AN ELASTIC GREEN MATRIX FOR A SEMISTRIP

Peradze J.
Iv. Javakhishvili Tbilisi State University

Abstract. A Green matrix is constructed for one mixed two-dimensional problem of the elasticity theory. The results of work [1] are thereby defined more correctly.

Key words: system of equations of elasticity theory, Green matrix

AMS subject classification 2000: 35Q72, 35C15

Assume that that we are seeking for a solution of the system of equations of the elasticity theory

$$
(\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x^2} + \mu \frac{\partial^2 u_1}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u_2}{\partial x \partial y} = f_1(x, y),
$$

$$
(\lambda + \mu) \frac{\partial^2 u_1}{\partial x \partial y} + \mu \frac{\partial^2 u_2}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2 u_2}{\partial y^2} = f_2(x, y)
$$

in a domain $-\infty < x \leq 0$, $0 \leq y \leq b$, under the following boundary conditions, interesting from the standpoint of application

$$
\mu \left( \frac{\partial u_1}{\partial y}(x, \alpha b) + \frac{\partial u_2}{\partial x}(x, \alpha b) \right) = 0, \quad u_2(x, \alpha b) = 0,
$$

$$
-\infty < x \leq 0, \quad \alpha = 0, 1,
$$

$$
u_1(0, y) = 0, \quad \mu \left( \frac{\partial u_1}{\partial y}(0, y) + \frac{\partial u_2}{\partial x}(0, y) \right) = 0, \quad 0 \leq y \leq b,
$$

and that for $x \to -\infty$ a solution is bounded.

Suppose that $f_i(x, y) \in L^1((-\infty, 0], [0, b])$ and $\int_{-\infty}^x \int_0^b f_i(\xi, \eta) d\xi d\eta = O((1-x)^{-1})$ as $x \to -\infty$, $i = 1, 2$.

To obtain the Green matrix of the posed problem we will use the method and notation of the paper [1]. Let us define the vectors $U(x, y) = (u_1(x, y), u_2(x, y))$ and $F(x, y) = (f_1(x, y), f_2(x, y))$ (here and in what follows the vector transposition sign is omitted). Assume that

$$
U(x, y) = \sum_{n=0}^{\infty} Q_n(y) U_n(x), \quad F(x, y) = \sum_{n=0}^{\infty} Q_n(y) F_n(x),
$$

$$
Q_n(y) = \text{diag}(\cos \nu y, \sin \nu y), \quad U_n(x) = (u_{1n}(x), u_{2n}(x)), \quad \nu = \pi n / b.
$$

Owing to such a representation, conditions (2.1) are fulfilled. From (1)–(3) we conclude that to find a pair of functions $u_{1n}, u_{2n}, n = 1, 2, \ldots$, it is required...
to solve the system of ordinary differential equations

\[(\lambda + 2\mu)u''_n - \mu \nu^2 u_{1n} + (\lambda + \mu)\nu u'_n = f_{1n},\]

\[-(\lambda + \mu)\nu u'_{2n} + \mu u''_{2n} - (\lambda + 2\mu)\nu^2 u_{2n} = f_{2n}\]

under the boundary conditions by which

\[u_{1n}(0) = 0, \quad u'_{2n}(0) = 0,\]

must be fulfilled at the point \(x = 0\), while for \(x \to -\infty\), \(u_{1n}\) and \(u_{2n}\) are bounded.

As to the case \(n = 0\), here we have only one sought for function \(u_{10}(x)\) satisfying the first equation in (4) and the first equality in (5) and bounded for \(x \to -\infty\).

To solve problem (4),(5), we use the well-known method [2]. First we consider the homogeneous system corresponding to (4). The system of its fundamental solutions forms the rectangular matrix \(\Phi_n = (\phi_{ij}^n)\), \(i = 1, 2, j = 1, 2, 3, 4\), whose elements are

\[\phi_{11}^n = -\phi_{21}^n = e^{\nu x}, \quad \phi_{12}^n = \phi_{22}^n = e^{-\nu x},\]

\[\phi_{13}^n = [(-1)^i(\lambda + \mu)\nu x + (i - 1)(\lambda + 3\mu)] e^{\nu x},\]

\[\phi_{14}^n = [((\lambda + \mu)\nu x - (i - 1)(\lambda + 3\mu)] e^{-\nu x}, \quad i = 1, 2.\]

Using fundamental solutions, we come to the conclusion that the general solution of system (4) has the form

\[U_n(x) = \int_{-\infty}^{x} S_n(x, \xi)\Phi_n(\xi) d\xi + \Phi_n(x)D_n, \quad n = 1, 2, \ldots, \]

where \(S_n(x, \xi) = (s_{ij}^n)\) is the second order matrix, \(i, j = 1, 2\), whose elements are defined by the relations

\[s_{1i}^n(x, \xi) = \frac{m}{2} \left( \frac{1}{\nu} (\lambda + 3\mu) \text{sh} \nu (x - \xi) \right.\]

\[\left. + (-1)^i(\lambda + \mu)(x - \xi)\text{ch} \nu (x - \xi) \right), \quad i = 1, 2,\]

\[-s_{12}^n(x, \xi) = s_{21}^n(x, \xi) = \frac{m}{2} (\lambda + \mu)(x - \xi) \text{sh} \nu (x - \xi),\]

\[m = (\mu(\lambda + 2\mu))^{-1},\]

and \(D_n = (d_i)\) is the column-matrix of arbitrary constants, \(i = 1, 2, 3, 4\).

Taking into account (6), let us choose elements \(D_n\) so that (5) and the condition of the boundedness of \(u_{1n}\) and \(u_{2n}\) for \(x \to -\infty\) be fulfilled. As a result, we obtain

\[d_1 = \frac{m}{2} \int_{-\infty}^{0} \left[ \left( \frac{1}{\nu} (\lambda + 3\mu) \text{sh} \nu \xi - (\lambda + \mu)\xi \text{ch} \nu \xi \right) f_1(\xi) \right.\]

\[\left. + (\lambda + \mu)\xi \text{sh} \nu \xi f_2(\xi) \right] d\xi,\]

\[d_2 = d_4 = 0, \quad d_3 = \frac{m}{2\nu} \int_{-\infty}^{0} (\text{sh} \nu f_1(\xi) - \text{ch} \nu f_2(\xi)) \; d\xi.\]
Formulas (6)–(8) imply
\[ U_n(x) = \int_{-\infty}^{0} g_n(x, \xi) F_n(\xi) \, d\xi, \tag{9} \]
where \( g_n(x, \xi) = (g^n_{ij}(x, \xi)) \) is the second order matrix, \( i, j = 1, 2 \), moreover,

\[ g^n_{ij}(x, \xi) = \begin{cases} \gamma^n_{ij}(x, \xi) + s^n_{ij}(x, \xi) & \text{for } x \geq \xi, \\ \gamma^n_{ij}(x, \xi) & \text{for } x \leq \xi, \end{cases} \tag{10} \]

and

\[ \begin{align*}
\gamma^1_{11}(x, \xi) &= m(p(x - \xi) - q(x - \xi) - p(x + \xi) + q(x + \xi)), \\
\gamma^1_{12}(x, \xi) &= m(p(x - \xi) + p(x + \xi)), \\
\gamma^1_{21}(x, \xi) &= m(p(x + \xi) - p(x - \xi)), \\
\gamma^1_{22}(x, \xi) &= -m(p(x - \xi) + q(x - \xi) + p(x + \xi) + q(x + \xi)),
\end{align*} \tag{11} \]

with the notation

\[ p(u) = \frac{1}{4}(\lambda + \mu)ue^{\nu u}, \quad q(u) = \frac{1}{4\rho}(\lambda + 3\mu)e^{\nu u}. \]

Formula (9) holds for \( n = 0 \) too, where \( g_0(x, \xi) = (g^n_{ij}(x, \xi)) \), \( i, j = 1, 2 \), and in the cases \( x \geq \xi \) and \( x \leq \xi \) \( g^0_{11} \) is equal respectively to \( m\mu x \) and \( m\mu \xi \), while for other elements of the matrix \( g_0(x, \xi) \) we have \( g^0_{12} = g^0_{21} = g^0_{22} = 0 \) in both cases.

By virtue of (3) and (9)
\[ U(x, y) = \int_{-\infty}^{0} \int_{0}^{b} G(x, y, \xi, \eta) F(\xi, \eta) \, d\xi \, d\eta, \]
where the sought Green matrix \( G(x, y, \xi, \eta) \) is defined by the relation

\[ G(x, y, \xi, \eta) = \frac{1}{\ell} \sum_{n=0}^{\infty} \varepsilon_n Q_n(y) g_n(x, \xi) Q_n(\eta). \tag{12} \]

Here \( \varepsilon_n \) is equal to 1 for \( n = 0 \), and to 2 in other cases. To obtain an explicit form of the elements \( G_{ij}(x, y, \xi, \eta) \) of the matrix \( G(x, y, \xi, \eta) \), \( i, j = 1, 2 \), we use (7), (10)–(12), the well-known formulas [3]

\[ \begin{align*}
\sum_{n=1}^{\infty} t^n \cos n\theta &= (1 - t \cos \theta)(1 - 2t \cos \theta + t^2)^{-1}, \\
\sum_{n=1}^{\infty} t^n n^{-1} \cos n\theta &= -\frac{1}{2} \ln(1 - 2t \cos \theta + t^2), \\
t^2 &< 1, \quad 0 < \theta < 2\pi,
\end{align*} \]
and the notation
\[ z = x + iy, \quad \zeta = \xi + i\eta, \quad \omega(u) = e^{\frac{2\pi u}{b}}, \quad P(u) = \text{Re}(1 - \omega(u)), \quad S(u) = \text{Im}\omega(u), \]
\[ E(u) = |1 - \omega(u)|^2, \quad Q(u) = P(u)E^{-1}(u), \quad T(u) = S(u)E^{-1}(u). \]

When \( x \leq \xi \), for the diagonal elements we have
\[
G_{ii}(x, y, \xi, \eta) = \frac{2 - i}{b} m\mu \text{Re}\zeta + \frac{m}{8\pi} (\lambda + 3\mu) \ln \left[ E(z - \zeta)E(z - \bar{\zeta}) \left( E(z + \zeta) \times E(z + \bar{\zeta}) \right)^{(-1)^i} \right] - \frac{m}{4b} (\lambda + \mu) \sum_{k=-1,+1} k^i \text{Re}(z + k\zeta) \left( Q(z + k\zeta) + Q(z + k\bar{\zeta}) \right),
\]
\[ i = 1, 2, \]
while the nondiagonal ones are defined by the formula
\[
G_{ij}(x, y, \xi, \eta) = \frac{m}{4b} (\lambda + \mu) \sum_{k=-1,+1} k^i \text{Re}(z + k\zeta) \left( T(z + k\zeta) - T(z + k\bar{\zeta}) \right),
\]
\[ i, j = 1, 2, \quad i \neq j. \]

For \( x \geq \xi \) these relations readily imply formulas for \( G_{ij}(x, y, \xi, \eta) \) if we use the well-known property of symmetry: in the diagonal and nondiagonal elements we should make the replacement \( x \leftrightarrow \xi \) and, in addition to this, the nondiagonal elements should be interchanged. We obtain
\[
G_{ii}(x, y, \xi, \eta) = \frac{2 - i}{b} m\mu \text{Re}\zeta + \frac{m}{8\pi} (\lambda + 3\mu) \ln \left[ E(-z + \zeta)E(-z + \bar{\zeta}) \times \left( E(z + \zeta)E(z + \bar{\zeta}) \right)^{(-1)^i} \right] - \frac{m}{4b} (\lambda + \mu) \sum_{k=-1,+1} k^{i+1} \text{Re}(z + k\zeta) \left( Q(kz + \zeta) + Q(kz + \bar{\zeta}) \right) + \]
\[ + Q(kz + \bar{\zeta}) \right), \quad i = 1, 2, \]
\[
G_{ij}(x, y, \xi, \eta) = \frac{m}{4b} (\lambda + \mu) \sum_{k=-1,+1} k^{i+1} \text{Re}(z + k\zeta) \left( T(kz + \zeta) - T(kz + \bar{\zeta}) \right), \quad i, j = 1, 2, \quad i \neq j.
\]

Acknowledgment. The author expresses his gratitude to Professor T. Tamadze and Candidate of Science P. Dvalishvili for useful discussions.

References


Received May, 18, 2007; revised October, 21, 2007; accepted December, 26, 2007.