

PURE MORPHISMS OF COMMUTATIVE RINGS ARE EFFECTIVE DESCENT MORPHISMS FOR MODULES – A NEW PROOF

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ABSTRACT. The purpose of this paper is to give a new proof of the Joyal-Tierney theorem (unpublished), which asserts that a morphism $f : R \rightarrow S$ of commutative rings is an effective descent morphism for modules if and only if f is pure as a morphism of R -modules.

Let R be a commutative ring with unit and $R\text{-mod}$ the category of R -modules. Since, for any R -module M , the group $C(M) = \text{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$ (where Ab is the category of abelian groups and \mathbb{Q}/\mathbb{Z} is the rational circle abelian group) becomes an R -module with the action of R on $C(M)$ by $(rf)(m) = f(rm)$, we can define a functor $C : (R\text{-mod})^{op} \rightarrow R\text{-mod}$, given by $C(M) = \text{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$. Since the abelian group \mathbb{Q}/\mathbb{Z} is an injective cogenerator in the category of abelian groups (see, for example, [1]), the functor C is exact and reflects isomorphisms. We say that a morphism $f : M \rightarrow M'$ of R -modules is pure if for any R -module N ,

$$1_N \otimes_R f : N \otimes_R M \rightarrow N \otimes_R M'$$

is monic. Let $f : M \rightarrow M'$ be a morphism of R -modules. The next lemma follows from the commutativity of the diagram

$$\begin{array}{ccc} \text{Hom}_R(C(M), C(M')) & \xrightarrow{\text{Hom}_R(C(M), C(f))} & \text{Hom}_R(C(M), C(M)) \\ \downarrow \approx & & \downarrow \approx \\ C(C(M) \otimes_R M') & \xrightarrow{C(1_{C(M)} \otimes_R f)} & C(C(M) \otimes_R M), \end{array}$$

where the vertical morphisms are the canonical isomorphisms.

1. LEMMA. *Let $f : M \rightarrow M'$ be a morphism of R -modules. The following conditions are equivalent:*

- (a) *f is a pure morphism of R -modules.*
- (b) *$C(f)$ is a split epimorphism of R -modules.*

Let $f : R \rightarrow S$ be a morphism of commutative rings. Recall that a descent datum on an object $M \in \text{Ob}(S\text{-mod})$ can be described as an S -module morphism $\theta : M \rightarrow S \otimes_R M$ such that θ makes

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$$\begin{array}{ccc}
 M & \xrightarrow{\theta} & S \otimes_R M \\
 & \searrow & \downarrow \mu \\
 & & M
 \end{array}$$

and

$$\begin{array}{ccc}
 M & \xrightarrow{\theta} & S \otimes_R M \\
 \theta \downarrow & & \downarrow 1 \otimes_R \theta \\
 S \otimes_R M & \xrightarrow{1 \otimes_R i_M} & S \otimes_R S \otimes_R M
 \end{array}$$

commutative, where μ denotes the S -module structure on M , and $i_M : M \rightarrow S \otimes_R M$ is an R -morphism given by $i_M(m) = 1 \otimes_R m$.

Let $\text{Des}(f)$ denote the category of pairs (M, θ) , θ descent datum on $M \in \text{Ob}(S\text{-mod})$, in which morphisms $(M, \theta) \rightarrow (M', \theta')$ are just morphisms $g : M \rightarrow M'$ in $S\text{-mod}$ which commute with descent data in the obvious sense (see, for example, [2]).

Any object $f^*(M) = (S \otimes_R M, i_M)$, $M \in \text{Ob}(R\text{-mod})$ can be equipped with descent data in a canonical way, and this gives rise to a commutative diagram

$$\begin{array}{ccc}
 R\text{-mod} & \xrightarrow{f^*} & \text{Des}(f) \\
 & \searrow S \otimes_R - & \downarrow U \\
 & & S\text{-mod},
 \end{array}$$

where U is the forgetful functor. f is said to be a descent morphism if f^* is full and faithful, and an effective descent morphism if f^* is an equivalence.

The functor f^* has a right adjoint f_* which is defined by requiring that

$$f_*(M, \theta) \xrightarrow{e} M \begin{array}{c} \xrightarrow{\theta} \\ \xrightarrow{i_M} \end{array} S \otimes_R M$$

is an equalizer of S -modules for each $(M, \theta) \in \text{Ob}(\text{Des}(f))$. The counit of this adjunction is defined by $\delta_M = \mu(1 \otimes_R e)$. The unit $\epsilon_M : M \rightarrow f_* f^*(M)$ is obtained from the diagram

$$\begin{array}{ccc}
 M & & \\
 \downarrow \epsilon_M & \searrow i_M & \\
 f_* f^*(M) & \xrightarrow{e} & M \begin{array}{c} \xrightarrow{\theta} \\ \xrightarrow{i_M} \end{array} S \otimes_R M \begin{array}{c} \xrightarrow{1_{S \otimes_R} i_M} \\ \xrightarrow{i_{S \otimes_R M}} \end{array} S \otimes_R S \otimes_R M.
 \end{array}$$

It does exist because i_M equalizes the two morphisms on the right hand side. From the description of ϵ and δ we obtain immediately the two following propositions.

2. PROPOSITION. $f : R \rightarrow S$ is a descent morphism if and only if f is pure as a morphism of R -modules.

3. PROPOSITION. A descent morphism f is effective if and only if $S \otimes_R -$ preserves the equalizer

$$f_*(M, \theta) \xrightarrow{e} M \begin{array}{c} \xrightarrow{\theta} \\ \xrightarrow{i_M} \end{array} S \otimes_R M$$

for each $(M, \theta) \in \text{Ob}(\text{Des}(f))$.

Let $f : R \rightarrow S$ be pure as a morphism of R -modules. Then by Lemma 1 there is an R -module morphism $g : C(R) \rightarrow C(S)$ such that $c(f)g = 1_{C(R)}$.

If $(M, \theta) \in \text{Ob}(\text{Des}(f))$, then we have a commutative diagram

$$\begin{array}{ccccc} f_*(M, \theta) & \xrightarrow{e} & M & \begin{array}{c} \xrightarrow{\theta} \\ \xrightarrow{i_M} \end{array} & S \otimes_R M \\ & & \downarrow i_M & & \downarrow i_{S \otimes_R M} \\ M & \xrightarrow{\theta} & S \otimes_R M & \begin{array}{c} \xrightarrow{1_{S \otimes_R} \theta} \\ \xrightarrow{1_{S \otimes_R} i_M} \end{array} & S \otimes_R S \otimes_R M, \end{array}$$

in which the rows are equalizer diagrams.

Applying the functor C to this diagram, we obtain the commutative diagram

$$\begin{array}{ccccc} C(S \otimes_R S \otimes_R M) & \begin{array}{c} \xrightarrow{C(1_{S \otimes_R} \theta)} \\ \xrightarrow{C(1_{S \otimes_R} i_M)} \end{array} & C(S \otimes_R M) & \xrightarrow{C(\theta)} & C(M) \\ C(i_{S \otimes_R M}) \downarrow & & \downarrow C(i_M) & & \\ C(S \otimes_R M) & \begin{array}{c} \xrightarrow{C(\theta)} \\ \xrightarrow{C(i_M)} \end{array} & C(M) & \xrightarrow{C(e)} & C(f_*(M, \theta)), \end{array}$$

in which the rows are coequalizer diagrams. Now, since for any R -module P we have the isomorphism of functors $C(P \otimes_R -) \rightarrow \text{Hom}_R(-, C(P))$, we obtain the commutative diagram

$$\begin{array}{ccccc} \text{Hom}_R(S \otimes_R M, C(S)) & \begin{array}{c} \xrightarrow{-\circ\theta} \\ \xrightarrow{-\circ i_M} \end{array} & \text{Hom}_R(M, C(S)) & & \\ -\circ g \uparrow \downarrow -\circ C(f) & & -\circ g \uparrow \downarrow -\circ C(f) & & \\ \text{Hom}_R(S \otimes_R M, C(R)) & \begin{array}{c} \xrightarrow{-\circ\theta} \\ \xrightarrow{-\circ i_M} \end{array} & \text{Hom}_R(M, C(R)) & \xrightarrow{-\circ e} & \text{Hom}_R(f_*(M, \theta), C(R)). \end{array}$$

Applying the above isomorphism of functors backwards, we deduce that there are R -morphisms h and h' , such that the diagram

$$\begin{array}{ccccc} C(S \otimes_R S \otimes_R M) & \begin{array}{c} \xrightarrow{C(1_{S \otimes_R} \theta)} \\ \xrightarrow{C(1_{S \otimes_R} i_M)} \end{array} & C(S \otimes_R M) & \xrightarrow{C(\theta)} & C(M) \\ h' \uparrow \downarrow C(i_{S \otimes_R M}) & & h \uparrow \downarrow C(i_M) & & \uparrow k \\ C(S \otimes_R M) & \begin{array}{c} \xrightarrow{C(\theta)} \\ \xrightarrow{C(i_M)} \end{array} & C(M) & \xrightarrow{C(e)} & C(f_*(M, \theta)) \end{array}$$

commutes. Since both left hand side squares commute, there is an R -morphism $k : C(f_*(M, \theta)) \rightarrow C(M)$ such that $C(\theta)h = kC(e)$. It means that the bottom row becomes a split coequalizer diagram [3] in the category of R -modules, which is split by the morphisms

$$C(f_*(M, \theta)) \xrightarrow{k} C(M) \xrightarrow{h} C(S \otimes_R M).$$

Since split coequalizers are preserved by any functor, its image under the functor $\text{Hom}(S, -)$ is a coequalizer diagram. So

$$\text{Hom}_R(S, C(S \otimes_R M)) \begin{array}{c} \xrightarrow{-\circ C(\theta)} \\ \xrightarrow{-\circ C(i_M)} \end{array} \text{Hom}_R(S, C(M)) \xrightarrow{-\circ C(e)} \text{Hom}_R(S, f_*(M, \theta))$$

is a coequalizer diagram, and hence so is

$$C(S \otimes_R S \otimes_R M) \begin{array}{c} \xrightarrow{C(1_S \otimes_R \theta)} \\ \xrightarrow{C(1_S \otimes_R i_M)} \end{array} C(S \otimes_R M) \xrightarrow{C(1_S \otimes_R e)} C(S \otimes_R f_*(M, \theta)).$$

The functor C is exact and reflects isomorphisms. Therefore it also reflects coequalizers. It follows that

$$S \otimes_R f_*(M, \theta) \xrightarrow{1_S \otimes_R e} S \otimes_R M \begin{array}{c} \xrightarrow{1_S \otimes_R \theta} \\ \xrightarrow{1_S \otimes_R i_M} \end{array} S \otimes_R S \otimes_R M$$

is an equalizer. But

$$M \xrightarrow{\theta} S \otimes_R M \begin{array}{c} \xrightarrow{1_S \otimes_R \theta} \\ \xrightarrow{1_S \otimes_R i_M} \end{array} S \otimes_R S \otimes_R M$$

is also an equalizer diagram. Thus we have an isomorphism $S \otimes_R f_*(M, \theta) \rightarrow M$.

We obtain

4. THEOREM. $f : R \rightarrow S$ is an effective descent morphism for modules if and only if f is pure as a morphism of R -modules.

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References

- [1] P. T. Johnstone, *Topos Theory*, Academic Press, New York 1977.
- [2] G. Janelidze and W. Tholen, *Facets of Descent I*, Appl. Categorical Structures **2**, 245–281(1994)
- [3] S. MacLane, *Categories for the Working Mathematician*, Springer, New York 1971.

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