

PROJECTIVE LINES AS GROUPOIDS WITH PROJECTION STRUCTURE

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ABSTRACT. The coordinate projective line over a field is seen as a groupoid with a further ‘projection’ structure. We investigate conversely to what extent such an, abstractly given, groupoid may be coordinatized by a suitable field constructed out of the geometry.

Introduction

Given a field k and a 2-dimensional vector space V over it, the corresponding projective line $P(V)$ is the set of 1-dimensional linear subspaces of V . However, it is not a structureless set, but rather, it is the set of objects of a groupoid $\mathbb{L}(V)$, whose morphisms are the linear isomorphisms between these subspaces. If A and B are distinct 1-dimensional subspaces of V , a linear isomorphism $A \rightarrow B$ consists in projection from A to B in the direction of a unique 1-dimensional linear subspace C , distinct from A and B . And vice versa, a 1-dimensional linear subspace C distinct from A and B gives rise to such a linear isomorphism.

This provides the groupoid $\mathbb{L}(V)$ with a certain combinatorial structure which we call a *projection structure*. The aim of the present article is to describe properties of abstract groupoids \mathbb{L} with such structure that will imply that \mathbb{L} is of the form $\mathbb{L}(V)$ for some vector space V over some field k . Note that the field k is to be constructed, unlike in [Kock, 2010a] where the field is presupposed.

The properties of \mathbb{L} to be described are presented in terms of four axioms, whose validity for the case of $\mathbb{L}(V)$ is easily verified algebraically, but to a certain extent have visible geometric content, for the case where k is the linear continuum; see e.g. the illustration to Axiom 4 below.

The present note is a completed form of the preprint [Kock, 2010b], and is a continuation of [Kock, 2010a]. The step from [Kock, 2010a] to the present note corresponds to going to the “deeper level” described by [Baer, 1952], Appendix I, where he says that the problem of coordinatizing projective geometry by means of linear algebra “*can be attacked on two essentially different levels: a deeper one where the projective geometry is given abstractly, where nothing is supposed to be known concerning the underlying linear manifold*”

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[vector space]; and the considerably simpler problem where the projective geometry is given as the projective geometry of subspaces of some definite linear manifold

Baer notes (loc.cit. III.4) that “a line . . . has no geometrical structure if considered as an isolated or absolute phenomenon But it receives a definite geometric structure, if embedded into a linear manifold of higher rank. Our contention here is that the geometric structure of the (projective) line can be described without explicit reference to such “higher rank” manifold, (the projective plane or 3-space), namely by considering it from a (geometrically natural) groupoid theoretic viewpoint. This viewpoint was (to our knowledge) first considered in [Kock, 1974], and developed further by Diers and Leroy [Diers-Leroy, 1994] and in [Kock, 2010a], and, independently, in [Cathelineau, 1995].

1. Projective lines in linear algebra

We collect here some classical facts from projective geometry when seen as an aspect of linear algebra. The emphasis is on the, less classical, groupoid theoretic aspect [Kock, 1974], [Diers-Leroy, 1994], [Cathelineau, 1995], [Kock, 2010a], as alluded to in the Introduction.

If A and B are mutually distinct 1-dimensional linear subspaces of a 2-dimensional vector space V , and if $\gamma : A \rightarrow B$ is a k -linear isomorphism, the set of vectors in V of the form $a - \gamma(a)$ for $a \in A$ form a 1-dimensional subspace C distinct from A and B . Conversely, a 1-dimensional linear subspace C distinct from A and B gives rise to such an isomorphism γ , with $\gamma(a)$ defined as the unique $b \in B$ with $a - b \in C$. This γ , we denote $C : A \rightarrow B$ or

$$A \xrightarrow{C} B.$$

This establishes a bijective correspondence between linear isomorphisms $A \rightarrow B$, and 1-dimensional linear subspaces C distinct from A and B .

We present some properties of this bijective correspondence, and the groupoid structure; we do it in a fully coordinatized situation, so we assume that the vector space V is k^2 . Let k be a field. It gives rise to a model $\mathbb{L}(k^2)$ of the axiomatics presented, namely the projective line over k . Its set of points is $P(k^2)$. So a point is a 1-dimensional linear subspace $A \subseteq k^2$, and it may be presented by any non-zero vector $a = (a_1, a_2) \in A$ with a_1 and $a_2 \in k$ (in traditional notation “ A is the point with homogeneous coordinates $(a_1 : a_2)$ ”). So a is a basis vector for A . The calculation of $C : A \rightarrow B$ in coordinates is the following. Here, A, B , and C are 1-dimensional subspaces of the 2-dimensional vector space k^2 . Pick basis vectors a for A , b for B and c for C . Then the linear map $C : A \rightarrow B$ has a matrix with respect to the bases $\{a\}$ for A and $\{b\}$ for B ; it is a 1×1 matrix, whose only entry is the scalar

$$\frac{|a \ c|}{|b \ c|}$$

where $|a \ c|$ denotes the determinant of the 2×2 matrix whose columns are $a \in k^2$, $c \in k^2$, and similarly for $|b \ c|$. Since composition of linear maps corresponds to multiplication of

matrices, which here is just multiplication of scalars, it follows that the composite map

$$A \xrightarrow{C} B \xrightarrow{D} A \quad (1)$$

is (multiplication by) the scalar

$$\frac{|a c|}{|b c|} \cdot \frac{|b d|}{|a d|} \quad (2)$$

where d is any basis vector for D ; and this scalar is independent of the choice of the four basis vectors a, b, c, d chosen, and it is the classical expression for cross ratios in $P(k^2)$. (This was also argued geometrically in [Kock, 1974] p. 3, and algebraically in [Diers-Leroy, 1994], Theorem 1-5-3.) The cross ratio (1) is denoted $(A, B; C, D)$. – Note that the expression (2) does not depend on c and d being linearly independent.

We note the following two trivial equations, which will be referred to later:

$$\frac{|a c|}{|b c|} \cdot \frac{|b c|}{|a c|} = 1; \quad \frac{|a c|}{|b c|} \cdot \frac{|b c|}{|d c|} = \frac{|a c|}{|d c|}. \quad (3)$$

This does not use any properties of determinants. Using that determinants are alternating, $|b c| = -|c b|$, we have also the equality

$$\frac{|a c|}{|b c|} \cdot \frac{|b d|}{|a d|} \cdot \frac{|a b|}{|c b|} = \frac{|a b|}{|c b|} \cdot \frac{|c a|}{|d a|} \cdot \frac{|d b|}{|c b|}, \quad (4)$$

because the factors on the left may be paired off with factors on the right (modulo four sign changes, which cancel). Similarly, the factors on the left in the following equation may be paired off modulo three sign changes, whence the minus sign:

$$\frac{|a c|}{|b c|} \cdot \frac{|b a|}{|c a|} \cdot \frac{|c b|}{|a b|} = -1 \quad (5)$$

Finally, we have the following well known relationship between cross-ratios

$$1 - (A, B; C, D) = (A, C; B, D) \quad (6)$$

in coordinates (with $a = (a_1, a_2)$ a basis vector for A , and similarly for b, c, d) the two sides to be compared are

$$1 - (A, B; C, D) = 1 - \frac{|a c|}{|b c|} \cdot \frac{|b d|}{|a d|} = \frac{|b c| \cdot |a d| - |a c| \cdot |b d|}{|b c| \cdot |a d|},$$

and

$$(A, C; B, D) = \frac{|a b|}{|c b|} \cdot \frac{|c d|}{|a d|} = \frac{|a b| \cdot |c d|}{|c b| \cdot |a d|}.$$

The denominators in these two expressions are the same except for sign, so it suffices to see that the numerators are the same except for sign. Using $|a b| = a_1 b_2 - a_2 b_1$ etc., we get for the first numerator

$$(b_1 c_2 - b_2 c_1) \cdot (a_1 d_2 - a_2 d_1) - (a_1 c_2 - a_2 c_1) \cdot (b_1 d_2 - b_2 d_1),$$

and multiplying out, we get eight terms; there are two terms $a_1b_1c_2d_2$ with opposite sign, and similarly two terms $a_2b_2c_1d_1$, and these terms cancel. The four remaining terms are then seen to be the terms of the numerator in the expression for $(A, C; B, D)$, except for sign. This proves $1 - (A, B; C, D) = (A, C; B, D)$, as desired.

2. The axiomatics

A *projection structure* on a groupoid \mathbb{L} consists in the following: for any two distinct objects A and B of \mathbb{L} , there is given a bijection between the set $\text{hom}_{\mathbb{L}}(A, B)$ and the set $|\mathbb{L}| \setminus \{A, B\}$, where $|\mathbb{L}|$ denotes the set of objects of \mathbb{L} . The arrow $A \rightarrow B$ corresponding to an object C (with $C \neq A, B$) is denoted $C : A \rightarrow B$ or

$$A \xrightarrow{C} B.$$

A functor $P : \mathbb{L} \rightarrow \mathbb{L}'$ between groupoids with projection structure is said to *preserve* these structures if it is injective on objects, and if

$$P(A \xrightarrow{C} B) = P(A) \xrightarrow{P(C)} P(B).$$

Note that the injectivity condition is needed to make sense to the right hand expression.

A functor P preserving projection structures in this sense is also called a *homomorphism* or a *projectivity* (whence the letter P). It is easy to see that a homomorphism, as described, is a *faithful* functor $\mathbb{L} \rightarrow \mathbb{L}'$.

If \mathbb{L} with a projection structure has only one or only two objects, it is a group, respectively a disjoint union of two groups, and in this case, a projection structure is a void concept. So henceforth, we assume that \mathbb{L} has at least three distinct objects.

If a groupoid \mathbb{L} with a projection structure has at least three objects, it is *connected* in the sense that for any pair A, B of objects, $\text{hom}_{\mathbb{L}}(A, B)$ is nonempty: any object $C \neq A, B$ defines a morphism $C : A \rightarrow B$.

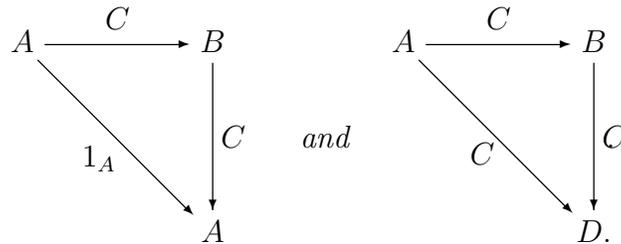
In a connected groupoid \mathbb{L} , all vertex groups $\text{hom}_{\mathbb{L}}(A, A)$ are isomorphic: conjugation by an arrow $A \rightarrow B$ provides an isomorphism $\text{hom}_{\mathbb{L}}(A, A) \rightarrow \text{hom}_{\mathbb{L}}(B, B)$. If one of these vertex groups is commutative, all vertex groups are commutative, and the isomorphism $\text{hom}_{\mathbb{L}}(A, A) \rightarrow \text{hom}_{\mathbb{L}}(B, B)$ defined by conjugation by an arrow $\alpha : A \rightarrow B$ does not depend on the choice of the arrow $\alpha : A \rightarrow B$. For brevity, we call such (connected) groupoids *commutative*¹. In such a groupoid, all vertex groups $\mathbb{L}(A, A)$ are *canonically* isomorphic, and may be identified with one single (commutative) group, the *group of abstract scalars* of \mathbb{L} ; the group $\text{hom}_{\mathbb{L}}(A, A)$ may be called *the group of scalars at A*; it is canonically isomorphic to the group of abstract scalars. If $\mu \in \text{hom}_{\mathbb{L}}(A, A)$ and $\mu' \in \text{hom}_{\mathbb{L}}(B, B)$ represent the same abstract scalar, we write $\mu \equiv \mu'$; this is the case if

¹or *abelian*, cf. e.g. [Mackenzie 1987], Definition 3.4. We prefer in the present context to use the term “commutative”, because multiplicative, rather than additive, notation is needed for the composition; ultimately it will be the multiplicative structure of a field.

there is an arrow $\alpha : A \rightarrow B$ conjugating μ to μ' , equivalently, with $\mu.\alpha = \alpha.\mu'$. (We compose from left to right in \mathbb{L} .) Sometimes we do not distinguish between abstract scalars, and scalars at A , and between $=$ and \equiv for scalars.

We present some axioms for groupoids \mathbb{L} with projection structure, with \mathbb{L} being a commutative groupoid with at least three objects. The objects of \mathbb{L} , we shall also call points of \mathbb{L} , because of the intended geometric interpretation of \mathbb{L} .

AXIOM 1. *The following two diagrams commute, where A, B, C , and D are mutually distinct points of \mathbb{L} :*



The validity of this axiom in the coordinate model $P(k^2)$ is expressed by (3).

We shall introduce the notion of *cross ratio*, and a matrix notation for it. Cross ratios are certain scalars. Consider four distinct points A, B, C, D of \mathbb{L} . We write

$$(A, B; C, D) \quad \text{or} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

for the scalar at A , or for the abstract scalar it represents, given as the composite

$$A \xrightarrow{C} B \xrightarrow{D} A.$$

and call it the *cross ratio* of the 4-tuple. The word *bi-rapport* will also be used.

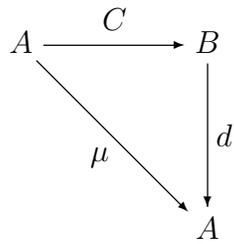
It is an immediate consequence of the notion of projectivity, i.e. a morphism P of groupoids with projection structure, that such projectivity P preserves cross ratios

$$(P(A), P(B); P(C), P(D)) = P((A, B; C, D)).$$

We note that if \mathbb{L} has only three points, there are no scalars that appear as cross ratios. On the other hand, if there are more than three points, every scalar except 1 appears as a cross ratio, more precisely

2.1. PROPOSITION. *Given $\mu \in \mathbb{L}(A, A)$ with $\mu \neq 1_A$. For any B and C , mutually distinct, and distinct from A , there is a unique D , distinct from A, B , and C , such that $\mu = (A, B; C, D)$.*

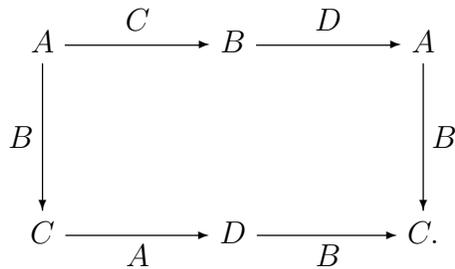
PROOF. Since \mathbb{L} is a groupoid, there is a unique arrow d making the triangle



commute. Since B and A are distinct, d is of the form $D : B \rightarrow A$ for a unique point D distinct from A and B ; and, by Axiom 1, $C = D$ would imply that we get $\mu = 1_A$, which we excluded.

In a cross ratio expression $(A, B; C, D)$, one might allow the possibility that $C = D$, in which case the first part of Axiom 1 would say that $(A, B; C, C) = 1_A$; however, this usage would make some of the statements to be made later more clumsy. This is why we consider only cross ratio expressions with four distinct entries.

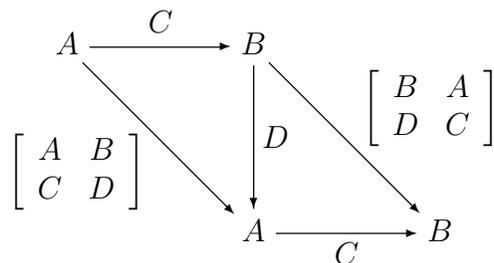
AXIOM 2. *The following diagram commutes, where A, B, C , and D are mutually distinct points of \mathbb{L} :*



The validity of this axiom in the coordinate model $P(k^2)$ is expressed by (4).

2.2. PROPOSITION. *In a cross ratio matrix, the columns may be interchanged, without changing the abstract scalar which the matrices represent. Also, the rows may be interchanged without changing the abstract scalar which the matrices represent.*

PROOF. For columns: consider the diagram



The two triangles commute by definition; hence, the total quadrangle commutes; but this can be expressed: $C : A \rightarrow B$ conjugates the scalar $(A, B; C, D)$ at A to the scalar $(B, A; D, C)$ at B . For rows, this is just a reformulation of Axiom 2, which in terms of cross ratios says that $B : A \rightarrow C$ conjugates $(A, B; C, D)$ to $(C, D; A, B)$.

2.3. PROPOSITION. *We have*

$$\begin{bmatrix} A & B \\ D & C \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} A & B \\ D & E \end{bmatrix} = \begin{bmatrix} A & B \\ C & E \end{bmatrix}.$$

PROOF. For the first assertion, we consider the composite

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} A & B \\ D & C \end{bmatrix}$$

which amounts to a four-fold composite

$$A \xrightarrow{C} B \xrightarrow{D} A \xrightarrow{D} B \xrightarrow{C} A.$$

The two middle factors give 1_B , by Axiom 1, and then the two remaining factors give 1_A , again by Axiom 1. – The proof of the second assertion is similar, just replace the last occurrence of C by E ; then the two middle factors again give 1_B , and then the two remaining factors give the defining composite for $(A, B; C, E)$.

We now state the third axiom which refers to “middle four interchange” in cross ratio expressions like $(A, B; C, D)$:

AXIOM 3. *If $(A, B; C, D) \equiv (A', B'; C', D')$, then also $(A, C; B, D) \equiv (A', C'; B', D')$.*

The validity of this axiom in the coordinate model $P(k^2)$ follows from (6), since if $(A, B; C, D) = (A', B'; C', D') = \mu$, say, then $(A, C; B, D) = 1 - \mu = (A', C'; B', D')$, by two applications of (6).

Middle four interchange is clearly an involution. Since every scalar $\mu \neq 1$ appears as a cross ratio of four distinct points, it follows from the Axiom that we have:

2.4. PROPOSITION. *There is an involution Φ on the set $G \setminus \{1\}$ of scalars $\neq 1$, such that for any distinct 4-tuple (A, B, C, D) , we have*

$$(A, C; B, D) = \Phi(A, B; C, D).$$

In a projective line over a field, $\Phi(\mu) = 1 - \mu$, and this involves the additive structure (in terms of the binary operation “minus”) of the field, which is not assumed in our context, but is rather something to be constructed.

We note that a projectivity $P : \mathbb{L} \rightarrow \mathbb{L}'$ of groupoids with projection structure, both of which satisfy Axioms 1-3, preserves the corresponding involutions Φ and Φ' : present a given scalar $\mu \neq 1$ of \mathbb{L} as $(A, B; C, D)$; then $\Phi(\mu) = (A, C; B, D)$, and so

$$P(\Phi(\mu)) = P(A, C; B, D) = (P(A), P(C); P(B), P(D))$$

$$= \Phi'(P(A), P(B); P(C), P(D)) = \Phi'(P(\mu)).$$

By a Φ -group, we understand a (multiplicatively written) commutative group G equipped with an involution Φ on the set $G \setminus \{1\}$; a morphism of such is an injective group homomorphism $P : G \rightarrow G'$ compatible with the Φ s, in the evident sense that for all $A \neq 1$ in G , we have $\Phi'(P(A)) = P(\Phi(A))$, where Φ and Φ' denote the assumed involutions on $G \setminus \{1\}$ and $G' \setminus \{1\}$, respectively.

The multiplicative group of nonzero elements in a field K is an example, with $\Phi(\mu) = 1 - \mu$. A homomorphism of fields is injective and is compatible with the respective Φ s.

2.5. PERMUTATION LAWS. Let $c : Q \rightarrow H$ be a surjection. If $\sigma : Q \rightarrow Q$ is an endo-map (we are only interested in permutations), then we say that σ descends along c if there exists an endomap $\bar{\sigma} : H \rightarrow H$ (necessarily unique, by surjectivity of c) such that the square

$$\begin{array}{ccc} Q & \xrightarrow{c} & H \\ \sigma \downarrow & & \downarrow \bar{\sigma} \\ Q & \xrightarrow{c} & H \end{array}$$

commutes. A composite of two endomaps which descend along c again descends along c . Clearly, σ descends along c iff $c(q) = c(q')$ implies $c(\sigma(q)) = c(\sigma(q'))$ for all q and q' in Q .

We have in mind the case where Q is the set $|\mathbb{L}|^{<4>}$ of 4-tuples of mutually distinct points of \mathbb{L} , and where c is cross ratio formation, with H the set of (abstract) scalars $\neq 1$ of \mathbb{L} . Then the following proposition distills some information already obtained or assumed; \mathbb{L} is a groupoid with projection structure, satisfying Axioms 1-3.

2.6. PROPOSITION. *Every permutation σ of $|\mathbb{L}|^{<4>}$ arising from a permutation $\sigma \in \mathfrak{S}_4$ (= the group of permutations of four letters) descends along the cross ratio formation map $c : |\mathbb{L}|^{<4>} \rightarrow G \setminus \{1\}$.*

PROOF. The 24 permutations on $|\mathbb{L}|^{<4>}$ (whose elements we, as above, write as 2×2 matrices) arising from \mathfrak{S}_4 are generated by the following four permutations: interchange of rows, interchange of columns, interchange of two lower entries, and middle four interchange. Interchange of rows, and interchange of columns descend to the identity map of $G \setminus \{1\}$, by Proposition 2.2; interchange of two lower entries descends to multiplicative inversion $(-)^{-1}$, by (the first assertion of) Proposition 2.3; and, finally, middle four interchange descends to Φ , by Proposition 2.4.

In a similar vein, let $p : G \rightarrow G'$ be a Φ -group homomorphism, let $P : |\mathbb{L}| \rightarrow |\mathbb{L}'|$, and let $p : G \rightarrow G'$ be a Φ -group homomorphism, where G and G' are the groups of scalars of \mathbb{L} and \mathbb{L}' , respectively. We assume Axioms 1-3. Then

2.7. PROPOSITION. *If P preserves the cross ratio of a given distinct 4-tuple (relative to p), then it also preserves any permutation instance of it.*

PROOF. It suffices to prove it for each of the four generating permutations mentioned above. The four cases are proved along the same lines, so let us just give the proof for the case of middle four interchange. So assume that a cross ratio $(A, B; C, D)$ is preserved, so $p((A, B; C, D)) = (P(A), P(B); P(C), P(D))$. Then

$$\begin{aligned} p((A, C; B, D)) &= p(\Phi(A, B; C, D)) = \Phi'(p(A, B; C, D)) = \\ &= \Phi'(P(A), P(B); P(C), P(D)) = (P(A), P(C); P(B), P(D)) \end{aligned}$$

where Φ and Φ' denote the involution arising from middle four interchange in \mathbb{L} and \mathbb{L}' , respectively.

2.8. TRI-RAPPORTS, AND THE SCALAR -1 . Cross ratios in projective geometry are also called *bi-rapports*. In our context, this is quite logical, since a cross ratio is an endomap presented as a composite of two maps (which themselves are not endomaps). Similarly, a tri-rapport in our context is an endomap presented as a composite of three (non-endo) maps,

$$A \xrightarrow{X} B \xrightarrow{Y} C \xrightarrow{Z} A,$$

and this scalar (at A) may be denoted in matrix form, in analogy with cross ratios (bi-rapports)

$$\begin{bmatrix} A & B & C \\ X & Y & Z \end{bmatrix}.$$

This usage (in our context) is found is e.g. [Diers-Leroy, 1994] (even generalized into n -rapports), or (vastly generalized) in [Cathelineau, 1995]. We may also denote tri-rapports in “one-line” form, by $(A, B, C; X, Y, Z)$, to save space.

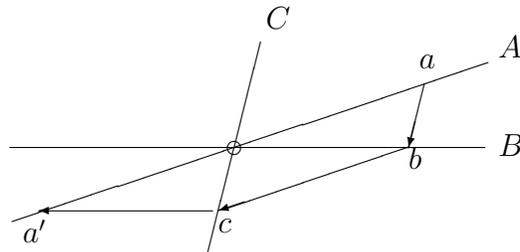
2.9. PROPOSITION. *The columns in a tri-rapport matrix may be permuted cyclically without changing its value.*

The proof is analogous to the proof of Proposition 2.2 (the first part), and is left to the reader.

Of particular importance are tri-rapports of the following form

$$A \xrightarrow{C} B \xrightarrow{A} C \xrightarrow{B} A, \tag{7}$$

so, in the matrix notation, the lower row is a cyclic permutation of the upper row. The geometric content of this scalar, in the classical geometric model of the real projective line, is that it is the scalar -1 . For, consider the picture (from [Kock, 1974])



Then given an element $a \in A$, applying consecutively the linear maps $C : A \rightarrow B$, $A : B \rightarrow C$, and $B : C \rightarrow A$ to a , as illustrated, gives reflection of a in O . But such reflection is independent of the choice of B and C . In our context, the reflection is an arrow $A \rightarrow A$, that is, a scalar at A , and we would like to denote it $(-1)_A$; to do this we need it to be independent of the choice of B and C . This independence seems not to be something that we can derive on the basis of the axioms given so far, so we need this as a further axiom. We shall see that it suffices to require independence of the choice of C , so we pose:

AXIOM 4. *Given four distinct points A, B, C, D , we have*

$$\begin{bmatrix} A & B & C \\ C & A & B \end{bmatrix} = \begin{bmatrix} A & B & D \\ D & A & B \end{bmatrix}.$$

The validity of this axiom in the coordinate model $P(k^2)$ follows from (5), since both sides of the equation in this case give -1 .

We can then deduce

2.10. PROPOSITION. *The scalar at A given by $(A, B, C; C, A, B)$ is independent of the choice of B and C , and may thus be denoted $(-1)_A$. Furthermore, $(-1)_A \equiv (-1)_B$ for any A and B , so they represent the same abstract scalar, denoted (-1) . Finally, $(-1).(-1) = 1$.*

PROOF. The independence of the choice of C is a reformulation of the Axiom. The independence of the choice of B then follows purely formally from Proposition 2.9. To prove $(-1)_A \equiv (-1)_B$, pick C distinct from A and B . By what is already proved, $(-1)_A$ may be presented by $(C : A \rightarrow B).(A : B \rightarrow C).(B : C \rightarrow A)$, and $(-1)_B$ may be presented by $(A : B \rightarrow C).(B : C \rightarrow A).(C : A \rightarrow B)$. Now consider the four-fold composite

$$A \xrightarrow{C} B \xrightarrow{A} C \xrightarrow{B} A \xrightarrow{C} B;$$

the composite of the three first of these maps is $(-1)_A$ and the composite of the three last is $(-1)_B$. Then the two ways of interpreting the four-fold composite, by the associative law, expresses that $C : A \rightarrow B$ conjugates $(-1)_A$ to $(-1)_B$, i.e. expresses $(-1)_A \equiv (-1)_B$. The fact that $(-1).(-1) = 1$ follows if we can prove that $(-1)_A.(-1)_A$ is the identity map at A . This follows by three applications of Axiom 1, by representing the first factor $(-1)_A$ by the tri-rapport $(A, B, C; C, A, B)$, and the second one by the tri-rapport $(A, C, B; B, A, C)$.

REMARK. Assume $-1 \neq 1$. Then the classical way of dealing with the scalar -1 , here defined as a tri-rapport, is in terms of *harmonic conjugates*: Given A, B, C, H . Then

$$(A, B; C, H) = -1$$

if and only if $H : B \rightarrow A$ equals the composite

$$B \xrightarrow{A} C \xrightarrow{B} A.$$

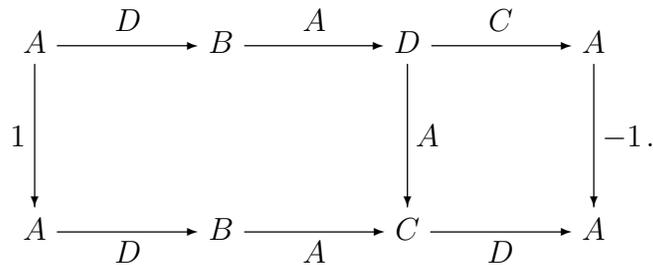
For, precomposing the composite with $C : A \rightarrow B$ gives -1 , and precomposing $H : B \rightarrow A$ with $C : A \rightarrow B$ gives -1 if and only if $(A, B; C, H) = -1$. The classical way of formulating this characterizing property of H is: H is the *harmonic conjugate* of C with respect to A, B .

For μ a scalar, $-\mu$ denotes of course $(-1).\mu (= \mu.(-1))$. We note the following sign change property of certain tri-rapports (with only four points occurring):

2.11. PROPOSITION. *Let A, B, C, D be mutually distinct points. Then*

$$(A, B, C; D, A, D) = -(A, B, D; D, A, C)$$

PROOF. Consider the diagram



The left hand rectangle commutes (insert a “North-East” arrow $A : B \rightarrow D$ and use Axiom 1); the right hand square commutes by (a variant of) the definition of the scalar -1 .

REMARK. One cannot conclude that -1 is distinct from 1 ; take $\mathbb{L}(k^2)$ for k a field of characteristic 2. I don’t know whether a four-point \mathbb{L} has $-1 \neq 1$. The projective line $P(k^2)$ over the field k with three elements has $-1 \neq 1$, and is up to isomorphism the only such groupoid with projection structure, satisfying the axioms 1-4 and $-1 \neq 1$. For, it can be seen by a straightforward exclusion argument that such \mathbb{L} is unique up to isomorphism: If the four points are A, B, C, D , the composition is uniquely determined; for instance $(C : A \rightarrow B).(A : B \rightarrow C)$ is necessarily $D : A \rightarrow B$; for, it cannot be $B : A \rightarrow C$, since post-composing with $B : C \rightarrow A$ would on the one hand give 1 , by Axiom 1, and on the other hand, it would be a three-fold composite defining the scalar -1 . See also the Example at the end of [Kock, 2010a].

3. Scalars as 2-cells

The purpose of the present Section is to give a proof of a relationship between bi-rapports (cross ratios) and tri-rapports; it is (to my knowledge) first proved in [Diers-Leroy, 1994], Proposition 2-4-3. We present an alternative and somewhat more conceptual proof.

Given a connected commutative groupoid \mathbb{L} , with group G of abstract scalars, there is a 2-dimensional groupoid with \mathbb{L} as underlying 1-dimensional groupoid: namely a 2-cell

$$\begin{array}{ccc}
 & f & \\
 A & \xrightarrow{\quad} & B \\
 & \Downarrow \alpha & \\
 & g &
 \end{array}$$

is a scalar α such that the scalar α_A at A which represents α satisfies $\alpha_A.g = f$, or equivalently, such that the scalar α_B at B which represents α satisfies $g.\alpha_B = f$. Both horizontal and vertical composition of 2-cells come about from the multiplication in G .

For two parallel arrows f and g from A to B , there is therefore a unique 2-cell $f \Rightarrow g$, (represented by the scalar $f.g^{-1}$ at A or by the scalar $g^{-1}.f$ at B). If $f : A \rightarrow A$ is a scalar at A , the unique 2-cell $f \Rightarrow 1_A$ is the abstract scalar associated to f .

There is thus nothing in the 2-dimensional structure which is not present in the information contained in the 1-dimensional structure, so the reader who wants a purely 1-dimensional proof of Proposition 3.1 below is referred to [Diers-Leroy, 1994]; however, the result and the proof seem more conceptual and compelling in the 2-dimensional formulation.

For the case of a groupoid \mathbb{L} with projection structure, satisfying Axiom 1, we see that the abstract scalar μ defined by the cross ratio $(A, B; C, D)$ (as an arrow $A \rightarrow A$) is a 2-cell μ

$$A \begin{array}{c} \xrightarrow{(A, B; C, D)} \\ \Downarrow \mu \\ \xrightarrow{1_A} \end{array} A \tag{8}$$

but as an abstract scalar, μ is also a 2-cell between two arrows which are not endo-arrows:

$$A \begin{array}{c} \xrightarrow{C} \\ \Downarrow \mu \\ \xrightarrow{D} \end{array} B$$

as can be seen by postcomposing μ in (8) by $D : A \rightarrow B$ and using Axiom 1. Identifying scalars at A with abstract scalars, we thus have the useful alternative diagrammatic way of defining cross ratios.

Consider now a (2-dimensional) diagram²

$$A \begin{array}{c} \xrightarrow{E} \\ \Downarrow \epsilon \\ \xrightarrow{E'} \end{array} B \begin{array}{c} \xrightarrow{F} \\ \Downarrow \phi \\ \xrightarrow{F'} \end{array} C \begin{array}{c} \xrightarrow{G} \\ \Downarrow \gamma \\ \xrightarrow{G'} \end{array} A.$$

The 2-cell obtained by horizontal composition is then the product in G of three cross ratios $\epsilon = (A, B; E, E')$, $\phi = (B, C; F, F')$, and $\gamma = (C, A; G, G')$. On the other hand, if the bottom composite is 1_A , the (unique) 2-cell from the top composite to the bottom one is the scalar represented by the top composite, i.e. the tri-rapport $(A, B, C; E, F, G)$. We have therefore proved the following (cf. [Diers-Leroy, 1994], Prop. 2-4-3):

3.1. PROPOSITION. *If $(A, B, C; E', F', G') = 1$, then*

$$(A, B, C; E, F, G) = (A, B; E, E').(B, C; F, F').(C, A; G, G').$$

²The point G occurring in the diagram is not to be confused with the group G of scalars.

We note that given distinct A, B, C , it is always possible to find points E', F', G' such that $(A, B, C; E', F', G') = 1$, provided there are at least five points in \mathbb{L} ; for, pick distinct E' and F' , both of them distinct from A, B , and C , then $(E' : A \rightarrow B).(F' : B \rightarrow C)$, as a non-scalar, is labelled by a unique G' distinct from A, C and from E', F' , and then E', F', G' will do the job (postcompose with $G' : C \rightarrow A$ and use Axiom 1).

3.2. PROPOSITION. *Assume that \mathbb{L} has at least five points, and that $P : |\mathbb{L}| \rightarrow |\mathbb{L}'|$, and $p : G \rightarrow G'$ are given, such that P preserves cross ratio formation (with respect to p), with p an injective group homomorphism. Then P preserves tri-rapports.*

PROOF. Consider a tri-rapport, say $(A, B, C; E, F, G)$. Pick, as argued above, E', F', G' with

$$(A, B, C; E', F', G') = 1.$$

Then applying Proposition 3.1 for \mathbb{L} , one rewrites the scalar $(A, B, C; E, F, G)$ as a product of three scalars, each of which is a bi-rapport, and hence is preserved by P , i.e. $p(A, B; E, E') = (P(A), P(B); P(E), P(E'))$ etc., and since p preserves products of scalars, it follows that $p(A, B, C; E, F, G) = (P(A), P(B), P(C); P(E), P(F), P(G))$, using now Proposition 3.1 for \mathbb{L}' .

4. Three-transitivity (“Fundamental Theorem”)

The following Section presents a version of the classical “three-transitivity” theorem for projectivities (called The Fundamental Theorem of Projective Geometry in e.g. [Coxeter, 1942] 2.8.5). We use Axioms 1-3, but not Axiom 4.

Recall that a morphism $P : \mathbb{L} \rightarrow \mathbb{L}'$ of groupoids with projection structure in particular is a faithful functor. Therefore, it gives rise to an injective group homomorphism $p : G \rightarrow G'$ on the corresponding groups of scalars, in particular, it restricts to a map $p : G \setminus \{1\} \rightarrow G' \setminus \{1\}$

4.1. PROPOSITION. *If \mathbb{L} and \mathbb{L}' satisfy the three axioms 1-3, this map p is compatible with the involutions Φ which we get by virtue of Axiom 3.*

PROOF. Given $\mu \neq 1$ in G . We have to prove that $\Phi'(p(\mu)) = p(\Phi(\mu))$, where Φ' is the involution on $G' \setminus \{1\}$. Since $\mu \neq 1$, we may assume that $\mu = (A, B; C, D)$ for four distinct points in \mathbb{L} , and so $\Phi(\mu) = (A, C; B, D)$, which by definition of cross ratios is the composite

$$A \xrightarrow{B} C \xrightarrow{D} A$$

in the groupoid \mathbb{L} . Applying the functor P thus gives

$$p(\Phi(\mu)) = P(A \xrightarrow{B} C).P(C \xrightarrow{D} A),$$

and since P is assumed to be a morphism of projection structure, this is the composite in \mathbb{L}'

$$P(A) \xrightarrow{P(B)} P(C) \xrightarrow{P(D)} P(A)$$

which in turn equals $(P(A), P(C); P(B), P(D)) = \Phi'(P(A), P(B); P(C); P(D))$. Now $(P(A), P(B); P(C), P(D))$ is $p(\mu)$, again because P is a functor, so that we get $\Phi'(p(\mu))$.

Let \mathbb{L} and \mathbb{L}' be groupoids with projection structure, satisfying axioms 1, 2, and 3, and let G and G' be the corresponding Φ -groups of scalars. Assume that \mathbb{L} has at least five points.

4.2. THEOREM. *Let $p : G \rightarrow G'$ be a (injective) homomorphism of Φ -groups. Then if A, B, C are distinct points in \mathbb{L} , and A', B', C' are distinct points in \mathbb{L}' , there exists a unique morphism of groupoids with projection structure $P : \mathbb{L} \rightarrow \mathbb{L}'$ which on scalars restricts to the given p and which satisfies $P(A) = A', P(B) = B', P(C) = C'$. If $p : G \rightarrow G'$ is an isomorphism, then so is the morphism P .*

PROOF. Part of the proof is from [Kock, 2010a] Section 3, but now it is in a more general setting. The construction, and the uniqueness, of P is forced: if $X \neq A, B, C$ in \mathbb{L} , then we are forced to define $P(X)$ to be the unique point of \mathbb{L}' (cf. Proposition 2.1 for \mathbb{L}') satisfying

$$p(A, B; C, X) = (A', B'; C', P(X)) \quad (= (P(A), P(B); P(C), P(X))).$$

The P thus defined is injective: if $P(X) = P(Y)$, we have $P(A, B; C, X) = P(A, B; C, Y)$, and from injectivity of $p : G \rightarrow G'$ then follows that $(A, B; C, X) = (A, B; C, Y)$. Proposition 2.1 for \mathbb{L} then gives that $X = Y$. - Since (P, p) thus preserves cross ratios of the form $(A, B; C, X)$, it follows from Proposition 2.7 that it also preserves all cross ratios of permutation instances thereof, i.e. cross ratios of 4-tuples, three of whose entries are A, B , and C . But then also (P, p) preserves all cross ratios of the form $(A, B; X, Y)$; for, by Proposition 2.3, $(A, B; X, Y) = (A, B; X, C).(A, B; C, Y)$, and since (P, p) preserves the two cross ratios on the right, it preserves their composite, being a group homomorphism $G \rightarrow G'$. From Proposition 2.7 now follows that (P, p) preserves all cross ratios, two of whose entries are A and B . But then also (P, p) preserves all cross ratios of the form $(A, X; Y, Z)$, using $(A, X; Y, Z) = (A, X; Y, B).(A, X; B, Z)$, and again therefore all cross ratios one of whose entries is A . And then finally by $(X, Y; Z, U) = (X, Y; Z, A).(X, Y; A, U)$, we see that (P, p) preserves all cross ratios.

We have now defined the value of P on all objects of \mathbb{L} . To define the values of P on the arrows of \mathbb{L} , we put

$$P(A \xrightarrow{C} B) := P(A) \xrightarrow{P(C)} P(B)$$

for arrows $A \rightarrow B$ with $A \neq B$; this then also ensures that P preserves the projection structure. Arrows $A \rightarrow A$ may canonically be identified with (abstract) scalars $\in G$, and the value on such are given canonically by the assumed $p : G \rightarrow G'$. Henceforth, we

don't distinguish notationally between P and p . It remains to prove that P preserves composition of arrows.

The P constructed commutes with composition (multiplication) of scalars, by assumption on p . Also, it commutes with composition of non-scalars; for, assume that $(E : A \rightarrow B).(F : B \rightarrow C) = (G : A \rightarrow C)$, or equivalently, by Axiom 1, that

$$A \xrightarrow{E} B \xrightarrow{F} C \xrightarrow{G} A = 1_A.$$

This means that the trirapport $(A, B, C; E, F, G)$ equals 1. Now since \mathbb{L} has at least five points, we may use Proposition 3.2 to conclude that P preserves tri-rapports, and so it follows that

$$(P(A), P(B), P(C); P(E), P(F), P(G)) = 1,$$

from which we again conclude from Axiom 1, and from the fact that P is compatible with the projection structure, that

$$P(E : A \rightarrow B).P(F : B \rightarrow C) = P(G : A \rightarrow C),$$

as claimed. It remains to be argued that P commutes with compositions of the form $\mu.(C : A \rightarrow B)$, with μ a scalar (and similarly with scalars multiplied on the right). Pick points D and E such that A, B, D , and E are mutually distinct. Then by Proposition 2.1, there is a (unique) F with $(A, D; E, F) = \mu$, so μ is the composite of the two leftmost arrows in

$$A \xrightarrow{E} D \xrightarrow{F} A \xrightarrow{C} B.$$

Rebracketing, one gets a composite of two non-scalars $A \rightarrow D$ and $D \rightarrow B$, and the $D \rightarrow B$ in question in turn is a composite of two non-scalars; so the whole three-fold product is preserved.

For the final assertion: if p is surjective, Proposition 2.1 for \mathbb{L} and for \mathbb{L}' , and the fact that (P, p) preserves cross ratios, implies surjectivity of P , by an easy argument.

5. Coordinatization

Given a field k , we get a Φ -group G by taking G to be the group of non-zero elements, and taking $\Phi(\mu) = 1 - \mu$ for $\mu \neq 1$. This Φ -group also has a specified element of order ≤ 2 , namely -1 . A morphism of fields $k \rightarrow k'$ (is injective and) induces a morphism $G \rightarrow G'$ of the corresponding Φ -groups, and it also preserves -1 . It is in fact a functor from the category of fields to the category of Φ -groups with specified -1 .

5.1. PROPOSITION. *This functor is full and faithful.*

PROOF. It is clear that the functor is faithful. To see that it is full, let $p : G \rightarrow G'$ be a morphism of Φ -groups with a specified -1 . If G and G' come from fields k and k' , respectively, the map $p \cup \{0\} : G \cup \{0\} \rightarrow G' \cup \{0\}$ is in fact a ring homomorphism: it clearly preserves multiplication, and it preserves addition, since addition can be reconstructed from the multiplication, Φ and -1 , by the formula

$$\lambda + \mu = \lambda \cdot \Phi((-1) \cdot \lambda^{-1} \cdot \mu) \quad (9)$$

(and $0 + \lambda = \lambda$, $0 \cdot \lambda = 0$).

In other words, it makes sense to ask whether for a Φ -group G , with specified -1 , the algebraic structure $G \cup \{0\}$ (with the described $+$ and \cdot etc.) is a *field* or not.

To give the following Coordinatization Theorem a more succinct formulation, we pose the following definition. Let \mathbb{L} be a groupoid with projection structure, satisfying Axioms 1-4; let its Φ -group of scalars with -1 be G . Then \mathbb{L} is called a *projective line groupoid* if $G \cup \{0\}$ (with the addition $+$ given by (9) etc.) is a field (called the *scalar field of \mathbb{L}*).

Then we have

5.2. THEOREM. *Every projective line groupoid \mathbb{L} with at least five³ points is isomorphic (as a groupoid with projection structure) to the groupoid $\mathbb{L}(k^2)$, where k is the field $G \cup \{0\}$.*

PROOF. We know already from Appendix that $\mathbb{L}(k^2)$ is a projective line groupoid with scalar field k . Pick three distinct points A, B, C of \mathbb{L} , and three distinct points A', B', C' of $\mathbb{L}(k^2)$. With p as the identity map $k \rightarrow k$, we can apply Theorem 4.2 (Fundamental Theorem on three-transitivity) to obtain a morphism of projective line groupoids $P : \mathbb{L} \rightarrow \mathbb{L}(k^2)$. It is surjective on objects by the last assertion in the Theorem 4.2.

6. The twelve scalars

It is well known that permuting the four entries in a cross ratio expression $(A, B; C, D)$ with value μ gives six classical scalars: $\mu, \mu^{-1}, 1 - \mu, (1 - \mu)^{-1}, 1 - \mu^{-1}$, and $(1 - \mu^{-1})^{-1}$, see e.g. [Struik 1953] p. 8. These relationships hold in the axiomatic setting, without the further assumptions needed for the coordinatization, in other words, they hold if $1 - x$ replaced by $\Phi(x)$ everywhere (recall the involution Φ defined in terms of middle four interchange of cross ratio expressions).

With the availability of the scalar -1 , each of six classical scalars derived from $\mu = (A, B; C, D)$, by permutation of A, B, C, D , may be multiplied by -1 , so that we get six further scalars. These, however, are not in general expressible by cross ratios (= bi-rapports) built from the four given points A, B, C, D , but they are expressible as tri-rapports built from them, using tri-rapport expressions for the six classical scalars. We refer to [Kock 2010b] for the full list of the twelve tri-rapport expressions, but we include for convenience the basic tool for compiling this list:

³The result also holds in the case where \mathbb{L} has precisely three points, or if it has precisely four points and $-1 \neq 1$, see the Remark at the end of Section 2.

6.1. PROPOSITION. *Let A, B, C, D be mutually distinct. Then*

$$(A, B; C, D) = (A, C, D; B, A, B),$$

and

$$(A, B, C; D, A, D) = -(A, B, D; D, A, C).$$

PROOF. In the diagram expressing Axiom 2, insert an arrow $B : D \rightarrow A$ from the D in the lower row to the upper right hand A . The diagram then decomposes into a triangle and a pentagon. The triangle commutes by Axiom 1, and hence the pentagon commutes as well, and it expresses the relationship between the bi-rapport and the tri-rapport claimed in the first equation. The second equation was proved in Proposition 2.11.

7. An open end

Of particular geometric interest is projective geometry over the geometric line, often identified with the field \mathbb{R} of real numbers. However, it can be argued that other models R of the geometric line exist, making differential geometry over manifolds based on R more combinatorial or synthetic in nature, see e.g. [Kock, 1981]. But such an R is not a field in the sense that the dichotomy: “for any $x \in R$, x is either 0 or is invertible”; rather R is a local ring (inside a topos, in fact).

From this point of view, the present note is unsatisfactory: it depends heavily on the dichotomy – here in the disguise that the arrows of the groupoid \mathbb{L} are *either* endomaps (scalars), *or* of the form $C : A \rightarrow B$ with A, B and C distinct objects.

Grassmannian manifolds, say projective spaces like $P(R^2)$, do exist and behave nicely when based on a local ring R , even in a topos, cf. e.g. [Kock-Reyes, 1977] or [Kock, 2010c], A.5. However, our present axiomatics is too strong to cover such manifolds.

An axiomatics that applies to this case is missing. For one thing, the \mathbb{L} in question should not be a groupoid; even over a field k , some of the formulations given in the present note could be made more homogeneous by including *all* linear maps between 1-dimensional linear subspaces of k^2 , not just the invertible ones, in other words, by including the zero-maps. In this case, \mathbb{L} satisfies the dichotomy: “for any two objects A and B , there is *exactly one* non-invertible map $A \rightarrow B$, and *at least one* invertible map $A \rightarrow B$ ”.

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