{3,2} + \varsigma_{11}E_2. \]

respectively;
the ninth of (5.5) is identically fulfilled.
From the first two of (5.1) it follows that

\[ \Phi_\alpha \equiv 0, \quad \alpha = 1, 2, \] 

the third of (5.1) will have the form

\[ X_{31,1} + X_{32,2} + \rho \sigma^2 u_3 = -\Phi_3; \]

while (5.2) will have the form

\[ D_{1,1} + D_{2,2} = f_e. \]

Substituting (5.7) and (5.8) into (5.12), and (5.9) and (5.10) into (5.13), we get

\[ (E_{2323}u_{3,1}),_1 + (E_{2323}u_{3,2}),_2 - (p_{113}E_1),_1 - (p_{113}E_2),_2 + \rho \sigma^2 u_3 = -\Phi_3, \]

and

\[ (p_{113}u_{3,1}),_1 + (p_{113}u_{3,2}),_2 + (\varsigma_{11}E_1),_1 + (\varsigma_{11}E_2),_2 = f_e, \]

respectively.
Taking into account

\[ E_\alpha = -\chi_{,\alpha}, \quad \alpha = 1, 2, \]

we obtain the following governing equations in the anti-plane piezoelectric state

\[ (E_{2323}u_{3,1}),_1 + (E_{2323}u_{3,2}),_2 + (p_{113}\chi_{,1}),_1 + (p_{113}\chi_{,2}),_2 + \rho \sigma^2 u_3 = -\Phi_3, \]

\[ (p_{113}u_{3,1}),_1 + (p_{113}u_{3,2}),_2 - (\varsigma_{11}\chi_{,1}),_1 - (\varsigma_{11}\chi_{,2}),_2 = f_e, \]
i.e.,

\[ (E_{2323}u_{3,1}),_\alpha + (p_{113}\chi_{,\alpha}),_\alpha + \rho \sigma^2 u_3 = -\Phi_3, \]
\[ (p_{113}u_{3,1}),_\alpha - (\varsigma_{11}\chi_{,\alpha}),_\alpha = f_e. \]
Let the plane domain of interest have the form given in Figure 5.1 and let

\[ E_{2323} = E_0 x_2^{\kappa_1}, \quad E_0 = \text{const} > 0, \quad \kappa_1 = \text{const} \geq 0; \]

\[ p_{113} = p_0 x_2^{\kappa_2}, \quad p_0 = \text{const} > 0, \quad \kappa_2 = \text{const} \geq 0; \]

\[ \varsigma_{11} = \varsigma_0 x_2^{\kappa_3}, \quad \varsigma_0 = \text{const} > 0, \quad \kappa_3 = \text{const} \geq 0, \]

then (5.14) and (5.15) take the forms

\[ E_0 (x_2^{\kappa_1} u_{3,\alpha}) + p_0 (x_2^{\kappa_2} \chi_{\alpha}) + \rho \sigma^2 u_3 = -\Phi_3, \quad (5.16) \]

and

\[ p_0 (x_2^{\kappa_2} u_{3,\alpha}) - \varsigma_0 (x_2^{\kappa_3} \chi_{\alpha}) = f_e, \quad (5.17) \]

respectively.

We will consider the following two cases. To this end first we state the following

**Theorem 5.1** (Jaiani, see [12]). If the coefficients \( a_{\alpha}, \alpha = 1,2, \) and \( c \) of the equation

\[ x_2^{\kappa_\alpha} u_{,\alpha\alpha} + a_{\alpha}(x_1, x_2) u_{,\alpha} + c(x_1, x_2) u = 0, \quad c \leq 0, \quad \kappa_\alpha = \text{const} \geq 0, \quad \alpha = 1,2, \]

are analytic in \( \overline{\omega} \), then

(i) if either \( \kappa_2 < 1 \), or \( \kappa_2 \geq 1 \),

\[ a_2(x_1, x_2) < x_2^{\kappa_2 - 1} \quad (5.18) \]

in \( \overline{\omega_\delta} \) for some \( \delta = \text{const} > 0 \), where

\[ \omega_\delta := \{(x_1, x_2) \in \omega : 0 < x_2 < \delta\}, \]

the Dirichlet problem (Problem D, \( u \in C^2(\omega) \cap C(\overline{\omega}) \)) is well-posed;
(ii) if \( \kappa_2 \geq 1 \),
\[
a_2(x_1, x_2) \geq x_2^{\kappa_2 - 1}
\]
(5.19)
in \( \omega \) and \( a_1(x_1, x_2) = O(x_2^{\kappa_1}) \), \( x_2 \to 0_+ \) (\( O \) is the Landau symbol), the Keldysh problem (Problem E, bounded \( u \in C^2(\omega) \cap C(\overline{\omega} \setminus \gamma^0) \)) is well-posed.

**Case 1.** \( \kappa_i = \kappa = \text{const} \geq 0 \), \( i = 1,3 \).

After some actions, from (5.16) and (5.17) we get
\[
(s_0 E_0 + p_0^2)(x_2^2 u_{3,\alpha})_\alpha + s_0 \rho \omega^2 u_3 = -s_0 \Phi_3 + p_0 f_e, \tag{5.20}
\]
and
\[
(p_0^2 + s_0 E_0)(x_2^2 \chi_\alpha)_\alpha + p_0 \rho \omega^2 u_3 = -p_0 \Phi_3 - E_0 f_e. \tag{5.21}
\]
(5.20) and (5.21) we rewrite as
\[
x_2 u_{3,\alpha} + \kappa u_{3,2} + s_0(s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho \omega^2 u_3 \\
= (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-s_0 \Phi_3 + p_0 f_e), \tag{5.22}
\]
and
\[
x_2 \chi_{\alpha} + \kappa \chi_{2} + p_0(s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho \omega^2 u_3 \\
= (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-p_0 \Phi_3 - E_0 f_e), \tag{5.23}
\]
respectively.

In the static case \( \rho = 0 \) and from (5.22), (5.23) we obtain separate equations
\[
x_2 u_{3,\alpha} + \kappa u_{3,2} = (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-s_0 \Phi_3 + p_0 f_e) \tag{5.24}
\]
\[
x_2 \chi_{\alpha} + \kappa \chi_{2} = (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-p_0 \Phi_3 - E_0 f_e), \tag{5.25}
\]
with respect to \( u_3 \) and \( \chi \), correspondingly.

We will consider

**Problem D.** Find solutions \( u_3, \chi \in C^2(\omega) \cap C(\overline{\omega}) \) of the system (5.24), (5.25) by their values prescribed on \( \partial \omega \).

and

**Problem E.** Find bounded solutions \( u_3, \chi \in C^2(\omega) \cap C(\omega \cup (\partial \omega \setminus \gamma^0)) \) of the system (5.24), (5.25) by their values prescribed only on the arc \( \partial \omega \setminus \gamma^0 \).

Now we are about to apply Theorem 5.1 but this theorem concerns the homogeneous equation whereas equations (5.24) and (5.25) are nonhomogeneous ones. If we find their particular solutions which are continuous on \( \overline{\omega} \), then in usual way we reduce the problems under consideration to BVPs for the homogeneous equations corresponding to equations (5.24), (5.25) with boundary data changed according to boundary values of the particular solutions. In case of Problem E it is sufficient to find a bounded particular solution continuous on \( \overline{\omega} \setminus \gamma^0 \). If \( \Phi_3 \) and \( f_e \) depend only on \( x_2 \), then we easily find such particular solutions, depending only on \( x_2 \). For another way of finding of desired particular solutions we refer the reader to [[23], pp. 75-78].

Further applying to nonhomogeneous equations Theorem 5.1 we will always have in mind this observation. According to Theorem 5.1 it follows

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Theorem 5.2. The values of \(u_3\) and \(\chi\) should be prescribed on the entire boundary (Problem D) for \(\kappa < 1\), while on the part of the boundary, where \(x_2 = 0\), should be freed at all of boundary conditions (Problem E) for \(\kappa \geq 1\). Both problems are uniquely solvable in the classical sense.

Proof. Indeed, for \(\kappa < 1\) and \(\kappa \geq 1\), correspondingly, (5.18) and (5.19) are fulfilled, which proves the theorem. \(\square\)

We now consider system (5.22), (5.23), assuming that the right-hand sides belong to \(L_2(\omega)\) and \(x_2^{1-\kappa} \rho \in C(\bar{\omega})\).

Equations (5.22) and (5.23) we rewrite as follows

\[
(x_2 u_{3,\alpha})_\alpha + (\kappa - 1) u_{3,2} + s_0 (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho^2 u_3 = (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} (-s_0 \Phi_3 + p_0 f_\epsilon),
\]

and

\[
(x_2 \chi, \alpha) = (\kappa - 1) x_2 = -p_0 (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho^2 u_3
\]

\[
+ (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} (-p_0 \Phi_3 - E_0 f_\epsilon),
\]

Intending to apply results of Section 9, we check for the first equation (5.26) of the system which contains only the unknown \(u_3\), the condition (9.15). In our case it looks like

\[
\frac{1}{2} E^{\kappa,\alpha} - C = \frac{1}{2} (\kappa - 1) - s_0 (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho^2 > c_0 > 0, \text{ on } \bar{\omega}.
\]

The last condition will be satisfied for

\[
\sigma^2 < \left[ \frac{1}{2} (\kappa - 1) - c_0 \right] (E_0 + s_0^{-1} p_0^2) \left[ \max_{(x_1, x_2) \in \omega} (x_2^{1-\kappa} \rho) \right]^{-1}
\]

provided

\[
\kappa > 1 + 2c_0,
\]

where \(c_0\) may be arbitrary small.

Therefore there exists an \(H\)-weak solution \(u_3\) of Problem E for equation (5.26). Substituting the found \(u_3\) into equation (5.27) we obtain the equation only with respect to \(\chi\). We now apply Theorem 5.1 to equation (5.27) and conclude that there exist unique classical solutions for Problem D if \(\kappa < 1\) and Problem E if \(\kappa \geq 1\). Evidently, classical solutions are \(H\)-weak solutions as well.

So, we have proved the following

Theorem 5.3. An \(H\)-weak solution \((u_3, \chi)\) of Problem E for system (5.26), (5.27) exists under conditions (5.28), (5.29). The last actually means

\[
\kappa > 1
\]

because of arbitrarily smallness of the constant \(c_0\).
Remark 5.1. If \( \bar{\omega} \) is a stripe \( \{-\infty < x_1 < +\infty, \ 0 \leq x_2 \leq L = const\} \) and all the quantities depend only on \( x_2 \) (it means that we consider cylindrical strain) then in the static case \( (o = 0) \) from (5.20) and (5.21) we obtain

\[
(x_2^\kappa u_3,2),2 = (s_0 E_0 + p_0^2)^{-1}(-s_0 \Phi_3 + p_0 f_e)
\]

and

\[
(x_2^\kappa \chi,2),2 = (s_0 E_0 + p_0^2)^{-1}(-p_0 \Phi_3 - E_0 f_e),
\]

respectively. Their general solutions have the forms

\[
u_3(x_2) = (s_0 E_0 + p_0^2)^{-1} \int_{x_2}^{x_2^2} \frac{d\tau}{\tau} \int_{\xi}^{\xi} [\Phi_3(t) + p_0 f_e(t)] dt + c_1^1 \\ln x_2 - \ln L \\
\]

for \( \kappa \neq 1 \) and

\[
\chi(x_2) = (s_0 E_0 + p_0^2)^{-1} \int_{x_2}^{x_2^2} \frac{d\tau}{\tau} \int_{\xi}^{\xi} [-p_0 \Phi_3(t) - E_0 f_e(t)] dt + c_2^2 \\ln x_2 - \ln L
\]

for \( \kappa = 1 \).

In the case under consideration BCs look like

\[
u_3(0) = c_0^1, \ \chi(0) = c_0^2; \ \nu_3(L) = c_1^1, \ \chi(L) = c_2^2 \ (\text{Problem D});
\]

\[
u_3(x_2) = O(1), \ \chi(x_2) = O(1), \ x_2 \to 0+; \ \nu_3(L) = c_1^1, \ \chi(L) = c_2^L \ (\text{Problem E}).
\]

From these BCs we easily calculate constants

\[
c_0^\alpha, \ \alpha, \beta = 1, 2, \ \text{for} \ \kappa < 1 \ (\text{Problem D})
\]

and

\[
c_2^\alpha, \ \alpha = 1, 2, \ \text{for} \ \kappa \geq 1 \ (\text{Problem E}),
\]

in the last case \( c_2^\alpha = 0, \ \alpha = 1, 2 \), (otherwise solutions will be unbounded) and some restrictions on \( \Phi_3(x_2), f_e(x_2) \) are required as well.

**Case 2.** \( \kappa_2 = \kappa_3 = \kappa = const \geq 0 \).

After some actions, from (5.16) and (5.17) we get

\[
((p_0^2 x_2^\kappa + s_0 E_0 x_2^\kappa_1) u_{3,\alpha}),_\alpha + s_0 \rho c_2^2 u_3 = -s_0 \Phi_3 + p_0 f_e, \quad (5.30)
\]

\[
s_0 (x_2^\kappa \chi,\alpha) = p_0 (x_2^\kappa u_{3,\alpha}),_\alpha - f_e. \quad (5.31)
\]

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So, for $\kappa_1 = 0$ and any $\kappa \geq 0$, equation (5.30) is not a degenerate one, while equation (5.31) is a degenerate one. If $\sigma = 0$, i.e., we deal with the static case and from (5.30), (5.31) we arrive at the system

\[(p_0^2 x_2^\kappa + \sigma_0 E_0)u_{3,\alpha},_\alpha = -\sigma_0 \Phi_3 + p_0 f_e, \quad (5.32)\]

\[\sigma_0 (x_2^\kappa \chi,_{\alpha,\alpha} = p_0 (x_2^\kappa u_{3,\alpha},_\alpha - f_e). \quad (5.33)\]

As (5.32) is not a degenerate equation, the values of $u_3$ should be prescribed on the entire boundary (Problem D), while, according to Theorem 5.1, the values of $\chi$ should be prescribed on the entire boundary (Problem D) for $0 \leq \kappa < 1$ and the part where $x_2 = 0$ should be freed of BCs (Problem E) for $\kappa \geq 1$. It will be clear if we rewrite (5.33) in the following form

\[x_2\chi,_{\alpha,\alpha} + \kappa \chi,_{2} = \sigma_0^{-1} [p_0 (x_2^\kappa u_{3,\alpha},_\alpha - f_e)]^{1-\kappa}.\]

Indeed, for $\kappa < 1$ and $\kappa \geq 1$, correspondingly, (5.18) and (5.19) are realized. So we have proved the following

**Theorem 5.4.** Problem D for equation (5.32) for all $\kappa \geq 0$ is uniquely solvable in the classical sense. Problem D for $0 \leq \kappa < 1$ and Problem E for $\kappa \geq 1$ for equation (5.33) are uniquely solvable in the classical sense. In other words Problem D for system (5.32), (5.33) has a unique classical solution, while Problem E has a unique classical solution for $\kappa \geq 1$.

**Remark 5.2.** Similarly to Case 1 we solve BVPs in the explicit form in the case of cylindrical strain (see Remark 5.1).

Now we come back to the time-harmonic motion and follow Section 9. Let $\kappa_1 > \kappa$, then we rewrite (5.30) as follows

\[(p_0^2 x_2 + \sigma_0 E_0 x_2^{\kappa_1-\kappa+1})u_{3,\alpha,\alpha} + (\kappa p_0^2 + \kappa_1 \sigma_0 E_0 x_2^{\kappa_1-\kappa})u_{3,2}
+ \sigma_0^2 \rho x_2^{1-\kappa}u_3 = x_2^{1-\kappa}(p_0 f_e - \sigma_0 \Phi_3). \quad (5.34)\]

Now, we write equation (5.34) in the following form

\[\((p_0^2 x_2 + \sigma_0 E_0 x_2^{\kappa_1-\kappa+1})u_{3,\alpha,\alpha} + (\kappa - 1)(p_0^2 + \sigma_0 E_0 x_2^{\kappa_1-\kappa})u_{3,2}
+ \sigma_0^2 \rho x_2^{1-\kappa}u_3 = x_2^{1-\kappa}(p_0 f_e - \sigma_0 \Phi_3). \quad (5.35)\]

Conditions (9.2) lead to the restrictions

\[
\begin{align*}
\kappa_1 &\geq \kappa + 1, \\
x_2^{1-\kappa} \rho &\in C(\bar{\omega}).
\end{align*}
\]

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Equation (5.35) has an order degeneracy by \(x_2 = 0\), i.e., on \(\gamma^0\). According to (9.5) and (9.6)
\[
\Phi|_{x_2=0} = E^1\nu_1|_{x_2=0} = E^2\nu_2|_{x_2=0} = E^2|_{x_2=0}
\]
\[
= [(\kappa - 1)p_0^2 - \varsigma_0 E_0 (1 - \kappa)x_2^{\kappa_1 - \kappa}]|_{x_2=0} = (\kappa - 1)p_0^2
\]
and consequently
\[
\gamma^0 = \begin{cases} 
\gamma_1 & \text{for } \kappa > 1, \\
\gamma_2 & \text{for } \kappa < 1, \\
\gamma_0 & \text{for } \kappa = 1,
\end{cases}
\]
which means that we have to do with the Dirichlet problem for \(\kappa < 1\) and with the Keldysh problem for \(\kappa \geq 1\), since \(\gamma_0 \cup \gamma_1\) should be freed of BC and at \(\gamma_0 \cup \gamma_1\) BC should be replaced by boundedness of \(u_3\) in a neighborhood of \(\gamma_0 \cup \gamma_1\).

Condition (9.15) looks like
\[
-\frac{1}{2}\varsigma_0 E_0 (1 - \kappa)(\kappa_1 - \kappa)x_2^{\kappa_1 - \kappa - 1} - \varsigma_0 \rho \sigma_2^{1-\kappa} > c_0 > 0 \text{ in } \omega,
\]
which is not valid since the limit of the left hand side, because of (5.36), tends to
\[
\begin{cases} 
-\infty & \text{for } \kappa > 1; \\
\varsigma_0 \rho^2 \sigma < 0 & \text{for } \kappa = 1, \kappa_1 \geq 2; \\
0 & \text{for } \kappa < 1, \kappa_1 > \kappa + 1; \\
-\frac{1}{2}\varsigma_0 E_0 (1 - \kappa) < 0 & \text{for } \kappa < 1, \kappa_1 > \kappa + 1,
\end{cases}
\]
as \(x_2 \to 0^+\).

To avoid this obstacle we have to apply an approach described in Remark 8.2. For an example of application of the above-mentioned approach we refer the reader to [13], where cusped elastic prismatic shells are considered.

Similarly, when \(\kappa > \kappa_1 + 1\), we first rewrite (5.30) as
\[
(p_0^2 x_2^{\kappa - \kappa_1 + 1} + \varsigma_0 E_0 x_2)u_{3,aa} + (\kappa p_0^2 x_2^{\kappa - \kappa_1} + \kappa_1 \varsigma_0 E_0)u_{3,2}
\]
\[
+ \varsigma_0 \rho \sigma x_2^{1-\kappa_1} u_3 = x_2^{1-\kappa_1} (p_0 f - \varsigma_0 \Phi_3), \quad (5.37)
\]
then as follows
\[
\left(p_0^2 x_2^{\kappa - \kappa_1 + 1} + \varsigma_0 E_0 x_2)u_{3,a}\right)_a + (\kappa_1 - 1)(p_0^2 x_2^{\kappa - \kappa_1} + \varsigma_0 E_0)u_{3,2}
\]
\[
+ \varsigma_0 \rho \sigma x_2^{1-\kappa_1} u_3 = x_2^{1-\kappa_1} (p_0 f - \varsigma_0 \Phi_3) \quad (5.38)
\]
and carry out the corresponding reasonings.

After substituting the found solutions of Problem E for equation (5.34), ((5.37)) into (5.31) we arrive at the already investigated problem for equation (5.31) with respect to \(\chi\) with the result that for \(\chi\) Problem E is well-posed for \(\kappa > 1 \, (\kappa > 2)\).
6 Antiplane Deformation of Piezoelectrics in N=0 Approximation

In the $N=0$ Approximation conditions of the antiplane state look like

1. $v_{10} \equiv 0$, $v_{20} \equiv 0$, $v_{30} \not\equiv 0$;  
2. $X_{130} \not\equiv 0$, $X_{230} \not\equiv 0$, $X_{\alpha 30} \equiv 0$, $\alpha, \beta = 1, 2$; $X_{330} \equiv 0$;  
3. $e_{130} \not\equiv 0$, $e_{230} \not\equiv 0$, $e_{\alpha 30} \equiv 0$, $\alpha, \beta = 1, 2$; $e_{330} \equiv 0$;  
4. $E_{10} \not\equiv 0$, $E_{20} \not\equiv 0$, $E_{30} \equiv 0$;  
5. $D_{10} \not\equiv 0$, $D_{20} \not\equiv 0$, $D_{30} \equiv 0$.

Then from (4.11)-(4.14) we have

\[ \frac{1}{2}(E_{\alpha 33 \delta} h v_{30, \delta}),_\alpha + \frac{1}{2}(E_{\alpha 3 \delta \gamma} h v_{30, \gamma}),_\alpha + (p_{\gamma \alpha 3} h \tilde{x}_{0, \gamma}),_\alpha + 0 = 0, \beta = 1, 2, \]

\[ \frac{1}{2}(E_{\alpha 33 \delta} h v_{30, \delta}),_\alpha + \frac{1}{2}(E_{\alpha 3 \delta \gamma} h v_{30, \gamma}),_\alpha + (p_{\gamma \alpha 3} h \tilde{x}_{0, \gamma}),_\alpha + 0 = \rho \tilde{v}_{30}, \]

\[ - (s_{\alpha \gamma} h \tilde{x}_{0, \gamma}),_\alpha + \frac{1}{2}(p_{\alpha 3 \delta} h v_{30, \delta}),_\alpha + \frac{1}{2}(p_{\alpha 3 \delta \gamma} h v_{30, \gamma}),_\alpha + 0 = D = f e_0. \]  

(6.1)

If we consider transversely isotropic piezoelectric materials, then

\[ E_{2323} = E_{1313} \not\equiv 0; \]
\[ E_{2222} = E_{1111} \not\equiv 0, \quad E_{1122} \not\equiv 0, \quad E_{2233} = E_{1133} \not\equiv 0, \quad E_{3333} \not\equiv 0; \]
\[ p_{223} = p_{113} \not\equiv 0; \]
\[ s_{22} = s_{11} \not\equiv 0, \quad s_{33} \not\equiv 0. \]  

(6.2)

Other elastic, piezoelectric, and dielectric permittivity constants are identically zero with regard to reciprocal symmetries.

Therefore, from (6.1), by virtue of (6.2), we obtain

\[ 0 = X_\beta, \beta = 1, 2; \]

\[ \frac{1}{2}(E_{1331} h v_{30, 1}),_1 + \frac{1}{2}(E_{2332} h v_{30, 2}),_2 + \frac{1}{2}(E_{1313} h v_{30, 1}),_1 + \frac{1}{2}(E_{2323} h v_{30, 2}),_2 \]

\[ + (p_{113} h \tilde{x}_{0, 1}),_1 + (p_{223} h \tilde{x}_{0, 2}),_2 + 0 = \rho \tilde{v}_{30} \]

and

\[ - (s_{11} h \tilde{x}_{0, 1}),_1 - (s_{22} h \tilde{x}_{0, 2}),_2 + \frac{1}{2}(p_{131} h v_{30, 1}),_1 + \frac{1}{2}(p_{232} h v_{30, 2}),_2 \]

\[ + \frac{1}{2}(p_{113} h v_{30, 1}),_1 + \frac{1}{2}(p_{223} h v_{30, 2}),_2 + 0 = f e_0. \]

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Whence, taking into account

\[ E_{1331} = E_{3131} = E_{2332} = E_{2323}, \]
\[ p_{223} = p_{113}, \quad \varsigma_{11} = \varsigma_{22}, \]

we get

\[ (E_{2323}h v_{30, \alpha})_\alpha + (p_{113}h \tilde{\chi}_{0, \alpha})_\alpha + X_3 = \rho \dot{v}_30. \tag{6.3} \]

and

\[ (p_{113}h v_{30, \alpha})_\alpha - (\varsigma_{11}h \tilde{\chi}_{0, \alpha})_\alpha + D = f_e0. \tag{6.4} \]

respectively.

Now assuming

\[ E_{2323}h = E_0 x_2^{\kappa_1}, \quad E_0 = \text{const} > 0, \quad \kappa_1 = \text{const} \geq 0; \]
\[ p_{113}h = p_0 x_2^{\kappa_2}, \quad p_0 = \text{const} > 0, \quad \kappa_2 = \text{const} \geq 0; \tag{6.5} \]
\[ \varsigma_{11}h = \varsigma_0 x_2^{\kappa_3}, \quad \varsigma_0 = \text{const} > 0, \quad \kappa_3 = \text{const} \geq 0, \]

in the case of time-harmonic vibration from (6.3), (6.4) we arrive at equations (5.16), (5.17); and (similarly to Section 5) at systems (5.20), (5.21); (5.22), (5.23); (5.26), (5.27); (5.31), (5.30); (5.31), for \( \kappa_1 > \kappa \) (5.34) i.e. (5.35); and (5.31), for \( \kappa > \kappa_1 \) (5.37) i.e. (5.38); of the antiplane state in the three-dimensional formulation, where \( u_3, \chi, \Phi_3, \) and \( f_e \) should be replaced by \( v_{30}, \tilde{\chi}_0, X_3, \) and \( f_e0 - \frac{D}{h} \) (the factor \( h \) should be put in before the zero moment of a weighted unknown \( v_{30} \)), respectively. Namely, we obtain the following systems (6.6), (6.7); (6.8), (6.9); (6.10), (6.11); (6.12), (6.13); (6.14), (6.15); (6.14), for \( \kappa_1 > \kappa \) (6.16) i.e. (6.17); and (6.14), for \( \kappa > \kappa_1 \) (6.18) i.e. (6.19):

\[ E_0(x_2^{\kappa_1}v_{30, \alpha})_\alpha + p_0(x_2^{\kappa_2}\tilde{\chi}_{0, \alpha})_\alpha + \rho \sigma^2 h v_{30} = -\frac{0}{X_3}, \tag{6.6} \]
\[ p_0(x_2^{\kappa_2}v_{30, \alpha})_\alpha - \varsigma_0(x_2^{\kappa_3}\tilde{\chi}_{0, \alpha})_\alpha = f_e0 - \frac{0}{D}; \tag{6.7} \]
\[ (\varsigma_0 E_0 + p_0^2)(x_2^{\kappa_2}v_{30, \alpha})_\alpha + \varsigma_0 \rho \sigma^2 v_{30} = -\varsigma_0 X_3 + p_0(f_e0 - \frac{0}{D}), \tag{6.8} \]
\[ (p_0^2 + \varsigma_0 E_0)(x_2^{\kappa_3}\tilde{\chi}_{0, \alpha})_\alpha + p_0 \rho \sigma^2 v_{30} = -p_0 X_3 - E_0(f_e0 - \frac{0}{D}); \tag{6.9} \]

\[ x_2 v_{30, \alpha} + \kappa v_{30, 2} + \varsigma_0 E_0 + p_0^2 - \varsigma_0 X_3 + p_0(f_e0 - \frac{0}{D}) \]
\[ = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \left[ -\varsigma_0 X_3 + p_0(f_e0 - \frac{0}{D}) \right]. \tag{6.10} \]
\[ x_2 \tilde{\chi}_{0,\alpha} + \kappa \tilde{\chi}_{0,2} + p_0 (s_0 E_0 + p_0^2) - \kappa x_2^{1-\kappa} \rho \delta^2 h v_{30} \]

\[ = (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \left[ -p_0 X_3 - E_0 (f_{e_0} - 0) \right]; \quad (6.11) \]

\[ (x_2 v_{30,\alpha})_{2} + (\kappa - 1) v_{30,2} + s_0 (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho \delta^2 h v_{30} \]

\[ = (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \left[ -s_0 X_3 + p_0 (f_{e_0} - 0) \right], \quad (6.12) \]

\[ (x_2 \tilde{\chi}_{0,\alpha})_{2} + (\kappa - 1) \tilde{\chi}_{0,2} = -p_0 (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho \delta^2 h v_{30} \]

\[ + (s_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \left[ -p_0 X_3 - E_0 (f_{e_0} - 0) \right]; \quad (6.13) \]

\[ s_0 (x_2 \tilde{\chi}_{0,\alpha})_{2} = p_0 (x_2^2 v_{30,\alpha})_{2} - (f_{e_0} - 0), \quad (6.14) \]

\[ ((p_0^2 x_2^2 + s_0 E_0 x_2^{\kappa_1}) v_{30,\alpha})_{2} + s_0 \rho \delta^2 v_{30} = -s_0 X_3 + p_0 (f_{e_0} - 0) \quad (6.15) \]

i.e. \((\kappa_1 \geq \kappa)\)

\[ (p_0^2 x_2 + s_0 E_0 x_2^{\kappa_1 - \kappa + 1}) v_{30,\alpha} + (\kappa p_0^2 + \kappa_1 s_0 E_0 x_2^{\kappa_1 - \kappa}) v_{30,2} \]

\[ + s_0 \rho \delta^2 x_2^{1-\kappa} h v_{30} = x_2^{1-\kappa} \left[ p_0 (f_{e_0} - 0) - s_0 X_3 \right] \quad (6.16) \]

i.e.

\[ ((p_0^2 x_2 + s_0 E_0 x_2^{\kappa_1 - \kappa + 1}) v_{30,\alpha})_{2} + (\kappa - 1) (p_0^2 + s_0 E_0 x_2^{\kappa_1 - \kappa}) v_{30,2} \]

\[ + s_0 \rho \delta^2 x_2^{1-\kappa} h v_{30} = x_2^{1-\kappa} [p_0 (f_{e_0} - 0) - s_0 X_3] \quad (6.17) \]

and (for \(\kappa > \kappa_1\))

\[ (p_0^2 x_2^{\kappa - \kappa_1 + 1} + s_0 E_0 x_2) v_{30,\alpha} + (\kappa p_0^2 x_2^{\kappa - \kappa_1} + \kappa_1 s_0 E_0) v_{30,2} \]

\[ + s_0 \rho \delta^2 x_2^{1-\kappa_1} h v_{30} = x_2^{1-\kappa_1} \left[ p_0 (f_{e_0} - 0) - s_0 X_3 \right] \quad (6.18) \]

i.e.
\[(p_0^2 x_2^{\kappa-\kappa+1} + \varsigma_0 E_0 x_2) v_{30,\alpha} + (\kappa_1 - 1)(p_0^2 x_2^{\kappa-\kappa+1} + \varsigma_0 E_0) v_{30,2} \]
\[+ \varsigma_0 p_0^2 x_2^{1-\kappa} h v_{30} = x_2^{1-\kappa} \left[ p_0(f e_0 - 0) - \varsigma_0 X_3 \right]. \quad (6.19)\]

In the static case \((o = 0)\) from (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.12), (6.13), (6.14), (6.15), (6.16), (6.17) and (6.18), (6.19) we have

\[E_0(x_2^{\kappa_1} v_{30,\alpha})_\alpha + p_0(x_2^{\kappa_2} \tilde{\chi}_{0,\alpha})_\alpha = -X_3; \quad (6.20)\]

\[p_0(x_2^{\kappa_2} v_{30,\alpha})_\alpha - \varsigma_0(x_2^{\kappa_3} \tilde{\chi}_{0,\alpha})_\alpha = f e_0 - 0; \quad (6.21)\]

\[(\varsigma_0 E_0 + p_0^2)(x_2^{\kappa_2} v_{30,\alpha})_\alpha = -\varsigma_0 X_3 + p_0(f e_0 - 0), \quad (6.22)\]

\[(p_0^2 + \varsigma_0 E_0)(x_2^{\kappa_2} \tilde{\chi}_{0,\alpha})_\alpha = -p_0 X_3 - E_0(f e_0 - 0); \quad (6.23)\]

\[x_2 v_{30,\alpha} + \kappa v_{30,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{-1-\kappa} \left[ -\varsigma_0 X_3 + p_0(f e_0 - 0) \right]; \quad (6.24)\]

\[x_2 \tilde{\chi}_{0,\alpha} + \kappa \tilde{\chi}_{0,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \left[ -p_0 X_3 - E_0(f e_0 - 0) \right]; \quad (6.25)\]

\[x_2 v_{30,\alpha} + (\kappa - 1) v_{30,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \left[ -\varsigma_0 X_3 + p_0(f e_0 - 0) \right]; \quad (6.26)\]

\[(x_2 \tilde{\chi}_{0,\alpha})_\alpha + (\kappa - 1) \tilde{\chi}_{0,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \left[ -p_0 X_3 - E_0(f e_0 - 0) \right]; \quad (6.27)\]

\[\varsigma_0(x_2^{\kappa_2} \tilde{\chi}_{0,\alpha})_\alpha = p_0(x_2^{\kappa_2} v_{30,\alpha})_\alpha - (f e_0 - 0); \quad (6.28)\]

\[(p_0^2 x_2^{\kappa_2} + \varsigma_0 E_0 x_2^{\kappa_2}) v_{30,\alpha} + (\kappa p_0^2 + \kappa_1 \varsigma_0 E_0 x_2^{\kappa_2}) v_{30,2} \]
\[= x_2^{1-\kappa} \left[ p_0(f e_0 - 0) - \varsigma_0 X_3 \right]. \quad (6.30)\]

i.e. (for \(\kappa_1 > \kappa\)
independent of space points i.e. Theorem 6.1.

\( (p_0^2 p_2 + s_0 E_0 x_2^\kappa) v_{30,\alpha} + (\kappa - 1)(p_0^2 + s_0 E_0 x_2^{\kappa-\kappa}) v_{30,2} \)

\( = x_2^{1-\kappa} \left[p_0 (f_0 - D) - s_0 X_3 \right] \) \hspace{1cm} (6.31)

and \((\kappa > \kappa_1)\)

\( (p_0^2 p_2^{\kappa-\kappa_1+1} + s_0 E_0 x_2) v_{30,\alpha} + (\kappa_p^2 p_2^{\kappa-\kappa_1} + \kappa_0 E_0) v_{30,2} \)

\( = x_2^{1-\kappa_1} \left[p_0 (f_0 - D) - s_0 X_3 \right] \) \hspace{1cm} (6.32)

i.e.

\( (p_0^2 p_2^{\kappa-\kappa_1+1} + s_0 E_0 x_2) v_{30,\alpha} + (\kappa - 1)(p_0^2 p_2^{\kappa-\kappa_1} + s_0 E_0) v_{30,2} \)

\( = x_2^{1-\kappa} \left[p_0 (f_0 - D) - s_0 X_3 \right] \) \hspace{1cm} (6.33)

respectively.

Let

\[ E_{2323} = \tilde{E}_0 x_2^{\kappa_1}, \quad h = h_0 x_2^{\kappa_1}, \quad \tilde{E}_0, h_0 = \text{const} > 0; \]

\[ p_{113} = \tilde{p}_0 x_2^{\kappa_2}, \quad \tilde{p}_0 = \text{const} > 0; \quad \tilde{s}_1 = \tilde{s}_0 x_2^{\kappa_3}, \quad \tilde{s}_0 = \text{const} > 0; \]

\[ \kappa_1 = \text{const} > 0; \quad \kappa_2 = \text{const} = \kappa_1^2 + \kappa_2^2 \geq 0; \quad \kappa_3 = const = const > 0. \]

Then

\[ E_{2323} h = E_0 x_2^{\kappa_1}, \quad \text{where } E_0 = \tilde{E}_0 h_0 = \text{const} > 0; \]

\[ p_{113} h = p_0 x_2^{\kappa_2}, \quad \text{where } p_0 = \tilde{p}_0 h_0 = \text{const} > 0; \] \hspace{1cm} (6.34)

\[ s_{11} h = \tilde{s}_0 x_2^{\kappa_3}, \quad \text{where } \tilde{s}_0 = \tilde{s}_0 h_0 = \text{const} > 0. \]

Taking into accounts (6.5), (6.34), from (6.1) it is evident that the following theorem is true.

**Theorem 6.1.** If s.c. elastic, piezoelectric, dielectric constants (coefficients) are independent of space points i.e. \( \kappa_i^1 = 0, \ i = \overline{1,3} \), then we have to do with cusped prismatic shells for \( \kappa_i = \kappa_i^2 > 0, \ i = \overline{1,3} \). In this case peculiarities, arising by setting BCs and those arising by prismatic shells of constant thickness i.e. \( \kappa_i^2 = 0 \) when the elastic, piezoelectric, and dielectric coefficients are changing according to (6.5) i.e. \( \kappa_i = \kappa_i^1 > 0, \ i = \overline{1,3} \), coincide. The stress-strain states coincide as well.

We now are in a position to reformulate results of Section 5 for the matter under consideration in Section 6.

Consequently, we may draw the following conclusions.

**Case 1.** \( \kappa_i = \kappa = \text{const} > 0, \ i = \overline{1,3} \), i.e. \( \kappa = \kappa_1^1 + \kappa_2^2 \), where \( \kappa_i^1 := \kappa_i^1, \ i = \overline{1,3} \).
Theorem 6.2. In the static case \((\theta = 0)\) for system (6.24), (6.25) according to Theorem 5.1 it follows that the values of \(v_{30}\) and \(\tilde{\chi}_0\) should be prescribed on the entire boundary (Problem D) for \(\kappa < 1\), while on the part of the boundary, where \(x_2 = 0\), should be freed of BCs (Problem E) for \(\kappa \geq 1\). Both problems are uniquely solvable in the classical sense.

Theorem 6.3. An \(H\)-weak solution \((v_{30}, \tilde{\chi}_0)\) of Problem E for system (6.12), (6.13) exists under the following conditions:

\[
\kappa > 1 + 2c_0.
\]

Remark 6.1. Similarly to Remark 5.1 we construct in the explicit form solutions of BVPs for system (6.22), (6.23) in the case of the cylindrical strain.

Case 2.) \(\kappa_2 = \kappa_3 = \kappa = \text{const} \geq 0\), i.e., \(\kappa_2^1 = \kappa_3^1\).

Theorem 6.4. For \(\kappa_1 = 0\) and any \(\kappa \geq 0\), the values of \(v_{30}\) (see (6.29)) should be prescribed on the entire boundary (Problem D), while (see (6.28)), according to Theorem 5.1, the values of \(\tilde{\chi}_0\) should be prescribed on the entire boundary (Problem D) for \(0 \leq \kappa < 1\) and the part where \(x_2 = 0\) should be freed of BCs (Problem E) for \(\kappa \geq 1\).

If either \(\kappa_1 \geq \kappa + 1\) or \(\kappa \geq \kappa_1 + 1\), \(\kappa_1 > 1\), we carry out reasonings for systems (6.31), (6.28) and (6.33), (6.28) similar to Section 5 for systems (5.35), (5.31) and (5.38), (5.31).

Remark 6.2. Just as in section 5 the problems D and E we solve in the explicit forms in the case of cylindrical strain (see Remark 5.1 and Remark 5.2). The governing equations obtained from (6.22), (6.23), (6.28), (6.29) have the following forms, correspondingly

\[
(s_0 E_0 + p_0^2)(x_2^0 v_{30,2})_2 = -s_0 X_3 + p_0 (f_{e0} - 0),
\]

\[
(p_0^2 + s_0 E_0)(x_2^0 \tilde{\chi}_{0,2})_2 = -p_0 X_3 - E_0 (f_{e0} - 0);
\]

\[
s_0 (x_2^0 \tilde{\chi}_{0,2})_2 = p_0 (x_2^0 v_{30,2})_2 - (f_{e0} - 0),
\]

\[
((p_0^2 x_2^0 + s_0 E_0 x_2^{\kappa_1}) v_{30,2})_2 = -s_0 X_3 + p_0 (f_{e0} - 0).
\]
\section{\textit{N} = 0 Approximation for Porous Isotropic Elastic Prismatic Shells}

In the case under consideration, assuming the constitutive coefficients \( \lambda \) and \( \mu \) (the Lamé constants), \( \tilde{\alpha} \), \( \tilde{b} \), and \( \tilde{\xi} \) to be constants from (4.11)-(4.14) we get the following governing system

\[
\mu \left[ (hv_{0,0})_{,\alpha} + (hv_{30,0})_{,\alpha} \right] + \lambda (hv_{0,0})_{,\beta} + \tilde{b} (hv_0)_{,\beta} + \frac{\partial}{\partial x_3} = \rho h \ddot{v}_{30}, \quad \beta = 1, 2; \quad (7.1)
\]

\[
\mu (hv_{30,0})_{,\alpha} + \frac{\partial}{\partial x_3} = \rho h \ddot{v}_{30}; \quad (7.2)
\]

\[
\tilde{\alpha} (hv_{0,0})_{,\alpha} - \tilde{b} h v_{0,0} - \tilde{\xi} h \psi_0 + \frac{\partial}{\partial H} = \rho h \ddot{\psi}_0 - F_0. \quad (7.3)
\]

BCs for the weighted displacements and the weighted volume fraction are non-classical in the case of cusped prismatic shells (see Figures 3.2, 3.3). Namely, we are not always able to prescribe them at cusped edges.

Let \( \omega \) be a domain bounded by a sufficiently smooth arc \((\partial \omega \setminus \gamma_0)\) lying in the half-plane \( x_2 > 0 \) and a segment \( \gamma_0 \) of the \( x_1 \)-axis (\( x_2 = 0 \)).

If the thickness looks like

\[
2h(x_1, x_2) = h_0 x_2^\kappa, \quad h_0, \kappa = \text{const} > 0, \quad (7.4)
\]

then we can prescribe the displacements and volume fraction at the cusped edge \( \gamma_0 \) if \( \kappa < 1 \), while we cannot do it if \( \kappa \geq 1 \).

Let us show it for the particular case of deformation when

\[
v_{0,0} \equiv 0, \quad \alpha = 1, 2; \quad v_{30} \neq 0.
\]

Then in the static case, taking into account (7.4), from (7.2), (7.3) we get

\[
x_2 v_{30,0,0} + \kappa v_{30,2} = 2(\mu h_0)^{-1} x_2^{1-\kappa} X_3, \quad (7.5)
\]

\[
x_2 \psi_{0,0,0} + \kappa \psi_{0,2} - \xi \alpha^{-1} x_2 \psi_0 = -2(\tilde{\alpha} h_0)^{-1} x_2^{1-\kappa} \left( \frac{\dot{\psi}_0}{H} + F_0 \right), \quad (7.6)
\]

respectively.

\textbf{Problem D (Dirichlet Problem: Find solutions}

\[
v_{30}, \psi_0 \in C^2(\omega) \cap C(\tilde{\omega})
\]

of (7.5), (7.6) by their values prescribed on \( \partial \omega \)

\textbf{Problem E (Keldysh Problem: Find bounded solutions}

\[
v_{30}, \psi_0 \in C^2(\omega) \cap C(\omega \cup (\partial \omega \setminus \gamma_0))
\]

of (7.5), (7.6) by their values prescribed only on the arc \( \partial \omega \setminus \gamma_0 \)

are uniquely solvable for equations (7.5), (7.6) by \( \kappa_2 < 1 \) and \( \kappa_2 \geq 1 \), correspondingly. It follows from Theorem 5.1. Indeed, from (5.18) and (5.19), it follows

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\( a_2(x_1, x_2) = \kappa < 1 \) for Problem D and \( a_2(x_1, x_2) = \kappa \geq 1 \) for Problem E, respectively, since \( \kappa_1 = \kappa_2 = 1 \).

To the general system (7.1)-(7.3) in the static and dynamical (time-harmonic motion) cases we apply results obtained for the more general system (see Section 8).

### 8 Systems of Elliptic Equations of the Second Order with an Order Degeneracy

Let \( G \subset \mathbb{R}^n, n \geq 2 \), be a bounded domain, \( x := (x_1, ..., x_n), \Gamma := \partial G \), and

\[
Lu := (A^{ij}u_{,i})_{,j} + E^i u_{,i} + Cu = F, \quad (8.1)
\]

i.e.,

\[
L_k u := (a^{ij}_{kl}u_{ ,i})_{,j} + e^i_{kl}u_{,i} + c_{kl}u_{,l} = F_k, \quad k = \overline{1,m},
\]

where

\[
u = (u_1, ..., u_m)\trans, \quad F := (F_1, ..., F_m)\trans, \quad A^{ij} := \|a_{kl}^{ij}\|, \quad E^i := \|e_{kl}^i\| \in C^1(\bar{G}); \quad (8.2)
\]

\[
C := \|c_{kl}\| \in C(\bar{G}), \quad i,j = \overline{1,n}, \quad k, l = \overline{1,m},
\]

where we use the Einstein convention, and indices after a comma, as usually., mean differentiation with respect to the corresponding variables.

Let

\[
\xi^{(i)} A^{(j)} \xi > 0, \quad x \in \bar{G} \setminus \Gamma^0, \quad (8.3)
\]

for all \( \xi^{(i)} := (\xi_1^{(i)}, ..., \xi_m^{(i)}), i = \overline{1,n} \), such that

\[
\sum_{i=1}^n \sum_{k=1}^m (\xi_k^{(i)})^2 > 0,
\]

where

\[
\Gamma^0 := \{ x \in \Gamma : A^{ij}(x) = 0, \quad i,j = \overline{1,n} \}. \quad (8.4)
\]

So, the system (8.1) is strongly elliptic on \( \bar{G} \setminus \Gamma^0 \) with the equations’ order degeneracy on \( \Gamma^0 \).

Let

\[
\Gamma = \bigcup_{k=1}^{p} \Gamma^{(k)},
\]

where \( \Gamma^{(k)}, \quad k = \overline{1,p} \), are smooth hypersurfaces with possibly common boundary points, and let the Gauss-Ostrogradsky formula be applicable to the domain \( G \).

At points of smoothness of \( \Gamma^0 \) let us consider matrix

\[
\Phi := E^i \nu_i, \quad (8.5)
\]
where $\nu := (\nu_1, ..., \nu_n)$ is the inward normal at the above boundary points. Let further

$$
\Gamma_0 := \{ x \in \Gamma^0 : \Phi(x) = 0 \}, \quad \Gamma_1 := \{ x \in \Gamma^0 : \Phi(x) > 0 \},
$$

$\Gamma_2 := \{ x \in \Gamma^0 : \Phi(x) < 0 \}, \quad \Gamma_3 : \Gamma \setminus \Gamma^0.
$$

(8.6)

Under matrix inequalities we mean inequalities for the corresponding quadratic forms on vectors with nonzero length.

Let $x_0 \in \Gamma^0$ be a common point of the hypersurfaces $\Gamma^{(i_1)}, ..., \Gamma^{(i_q)}$ for certain $i_1, ..., i_q \in \{1, ..., p\}$, $q = \frac{p(1-p)}{2}$, $i_k \neq i_q$, $k \neq l$. If there exists such a neighbourhood $\omega_{\hat{i}}$, $\hat{i} := (i_1, ..., i_q)$, of the point $x_0$, that

$$
(\omega_{\hat{i}} \cap \bigcup_{l=1}^{q} \Gamma^{(i_l)}) \subset (\Gamma_2 \cup \Gamma_3),
$$

then the above point will be added to $\Gamma_2 \cup \Gamma_3$, otherwise it will be added to $\Gamma_0 \cup \Gamma_1$.

Let further $\Gamma^0 = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ and

$$
u|_{\Gamma_2 \cup \Gamma_3} = 0.
$$

(8.7)

**Definition 8.1.** Let $C_L$ be the class of bounded vectors $u$ such that

$$
u \in C^2(G) \cap C(G \cup \Gamma_2 \cup \Gamma_3),$$

$$A^{ij}u_{,i}, \ A^{ij}u, \ E^i u \in C^1(\bar{G}),$$

$$(A^{ij}u_{,i})|_{\Gamma^0} = 0, \ Lu \text{ be bounded on } G.
$$

(8.8)

**Definition 8.2.** A vector $u \in C_L$ satisfying system (8.1) and BC (8.7) will be called a regular solution of the BVP (8.1), (8.7).

**Definition 8.3.** A vector $u \in L_2(\Gamma)$ will be called an $L_2(G)$-weak solution of the BVP (8.1), (8.7) if $F \in L_2(G)$ and

$$
\int_G v F dx = \int_G L^* v \cdot u dx
$$

(8.9)

is valid for any $v \in C_{L^*}$, satisfying the condition

$$
v|_{\Gamma_1 \cup \Gamma_3} = 0,
$$

(8.10)

(the space of such vectors $v$ will be denoted by $V$, if for $v$ and $u \in C_L$ with (8.7) equality (8.9) is valid), $L^*$ is the adjoint operator to $L$:

$$
L^* v := (v_{,i} A^{ij})_{,j} - (v E^i)_{,i} + vC,
$$

i.e.,

$$
L^*_l v := (v_{k,l} A^{ij}_{kl})_{,j} - (v_{k,l} E^i_{kl})_{,i} + v_{k,l} C_{kl}, \quad l = \Gamma_{\Gamma},
$$

and $C_{L^*}$ is defined for the operator $L^*$ like $C_L$ for the operator $L$. 

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It is easily seen that if there exists a regular solution of problem (8.1), (8.7), it will be also a weak solution. Indeed, if \( v \in C^2(G) \) and \( u \in C_L \), then we have the following Green formula

\[
\int_G v L u \, dx - \int_G L^* v \cdot u \, dx = \int_{\Gamma} [ (v, A^{ij} u - v A^{ij} u_j) \nu_j - v \Phi u] \, d\Gamma , \quad A^{ij} = A^{ji} . \tag{8.11}
\]

Presenting \( \Gamma \) as \( \Gamma_3 \cup \Gamma_0 \), then the first sum under the integral will be equal to zero on \( \Gamma_3 \) because of (8.7) and on \( \Gamma_0 \) because of (8.4). The second sum will be equal to zero on \( \Gamma_0 \), by virtue of (8.8), and on \( \Gamma_3 \), by virtue of (8.10). We now present \( \Gamma \) as \( \Gamma_3 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_0 \), then on \( \Gamma_3 \cup \Gamma_1 \cup \Gamma_2 \) the third summand becomes zero because of (8.7), (8.10) and on \( \Gamma_0 \) because of (8.6). Thus, the right hand side is zero and, since \( Lu = F \), from (8.11) it follows (8.9).

**Theorem 8.1.** Let \( A^{ij} = A^{ji} \), let the matrices \( E^i, i = 1, n, \) be symmetric, and

\[
E^i, i - 2C \geq 0, \quad x \in G , \quad (8.12)
\]

then homogeneous BVP corresponding to the BVP (8.1), (8.7) has only the trivial solution in \( C_L \) if

\[
u A^{ij} u, j + \frac{1}{2} u E^i u \in C^1(G), \quad \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \emptyset .
\]

**Proof.** Integrating the equality (see [24])

\[
u L u + u, i A^{ij} u, j + \frac{1}{2} u (E^i, i - 2C) u = (u A^{ij} u, j + \frac{1}{2} u E^i u), i \tag{8.13}
\]

on \( G \) and using the Gauss-Ostrogradsky formula, by virtue of (8.6)-(8.8) and \( Lu = 0 \), we obtain

\[
\int_G [ u, i A^{ij} u, j + \frac{u}{2} (E^i, i - 2C) u ] \, dG + \frac{1}{2} \int_{\Gamma_1} u \Phi u \, d\Gamma_1 = 0.
\]

Hence, in view of (8.3), (8.12), and \( \Phi > 0 \) on \( \Gamma_1 \), we arrive at

\[
u, i A^{ij} u, j = 0, \quad x \in G .
\]

But, according to (8.3), it is admissible only if

\[
u, i = 0, \quad i = 1, n, \quad x \in G .
\]

Therefore, \( u = \text{const} \) and if \( \Gamma_2 \cup \Gamma_3 \neq \emptyset \), then by virtue of (8.7),

\[
u = 0 . \tag{8.14}
\]

If \( \Gamma_2 \cup \Gamma_3 = \emptyset \) but \( \Gamma_1 \neq \emptyset \), then (8.14) is valid since

\[
u|_{\Gamma_1} = 0
\]

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because of
\[ u\Phi|_{\Gamma_1} = 0 \text{ and } \Phi|_{\Gamma_1} > 0. \]
If
\[ \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \emptyset, \]
but inequality (8.12) is strong, then it is easily seen that
\[ u(E_i - 2C)u = 0, \quad x \in G. \]
Therefore, (8.14) is valid.

\textbf{Theorem 8.2.} Let \( A^{ij} = A^{ji} \), let the matrices \( E_i \), \( i = 1, n \), be symmetric, and
\[ \xi(E_i - 2C)\xi \geq c_0 \sum_{k=1}^{m} \xi_k^2, \quad x \in G, \quad c_0 = \text{const} > 0, \quad \xi := (\xi_1, ..., \xi_m), \quad (8.15) \]
then there exists a weak solution of BVP (8.1), (8.7).

\textit{Proof.} According to the Abstract Existence Principle (G. Fichera, see Theorem 8.3 below and [15], and also [16]) it follows that for the existence of a weak solution the fulfilment of the following inequality is sufficient
\[ \|v\|_2 \leq K\|L^*v\|_2, \quad (8.16) \]
where
\[ \quad K = \text{const}, \quad \|v\|_2 := \left( \int_G \sum_{k=1}^{m} v_k^2 dG \right)^{\frac{1}{2}}. \]
Indeed, let \( V \) be an abstract linear manifold of the real field; let \( B_\alpha, \alpha = 1, 2, \) be real Banach spaces; let \( M_\alpha, \alpha = 1, 2, \) be linear homomorphisms of \( V \) into \( B_\alpha \); let \( F \) and \( u \) be vectors of conjugate spaces \( B_1^* \) and \( B_2^* \), respectively. Let us consider for any \( v \in V \) the following functional equation
\[ \langle F, M_1(v) \rangle = \langle u, M_2(v) \rangle \quad (8.17) \]
where \( F \) is a given, and \( u \) is an unknown vector. We denote: by \( V_2 \) a kernel of the homomorphism \( M_2 \)
\[ \quad V_2 := \{ v \in V : M_2(v) = 0 \}; \]
by \( M_1(V_2) \) an image of \( V_2 \) on \( B_1 \) by homomorphism \( M_1 \); and by \( \overline{M_1(V_2)} \) its closure.
Let us consider a factor space
\[ Q := B_1/\overline{M_1(V_2)}. \]
Let \( M_1 \) be a homomorphism, which maps \( v \in V \) on a class of equivalence \([M_1(v)]\) of the factor space \( Q \). A given vector \( F \) should fulfil the following necessary condition
\[ \langle F, M_1(v_2) \rangle = 0, \quad v_2 \in V_2. \quad (8.18) \]
The following existence principle holds:
Theorem 8.3 (Abstract Existence Principle (G. Fichera)). A solution \( u \) of functional equation (8.17) exists for any fixed \( F \) satisfying the condition (8.18) if and only if there exists a constant \( K \) such that for any \( v \in V \) the inequality

\[
\|M_1(v)\|_Q \leq K\|M_2(v)\|_{B_2}
\]  

(8.19)

holds.

Let \( A \) be a closed subspace of the space \( B_2^* \), consisting of the vectors \( u_0 \), solutions of the “homogeneous” problem

\[
<u_0, M_2(v)> = 0, \quad v \in \mathcal{V}.
\]

We denote by \( \mathcal{F} \) the Banach factor space \( \mathcal{F} := B_2^*/A \). For any \( F \in B_1^* \), satisfying the compatibility conditions (8.18) (that is to say for any element of the adjoint space \( Q^* \)), there exists uniquely defined \( U \in \mathcal{F} \) such that, if \( u \) is any element in the equivalence class \( U \), then \( u \) is a solution of equation (8.17) with

\[
\|U\|_{\mathcal{F}} \leq K\|F\|_{B_1^*}.
\]

(8.20)

Inequality (8.20) is said to be a dual inequality of (8.19).

In our case under consideration instead of the functional equation (8.17) we have equation (8.9), \( V \equiv \mathcal{V} \), \( M_1 \) is an identical operator, \( M_2 \equiv L^* \), \( B_1 \equiv B_2 \equiv B_2^* \equiv L_2(G) \), \( Q \equiv L_2(\omega) \backslash \bar{V}_2 \),

where

\[
V_2 := \{ v \in V : L^*(v) = 0 \},
\]

\[
\|M_1(v)\|_Q \equiv \|[M_1(v)]\|_Q \equiv \|[v]\|_Q := \inf_{v_2 \in V_2} \|v + v_2\|_2,
\]

\[
\mathcal{F} \equiv L_2/A, \quad \mathcal{A} := \{ u_0 \in L_2(\omega) : (u_0, L^*v) = 0 \};
\]

\[
\|U\|_\mathcal{F} := \inf_{u_0 \in \mathcal{A}} \|u + u_0\|_2,
\]

and Fichera’s Abstract Existence Principle takes the form:

**Theorem 8.4.** A solution \( u \in L_2(G) \) of the functional equation (8.9) exists for any \( F \in L_2(G) \), satisfying condition

\[
\int_G Fv_2dx = 0, \quad v_2 \in V_2,
\]

(8.21)

if and only if there exists a constant \( K \) such that for any \( v \in V \) the inequality

\[
\|[v]\|_Q = \inf_{v_2 \in V_2} \|v + v_2\|_2 \leq K\|L^*(v)\|_2,
\]

(8.22)

is valid; moreover

\[
\|U\|_\mathcal{F} = \inf_{u_0 \in \mathcal{A}} \|u + u_0\|_2 \leq K\|F\|_2.
\]
Let now (8.16) be valid, then from

\[ L^*(v_2) = 0 \]

it follows that \( v_2 \equiv 0 \), i.e., \( V_2 \equiv \{0\} \). Therefore,

\[ \inf_{v_2 \in V_2} \|v + v_2\|_2 = \|v\|_2, \]

and (8.22) holds, by virtue of (8.16). Hence, there exists a solution of the functional equation (8.9), i.e., a weak solution of problem (8.1), (8.7). It is clear, that in this case condition (8.21) is fulfilled for any \( F \in L_2(G) \).

In order to establish equality (8.16) we write identity (8.13) for the operator \( L^* \):

\[ L^*v + v_i A_{ij} v_j + \frac{v}{2}(E_i^i - 2C)v = (v_j A_{ij} v - \frac{v}{2} E_i v)_i. \]

After integrating the last equality on \( G \) and using the Gauss-Ostrogradsky formula, by virtue of (8.4), (8.5), (8.10) we have

\[ \int_Q L^*v \cdot vdx + \int_G [v_i A_{ij} v_j + \frac{v}{2}(E_i^i - 2C)v]dG = \frac{1}{2} \int_{\Gamma_2} v\Phi v d\Gamma_2 = 0. \]

Therefore, in view of (8.3), (8.6), (8.15), we have

\[ -\int_G L^*v \cdot vdx \geq \frac{1}{2} \int_G v(E_i^i - 2C)v dG \geq \frac{c}{G} \sum_{k=1}^{m} v_k^2 dG = \|v\|_2^2, \quad c = \text{const}. \]

On the other hand, applying the Hölder inequality, we get

\[ \int_G (-L^*v)vdx \leq \| - L^*v\|_2 \cdot \|v\|_2 = \|L^*v\|_2 \cdot \|v\|_2. \]

Hence,

\[ c \|v\|_2^2 \leq \|L^*v\|_2 \cdot \|v\|_2, \]

i.e, (8.16) is valid.

\[ \square \]

**Remark 8.1.** The system with nonsmooth coefficients (they are infinite on a subset of \( \Gamma^0 \)) we reduce to the present case by means of smoothing factors.

**Remark 8.2.** Let us introduce new unknown functions \( \omega_i, \ i = 1, m \), by relations

\[ u_i = \tilde{\psi} w_i, \quad i = 1, m, \quad \tilde{\psi} \in C^2(\overline{G}), \quad \tilde{\psi} > 0 \text{ on } \overline{G}. \]

(8.23)

Note, that the matrix \( \Phi \) is invariant to change of unknown functions (8.23), in the sense of change of sign since for the system obtained

\[ \hat{L}_w := (\hat{A}^{ij} w_{,i})_{,j} + \hat{E}^i w_i + \hat{C} w = F, \quad w = (w_1 \cdots w_m), \]

(8.24)

where

\[ \hat{A}^{ij} = \tilde{\psi} A^{ij}, \quad \hat{E}^i = \tilde{\psi} E^i + \tilde{\psi}_j A^{ij}, \quad \hat{C} = \tilde{\psi} C, \]

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by virtue of (8.4), it looks like
\[ \Phi = \hat{E}^i u_i = (\tilde{\psi} E^i + \tilde{\psi}_j A^{ij}) u_i = \tilde{\psi} E^i u_i = \tilde{\psi} \Phi. \]
If the function \( \psi \) is such that for the system (8.24) the condition (8.15) is valid, then under assumptions of Theorem 8.2 there exists a weak solution of the problem
\[ \hat{L} w = F, \quad w|_{\Gamma_2 \cup \Gamma_3} = 0, \]
i.e.,
\[ (v, F) = (\hat{L}^* v, w), \]
where \( \hat{L}^* \) is the operator conjugate to the operator \( \hat{L} \). But
\[ (\hat{L}^* v, w) = (\tilde{\psi}^{-1} \hat{L}^* v, u) = (\tilde{\psi}^{-1} \hat{L}^* v, u), \]
since if \( A^{ij} = A^{ji} \), then
\[ \tilde{\psi}^{-1} \hat{L}^* v = \hat{L}^* v. \]
Thus,
\[ (v, F) = (L^* v, u), \]
i.e, a weak solution of problem (8.1), (8.7) exists and according to (8.23), we get it, from the weak solution of problem (8.25).

9 The \( \mathcal{H} \)-weak Solution for a Single Equation

For the convenience of the reader we repeat the relevant material from [15] (compare with [16] Chapter I, §4) with proofs in a slightly changed form, thus making our exposition of the present work self-contained.

Let \( G \subset \mathbb{R}^n \) be a domain, \( \Gamma := \partial G \), and
\[ Lu := (A^{ij} u_{,i})_{,j} + E^i u_{,i} + Cu = F, \quad (9.1) \]
where real functions
\[ A^{ij}, \quad E^i \in C^1(\bar{G}); \quad C \in C(\bar{G}), \quad i, j = \overline{1,n}. \]
(9.2)
As usual we use the Einstein’s summation convention and indices after a comma mean differentiation with respect to the corresponding variables.

Let
\[ \xi_i A^{ij} \xi_j > 0, \quad x \in \bar{G} \setminus \Gamma^0, \quad (9.3) \]
for any \( \xi_i \in \mathbb{R}^1 \), \( i = \overline{1,n} \), such that
\[ \sum_{i=1}^n \xi_i^2 > 0, \]
where
\[ \Gamma^0 := \{ x \in \Gamma : A^{ij}(x) = 0, \quad i, j = \overline{1,n} \}. \]
(9.4)
So, equation (9.1) is elliptic on \( \bar{G} \setminus \Gamma^0 \) with order degeneracy on \( \Gamma^0 \).
Let
\[ \Gamma = \bigcup_{k=1}^{p} \Gamma^{(k)}, \]
where \( \Gamma^{(k)} \), \( k = \Gamma, p \), are smooth hypersurfaces with possibly common boundary points, and to the domain \( G \) the Gauss-Ostrogradsky formula is applicable.

In points of smoothness of \( \Gamma^0 \) let us consider the function, which we shall call Fichera’s function,
\[ \Phi := E^i \nu_i, \quad (9.5) \]
where \( \nu := (\nu_1, ..., \nu_n) \) is the inward normal at the above boundary points. Let further
\[ \Gamma_0 := \{ x \in \Gamma^0 : \Phi(x) = 0 \}, \quad \Gamma_1 := \{ x \in \Gamma^0 : \Phi(x) > 0 \}, \]
\[ \Gamma_2 := \{ x \in \Gamma^0 : \Phi(x) < 0 \}, \quad \Gamma_3 : \Gamma \setminus \Gamma^0. \quad (9.6) \]

Let \( x_0 \in \Gamma^0 \) be a common point of the hypersurfaces \( \Gamma^{(i_1)}, ..., \Gamma^{(i_q)} \) for certain \( i_1, ..., i_q \in \{1, ..., p\}, 2 \leq q \leq p \), \( i_k \neq i_q, k \neq l \). If there exists a neighbourhood \( \omega_i, i := (i_1, ..., i_q) \), of the point \( x_0 \), such that
\[ (\omega_i \cap \bigcup_{l=1}^{q} \Gamma^{(i_l)}) \subset (\Gamma_2 \cup \Gamma_3), \]
then the above point will be added to \( \Gamma_2 \cup \Gamma_3 \), otherwise it will be added to \( \Gamma_0 \cup \Gamma_1 \).

Let further \( \Gamma^0 = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \),
\[ u|_{\Gamma_2 \cup \Gamma_3} = 0, \quad (9.7) \]

**Definition 9.1.** Let \( C_L \) be the class of bounded functions \( u \) such that
\[ u \in C^2(G) \cap C(G \cup \Gamma_2 \cup \Gamma_3), \]
\[ A^{ij}u, \ A^{ij}u, \ E^i u \in C^1(\bar{G}), \]
\[ (A^{ij}u, i)|_{\Gamma^0} = 0, \quad Lu \text{ be bounded in } G. \quad (9.8) \]

**Definition 9.2.** A function \( u \in C_L \) satisfying system (9.1) and BC (9.7) will be called a regular solution of the BVP (9.1), (9.7).

For a function \( u \in C_L \) and a function \( v \in C^1(\bar{G}) \) the integral identity
\[ \int_G vLudx = - \int_G [A^{ij}v, j u, i + uE^i v, i + (E, i - C)uv]dx \]
\[ - \int_{\Gamma_3} vA^{ij}u, i n, j d\Gamma - \int_{\Gamma} uv \Phi d\Gamma, \quad (9.9) \]
where \( n \) is an inward normal, holds.
Let $W$ be the class of functions belonging to $C^1(\bar{G})$ and vanishing on $\Gamma_3$ (when not empty). If $u \in C_L$ and vanishes a.e. on $\Gamma_2 \cup \Gamma_3$, then for any $v \in W$ the identity

$$\int_G vLudx = -\int_G [A^{ij}v_{,j}u_{,i} + uE^i v_{,i} + (E_{,i} - C)uv]dx - \int_{\Gamma_1} uv\Phi d\Gamma.$$  \hspace{1cm} (9.10)

is satisfied (see (9.9)).

Let us introduce a scalar product in $W$ in the following way

$$\langle u, v \rangle_H = \int_G (A^{ij}u_{,i}v_{,j} + uv)dx + \int_{\Gamma_1 \cup \Gamma_2} uv|\Phi|d\Gamma.$$  \hspace{1cm} (9.11)

The space $H$ will be the Hilbert space, obtained by functional completion from $W$ with the introduced scalar product.

Let us consider for $u, v \in W$ the bilinear form

$$B(u, v) = -\int_G [A^{ij}v_{,j}u_{,i} + uE^i v_{,i} + (E_{,i} - C)uv]dx - \int_{\Gamma_1} uv\Phi d\Gamma.$$  \hspace{1cm} (9.12)

It is easily seen that

$$|B(u, v)| \leq K \left( \int_G |\text{grad}v|^2 + v^2 dx + \int_{\Gamma_1} |v|^2 d\Gamma \right)^{\frac{1}{2}} \|u\|_H,$$

where $K$ is a constant, depending on the coefficients of $L$. For any fixed $v \in W$, $B(u, v)$ can be considered as a linear bounded functional of $u$, defined on $H$.

**Definition 9.3.** A function $u \in H$ will be called an $H$-weak solution of the BVP (9.1), (9.7) if $F \in L^2(G)$ and

$$\int_G vFdx = B(u, v)$$  \hspace{1cm} (9.13)

is valid for any $v \in W$.

Let $V$ (see Section 8 and take $m = 1$, then we get corresponding to equation (9.1) results) coincide with the class of functions $C_L^*$ satisfying a.e. the BC

$$v|_{\Gamma_1 \cup \Gamma_3} = 0,$$  \hspace{1cm} (9.14)

(the space of such vectors $v$ will be denoted by $V$, if for $v$ and $u \in C_L$ with (9.7) equality (9.12) is valid), $L^*$ is the adjoint operator to $L$:

$$L^* v := (v_{,i} A^{ij})_{,j} - (vE^i)_{,i} + vC,$$

then any $H$-weak solution is an $L^*$-weak solution.

According to the representation theorem of linear functionals in Hilbert spaces, we have for $u \in H, v \in W$:

$$B(u, v) = \langle u, T(v) \rangle_H,$$

where $T(v)$ is a linear transformation defined in $W$ and with range in $H$.  

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For $u \in W$, and $v \in W$, from (9.11) we have

$$B(u, v) = -\int_G \left[ A_{ij} v_{,j} u_{,i} + \frac{1}{2} u E^i v_{,i} - \frac{1}{2} v E^i u_{,i} + \left( \frac{1}{2} E^i_{,i} - C \right) uv \right] dx$$

$$- \frac{1}{2} \int_{\Gamma_1} uv \Phi d\Gamma + \frac{1}{2} \int_{\Gamma_2} uv \Phi d\Gamma,$$

(9.14)

since

$$\int_G u E^i v_{,i} dx = \frac{1}{2} \int_G u E^i v_{,i} dx + \frac{1}{2} \int_G u E^i v_{,i} dx$$

$$= \frac{1}{2} \int_G \left[ u E^i v_{,i} + \frac{1}{2} \left( u E^i (v)_{,i} - \frac{1}{2} u_{,i} E^i v - \frac{1}{2} u E^i v \right) \right] dx$$

$$= \frac{1}{2} \int_G u E^i v_{,i} dx - \frac{1}{2} \int_{\Gamma_1 \cup \Gamma_2} u \Phi v d\Gamma - \frac{1}{2} \int_G u_{,i} E^i v dx - \frac{1}{2} \int_G u E^i_{,i} v dx.$$

Let us suppose that

$$\frac{1}{2} E^i_{,i} - C > c_0 = const > 0 \text{ in } G \cup \Gamma.$$  (9.15)

This condition is satisfied if we assume $C$ negative and $|C|$ large enough. If (9.15) is satisfied we easily get from (9.14) for $v \in W$:

$$|B(v, v)| = \int_G \left[ A_{ij} v_{,j} v_{,i} + \left( \frac{1}{2} E^i_{,i} - C \right) v^2 \right] dx + \frac{1}{2} \int_{\Gamma_1} v^2 \Phi d\Gamma$$

$$+ \frac{1}{2} \int_{\Gamma_2} v^2 \Phi d\Gamma \geq \lambda_0 \| v \|_{\mathcal{H}}^2 \quad (\lambda_0 > 0).$$

Therefore,

$$\| v \|_{\mathcal{H}}^2 \leq \frac{1}{\lambda_0} |B(v, v)| \leq \frac{1}{\lambda_0} \| (v, T(v)) \| \| T(v) \|_{\mathcal{H}},$$

i.e.,

$$(\int_G v^2 dx)^{\frac{1}{2}} \leq \| v \|_{\mathcal{H}} \leq \frac{1}{\lambda_0} \| T(v) \|_{\mathcal{H}}.$$  (9.16)

Hence, from Fichera’s Abstract Existence Principle (see Theorem 8.3) we deduce:

**Theorem 9.1.** If condition (9.15) is satisfied, for any $F \in L_2(G)$ an $\mathcal{H}$-weak solution of problem (9.1), (9.7) exists.

The uniqueness of the $\mathcal{H}$-weak solution is connected with the continuity of the bilinear form $B(u, v)$ with respect to the pair $(u, v)$. When this is the case then, since $B(u, v)$ can be extended by continuity in $\mathcal{H} \times \mathcal{H}$, from (9.16) uniqueness of the $\mathcal{H}$-weak solution follows easily.

**Remark 9.1.** $B(u, v)$ is continuous with respect to the pair $(u, v)$ in the case when $L$ is self-adjoint, i.e., when $E^v = 0$. This is easily seen from (9.14). In this case, if $C$ is negative in $G \cup \Gamma$, an $\mathcal{H}$-weak solution exists for any given $f \in L_2(G)$ and is unique.
10 Mathematical Moments

Let \( f(x_1, x_2, x_3) \) be a given function in \( \Omega \) having integrable partial derivatives, let \( f_r \) be its \( r \)-th order moment defined as follows

\[
f_r(x_1, x_2) := \int f(x_1, x_2, x_3) P_r(ax_3 - b)\, dx_3,
\]

where (see Section 3)

\[
a(x_1, x_2) := \frac{1}{h(x_1, x_2)}, \quad b(x_1, x_2) := \frac{\hat{h}(x_1, x_2)}{h(x_1, x_2)},
\]

\[
2h(x_1, x_2) = (+) h(x_1, x_2) - (-) h(x_1, x_2) > 0,
\]

\[
2\hat{h}(x_1, x_2) = (+) h(x_1, x_2) + (-) h(x_1, x_2) > 0,
\]

and

\[
P_r(\tau) = \frac{1}{2^r r!} \frac{d^r (\tau^2 - 1)^r}{d\tau^r}, \quad r = 0, 1, \ldots,
\]

are the \( r \)-th order Legendre polynomials with the orthogonality property

\[
\int_{-1}^{+1} P_m(\tau) P_n(\tau)\, d\tau = \frac{2}{2m + 1} \delta_{mn}.
\]

From here, substituting

\[
\tau = ax_3 - b = \frac{2}{h(x_1, x_2)} x_3 - \frac{(+)}{h(x_1, x_2)} h(x_1, x_2) - \frac{(-)}{h(x_1, x_2)} h(x_1, x_2),
\]

we have

\[
(m + \frac{1}{2}) a \int_{h(x_1, x_2)} P_m(ax_3 - b) P_n(ax_3 - b)\, dx_3 = \delta_{mn}.
\]

Using the well-known formulas of integration by parts (with respect to \( x_3 \)) and differentiation with respect to a parameter of integrals depending on parameters \( (x_\alpha) \), taking into account \( P_r(1) = 1, P_r(-1) = (-1)^r \), we deduce

\[
\int\limits_{+}^{(-)} h(x_1, x_2) P_r(ax_3 - b) f_{,3}\, dx_3 = -a \int\limits_{+}^{(-)} h(x_1, x_2) P_r(ax_3 - b) f\, dx_3 + \frac{(+)}{(-)} f - (-1)^r f, \quad (10.1)
\]

\[
\int\limits_{+}^{(-)} h(x_1, x_2) P_r(ax_3 - b) f_{,3}\, dx_3 = -a \int\limits_{+}^{(-)} h(x_1, x_2) P_r(ax_3 - b) f\, dx_3 + \frac{(+)}{(-)} f - (-1)^r f, \quad (10.1)
\]
that a, and in view of corresponding variables, \( \frac{\partial}{\partial x} \) where superscript prime means differentiation with respect to the argument \( ax_3 - b \), subscripts preceded by a comma mean partial derivatives with respect to the variables.

where superscript prime means differentiation with respect to the argument \( ax_3 - b \), subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables, \( f := f(x_1, x_2, h(x_1, x_2)) \). Applying the following relations from the theory of the Legendre polynomials (see e.g. [25], p. 299 or p. 338-339 of the second edition)

\[
P_r'(\tau) = \sum_{s=0}^{r} (2s + 1) \frac{1}{2} P_s(\tau)^{ii},
\]

\[
\tau P_r'(\tau) = r P_r(\tau) + P_{r-1}'(\tau) = r P_r(\tau) + \sum_{s=0}^{r-1} (2s + 1) \frac{1}{2} P_s(\tau)^{ii} \tag{10.3}
\]

and, in view of \( \frac{a_a}{h_a} = (\ln a)' = \frac{h_{a\alpha}}{h} \), \( a_{a\alpha} b = \tilde{h}_{a\alpha} \), \( b_{\alpha\alpha} = (\tilde{h} a\alpha)_{\alpha} \), it is easily seen that

\[
P_r'(ax_3 - b)(a_{\alpha\alpha} x_3 - b_{\alpha\alpha}) = \frac{a_{a\alpha}}{h} (ax_3 - b) P_r'(ax_3 - b) + (\frac{a_{a\alpha} b - b_{\alpha\alpha}}{h}) P_r'(ax_3 - b)
\]

\[
= -h_{a\alpha} h^{-1}(ax_3 - b) P_r'(ax_3 - b) - \tilde{h}_{a\alpha} h^{-1} P_r'(ax_3 - b)
\]

\[
= -a_{a\alpha} x_3 - b - \sum_{s=0}^{r-1} a_{a\alpha} P_s(ax_3 - b)^{iv}, \tag{10.4}
\]

ii on the top of the symbol \( \sum \) both \( r - 1 \) and \( r \) are true since the last term equals zero.

iii on the top of the symbol \( \sum \) both \( r - 2 \) and \( r - 1 \) are true since the last term equals zero.

iv since

\[
\sum_{s=0}^{r-1} (2s + 1) \left[ \frac{h_{a\alpha} + (-1)^{r+s} h_{a\alpha}}{2h} + \frac{\tilde{h}_{a\alpha} - (-1)^{r+s} h_{a\alpha}}{2h} \right] P_s(ax_3 - b)
\]

\[
= \sum_{s=0}^{r-1} \frac{(2s + 1) (h_{a\alpha} + (-1)^{r+s} h_{a\alpha})}{2h} + \frac{\tilde{h}_{a\alpha} - (-1)^{r+s} h_{a\alpha}}{2h} \right) P_s(ax_3 - b)
\]

\[
= \sum_{s=0}^{r-1} \frac{(2s + 1) (h_{a\alpha} + (-1)^{r+s} h_{a\alpha})}{2h} P_s(ax_3 - b)
\]

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where
\[ r_{αs} := \frac{h_α}{h}, \quad a_{αs} := (2s + 1)\frac{(+)^{-r} + (-)^{(r+s)}}{2h}, \quad s \neq r. \]

Now, bearing in mind (10.4) and (10.3), from (10.1) and (10.2) we have
\[
\begin{align*}
\int_{(+)}^h (x_1, x_2) \left( P_r(ax_3 - b)f_{+, \alpha} \right) dx_3 \\
\int_{(-)}^h (x_1, x_2) \left( P_r(ax_3 - b)f_{-, \alpha} \right) dx_3 &= f_{+, \alpha} + \sum_{s=0}^r a_{αs} f_s - f_{-, \alpha} + (-1)^r \left( (-)^{r}\frac{(-)^{s} + (-1)^{r}}{2h}, \quad α = 1, 2, \right. (10.5) \\
\int_{(+)}^h (x_1, x_2) \left( P_r(ax_3 - b)f_{+, \alpha} \right) dx_3 &= \sum_{s=0}^r a_{3s} f_s + f_{-, \alpha} - (-1)^r f_{-, \alpha}, \quad (10.6)
\end{align*}
\]
respectively. Here
\[ a_{3s} := -(2s + 1)\frac{1 - (-1)^{s} + r}{2h}. \]

Let
\[ f(x_1, x_2, x_3) = \sum_{r=0}^∞ \left( r + \frac{1}{2} \right) f_r(x_1, x_2)P_r(ax_3 - b), \] (10.7)
then
\[
\begin{align*}
\int_{(+)}^h (x_1, x_2) \left( P_r(ax_3 - b)f_{+, \alpha} \right) dx_3 &= \sum_{s=0}^∞ \left( s + \frac{1}{2} \right) f_s(±1)^s \\
&= \sum_{s=0}^{∞} (±1)^s(2s + 1)\frac{f_s}{2h}, \quad i = 1, 3, (10.8)
\end{align*}
\]
whence
\[
\begin{align*}
&f_{+, \alpha} - (-1)^r f_{-, \alpha} = -\sum_{s=0}^∞ a_{3s} f_s, \quad i = 1, 3, \quad (10.9) \\
&f_{+, \alpha} - (-1)^r f_{-, \alpha} = \sum_{s=0}^∞ a_{αs} f_s, \quad i = 1, 3, \quad α = 1, 2, (10.10)
\end{align*}
\]
where
\[ a_{αs} = a_{αs}, \quad s \neq r, \quad a_{αr} = (2r + 1)\frac{h_α}{h}. \]

Substituting (10.10) and (10.9) into (10.5) and (10.6), respectively, we get
\[
\begin{align*}
\int_{(+)}^h (x_1, x_2) \left( P_r(ax_3 - b)f_{+, \alpha} \right) dx_3 &= f_{+, \alpha} + \sum_{s=0}^r a_{αs} f_s - \sum_{s=0}^∞ a_{αs} f_s \\
&= f_{+, \alpha} + \sum_{s=0}^∞ b_{αs} f_s, \quad (10.11)
\end{align*}
\]
where
\[ r_{js} := -a_{js}, \quad s > r; \quad r_{js} = 0, \quad s < r; \]
\[ r_{\alpha r} := a_{\alpha r} - r_{\alpha r} = -(r + 1) \frac{h_{\alpha r} - h_{\alpha r}}{2h}, \quad b_{3s} = 0, \]
and
\[
\begin{align*}
\frac{(+)}{h} \int_{(x_1,x_2)} P_r(ax_3 - b)f,3 \, dx_3 & = \sum_{s=0}^{r} r_{3s}f_s - \sum_{s=0}^{\infty} r_{3s}f_s \\
& = - \sum_{s=r+1}^{\infty} r_{3s}f_s,
\end{align*}
\]
respectively.

If \( f \) and \( f \) are known (prescribed), then from (10.5) and (10.6), correspondingly, we obtain
\[
\begin{align*}
\frac{(+)}{h} \int_{(x_1,x_2)} P_r(ax_3 - b)f,3 \, dx_3 & = f_{r,\alpha} + \sum_{s=0}^{r} a_{3s}f_s \\
& + f_{n,\alpha} \sqrt{1 + (h_{,1})^2 + (h_{,2})^2} + (-1)^r f_{n,\alpha} \sqrt{1 + (h_{,1})^2 + (h_{,2})^2},
\end{align*}
\]
and
\[
\begin{align*}
\frac{(+)}{h} \int_{(x_1,x_2)} P_r(ax_3 - b)f,3 \, dx_3 & = \sum_{s=0}^{r} a_{3s}f_s \\
& + f_{n,\alpha} \sqrt{1 + (h_{,1})^2 + (h_{,2})^2} + (-1)^r f_{n,\alpha} \sqrt{1 + (h_{,1})^2 + (h_{,2})^2},
\end{align*}
\]
since
\[
\frac{(+)}{n,\alpha} = \pm h_{,\alpha}, \quad \frac{(+)}{n,\alpha} = \pm 1 \\
\sqrt{1 + (h_{,1})^2 + (h_{,2})^2}, \quad \sqrt{1 + (h_{,1})^2 + (h_{,2})^2}.
\]

11 Conclusions

1. Differential hierarchical models for piezoelectric nonhomogeneous viscoelastic Kelvin-Voigt prismatic shells with voids are constructed. The ways of investigation of boundary value problems and initial boundary value problems, including the case of cusped prismatic shells [4], are indicated and some preliminary results are presented.
2. It is shown that in the case of hierarchical models of cusped prismatic shells, depending on the character of vanishing of the thickness at the lateral boundary of the prismatic shell, for well-posedness of the boundary value and initial boundary value problems the setting of boundary conditions is nonclassical, in general. Namely, in the case of nonclassical setting of boundary conditions they should be either weighted ones or the cusped edge should be freed from boundary conditions. In other words, at cusped edges: in the case of piezoelectric viscoelastic materials the displacements, volume fraction, and electric potential cannot always be prescribed.

3. If either elastic, piezoelectric, and dielectric constitutive coefficients are independent of the space points while the thickness of the prismatic shell vanishes in some way at some part of the boundary of the prismatic shell or the thickness of the prismatic shell is constant while the elastic, piezoelectric, and dielectric constitutive coefficients vanish in the same way at the same part of the boundary of the prismatic shell, then peculiarities of setting the boundary conditions for the displacement in the first case and those arising for the volume fraction function and the electric potential in the second case coincide. The stress-strain states coincide as well.

4. Antiplane deformation of piezoelectric nonhomogeneous transversely isotropic materials in the three-dimensional formulation and in \( N = 0 \) approximation is analysed. Some boundary value problems are solved in explicit forms in concrete cases.
References


Previous Issues

1. **W.-L. Schulze.** *Pseudo-differential Calculus and Applications to Non-smooth Configurations.* Volume 1, 2000,


8. Volume 8, 2007 was dedicated to the Centenary of Ilia Vekua.
   It contains the following articles:
   - R.P. Gilbert, G.V. Jaiani. *Ilia Vekua’s Centenary*  
   - V. Kokilashvili, V. Paataashvili. *On the Riemann-Hilbert Problem in Weighted Classes of Cauchy Type Integrals with Density from $L^p(\Gamma)$.* pp. 43-52  
   - Tavkhelidze. *Classification of a Wide Set of Geometric Figures.* pp. 53-61


17. Volume 17, 2016 contains the following articles:

- **Alexander Meskhi.** *Multilinear Integral Operators in Weighted Function Spaces,* pp. 5-18
- **Alice Fialowski.** *The Moduli Space and Versal Deformations Of Algebraic Structures,* pp. 19-34
- **Reinhold Kienzler.** *Material Conservation and Balance Laws in Linear Elasticity with Applications,* pp. 35-65

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9. The decision on the acceptance is taken after a peer-reviewing procedure.

10. Authors submit their papers on the condition that they have not been published previously and are not under consideration for publication elsewhere.

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