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# SEMIUNITAL SEMIMONOIDAL CATEGORIES (APPLICATIONS TO SEMIRINGS AND SEMICORINGS)

JAWAD ABUHLAIL

Abstract. The category ${}_A\mathbb{S}_A$ of bisemimodules over a semialgebra $A$, with the so called *Takahashi's tensor-like product* $-\boxtimes_A -$, is semimonoidal but *not* monoidal. Although not a unit in ${}_A\mathbb{S}_A$, the base semialgebra $A$ has properties of a *semiunit* (in a sense which we clarify in this note). Motivated by this interesting example, we investigate *semiunital semimonoidal categories* $(\mathcal{V}, \bullet, \mathbf{I})$ as a framework for studying notions like *semimonoids* (*semicomonoids*) as well as a notion of monads (comonads) which we call $\mathbb{J}$-*monads* ($\mathbb{J}$-*comonads*) with respect to the endo-functor $\mathbb{J} := \mathbf{I}\bullet - \simeq -\bullet\mathbf{I} : \mathcal{V} \longrightarrow \mathcal{V}$. This motivated also introducing a more generalized notion of monads (comonads) in arbitrary categories with respect to arbitrary endo-functors. Applications to the semiunital semimonoidal variety $({}_A\mathbb{S}_A, \boxtimes_A, A)$ provide us with examples of semiunital $A$-semirings (semicounital $A$-semicorings) and semiunitary semimodules (semicounitary semicomodules) which extend the classical notions of unital rings (counital corings) and unitary modules (counitary comodules).

## 1. Introduction

A *semiring* is, roughly speaking, a ring not necessarily with subtraction. The first natural example of a semiring is the set $\mathbb{N}_0$ of non-negative integers. Other examples include the set Ideal$(R)$ of (two-sided) ideals of every associative ring $R$ and distributive complete lattices. A *semimodule* is, roughly speaking, a module not necessarily with subtraction. The category of Abelian groups is nothing but the category of modules over $\mathbb{Z}$; similarly, the category of commutative monoids is nothing but the category of semimodules over $\mathbb{N}_0$.

Semirings were studied by many algebraists beginning with Dedekind [Ded1984]. Since the sixties of the last century, they were shown to have significant applications in several areas as Automata Theory, Optimization Theory, Tropical Geometry and Idempotent Analysis (for more, see [Gol1999], Gal2002). Recently, Durov [Dur2007] demonstrated that semirings are in one-to-one correspondence with the *algebraic additive monads* on the category **Set** of sets. The theory of semimodules over semirings was developed by

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many authors including Takahashi, Patchkoria and Katsov (*e.g.* [Tak1982], [Tak1982], [Pat2006], [Kat1997]).

A strong connection between corings [Swe1975] over a ring $A$ (coalgebras in the monoidal category ${}_A\mathbf{Mod}_A$ of bimodules over $A$) and their comodules on one side and comonads induced by the tensor product $-\otimes_A-$ and their comodules on the other side has been realized by several authors (*e.g.* [BW2003]). Moreover, the theory of monads and comonads in (autonomous) monoidal categories received increasing attention in the last decade and extensions to arbitrary categories were carried out in several recent papers (*e.g.* [BW2009]).

Using the so called Takahashi's *tensor-like product* $-\boxtimes_A-$ of semimodules over an associative semiring $A$ [Tak1982], notions of *semiunital semirings* and *semicounital semicorings* were introduced by the author in 2008. However, these could not be realized as monoids (comonoids) in the category ${}_A\mathbb{S}_A$ of $(A,A)$-bisemimodules. This is mainly due to the fact that the category $({}_A\mathbb{S}_A,\boxtimes_A,A)$ is *not* monoidal in general (an alternative tensor product $-\otimes_A-$ was recalled by Katsov in [Kat1997]; in fact $({}_A\mathbb{S}_A,\otimes_A,A)$ is monoidal. For the relation between $-\otimes_A-$ and $-\boxtimes_A-$, see [Abu]). Motivated by the desire to fix this defect, we introduce and investigate a notion of *semiunital semimonoidal categories* with prototype $({}_A\mathbb{S}_A,\boxtimes_A,A)$ and investigate *semimonoids* (*semicomonoids*) in such categories as well as their categories of *semimodules* (*semicomodules*). In particular, we realize our semiunital $A$-semirings (semicounital $A$-semicorings) as semimonoids (semicomonoids) in $({}_A\mathbb{S}_A,\boxtimes_A,A)$. Moreover, we introduce and study $\mathbb{J}$-*monads* ($\mathbb{J}$-*comonads*) in an arbitrary category $\mathfrak{A}$, where $\mathbb{J}:\mathfrak{A}\longrightarrow\mathfrak{A}$ is an endo-functor, and apply them to semiunital semimonoidal categories in general and to ${}_A\mathbb{S}_A$ in particular. Our results extend recent ones on monoids (comonoids) in monoidal categories as well as monads (comonads) in arbitrary categories to semimonoids (semicomonoids) in semiunital semimonoidal categories as well as $\mathbb{J}$-monads ($\mathbb{J}$-comonads) in arbitrary categories.

Throughout, $\mathbb{I}$ denotes the identity endo-functor on the category under consideration. The paper is organized as follows. After this introduction, we present in Section 2 our (generalized) notion of $\mathbb{J}$-monads and $\mathbb{J}$-comonads in arbitrary categories. In Section 3, we introduce and investigate *semiunits* in semimonoidal categories. In Section 4, we introduce semimonoids (semicomonoids) in semiunital semimonoidal categories as well as their categories of semimodules (semicomodules). Moreover, we present two *reconstruction* results, namely Theorems 4.8 and 4.17. In Section 5, we consider the semiunital semimonoidal category (variety) of bisemimodules ${}_A\mathbb{S}_A$ over a semialgebra $A$ which provides us with a rich source of concrete examples for applying our results. As mentioned above, these concrete examples were the main motivation behind introducing all the abstract notions in this paper. Further investigations of $\mathbb{J}$-*bimonads* and *Hopf* $\mathbb{J}$-*monads* as well as *bisemimonoids* and *Hopf semimonoids* in semiunital semimonoidal categories will be the subject of a forthcoming paper.

## 2. Monads and Comonads

Recall first the so called *Godement product* of natural transformations between functors:

2.1. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be arbitrary categories. Any natural transformations $\psi : F \longrightarrow G$ and $\psi' : F' \longrightarrow G'$ of functors $\mathfrak{A} \overset{F,G}{\longrightarrow} \mathfrak{B} \overset{F',G'}{\longrightarrow} \mathfrak{C}$ can be multiplied using the *Godement product* to yield a natural transformation $\psi'\psi : F'F \longrightarrow G'G$, where

$$\psi'_{G(X)} \circ F'(\psi_X) = (\psi'\psi)_X = G'(\psi_X) \circ \psi'_{F(X)} \text{ for every } X \in \mathfrak{A}. \tag{1}$$

Moreover, if $\mathfrak{A} \overset{H}{\longrightarrow} \mathfrak{B} \overset{H'}{\longrightarrow} \mathfrak{C}$ are functors and $\phi : G \longrightarrow H$, $\phi' : G' \longrightarrow H'$ are natural transformations, then the following *interchange law* holds

$$(\phi \circ \psi)(\phi' \circ \psi') = (\phi'\phi) \circ (\psi'\psi). \tag{2}$$

2.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories, $L : \mathfrak{A} \longrightarrow \mathfrak{B}$, $R : \mathfrak{B} \longrightarrow \mathfrak{A}$ be functors and $\mathbb{J} : \mathfrak{A} \longrightarrow \mathfrak{A}$, $\mathbb{K} : \mathfrak{B} \longrightarrow \mathfrak{B}$ be endo-functors such that $R\mathbb{K} \simeq \mathbb{J}R$ and $L\mathbb{J} \simeq \mathbb{K}L$. We say that $(L, R)$ is a $(\mathbb{J}, \mathbb{K})$-*adjoint pair* iff we have natural isomorphisms in $X \in \mathfrak{A}$ and $Y \in \mathfrak{B}$ :

$$\mathfrak{B}(L\mathbb{J}(X), \mathbb{K}(Y)) \simeq \mathfrak{A}(\mathbb{J}(X), R\mathbb{K}(Y)).$$

For the special case $\mathbb{J} = \mathbb{I}_{\mathfrak{A}}$ and $\mathbb{K} = \mathbb{I}_{\mathfrak{B}}$, we recover the classical notion of adjoint pairs.

Till the end of this section, $\mathfrak{A}$ is an arbitrary category.

2.3. Let $\mathbb{T} : \mathfrak{A} \longrightarrow \mathfrak{A}$ be an endo-functor. An object $X \in \mathrm{Obj}(\mathfrak{A})$ is said to have a $\mathbb{T}$-*action* or to be a $\mathbb{T}$-*act* iff there is a morphism $\varrho_X : \mathbb{T}(X) \longrightarrow X$ in $\mathfrak{A}$. For two objects $X, X'$ with $\mathbb{T}$-actions, we say that a morphism $\varphi : X \longrightarrow X'$ in $\mathfrak{A}$ is a *morphism of* $\mathbb{T}$-*acts* iff the following diagram is commutative

$$\begin{array}{ccc} \mathbb{T}(X) & \xrightarrow{\varrho_X} & X \\ {\scriptstyle \mathbb{T}(\varphi)}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{T}(X') & \xrightarrow[\varrho_{X'}]{} & X' \end{array}$$

The *category of* $\mathbb{T}$-*acts* is denoted by $\mathbf{Act}_{\mathbb{T}}$. Dually, one can define the category $\mathbf{Coact}^{\mathbb{T}}$ of $\mathbb{T}$-*coacts.*

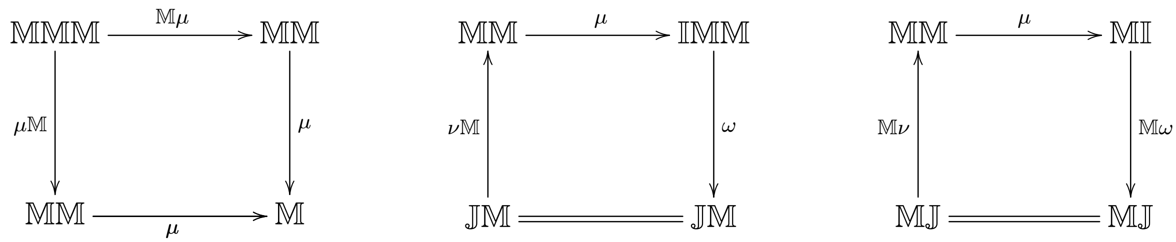
2.4. Remark. The objects of $\mathbf{Coact}^{\mathbb{F}}$, where $\mathbb{F} : \mathbf{Set} \longrightarrow \mathbf{Set}$ is an endo-functor, play an important role in logic and theoretical computer science. They are called $\mathbb{F}$-*systems* (*e.g.* [Rut2000]). Some references call these $\mathbb{F}$-*coalgebras* (*e.g.* [Gum1999]). For us, coalgebras are always coassociative and counital unless something else is explicitly specified.

$\mathbb{J}$-Monads.

2.5. Let $\mathbb{J} : \mathfrak{A} \longrightarrow \mathfrak{A}$ be an endo-functor. With a $\mathbb{J}$-*monad* on $\mathfrak{A}$ we mean a datum $(\mathbb{M}, \mu, \omega, \nu; \mathbb{J})$ consisting of an endo-functor $\mathbb{M} : \mathfrak{A} \longrightarrow \mathfrak{A}$ associated with natural transformations

$$\mu : \mathbb{M}\mathbb{M} \longrightarrow \mathbb{M}, \ \omega : \mathbb{I} \longrightarrow \mathbb{J} \text{ and } \nu : \mathbb{J} \longrightarrow \mathbb{M}$$

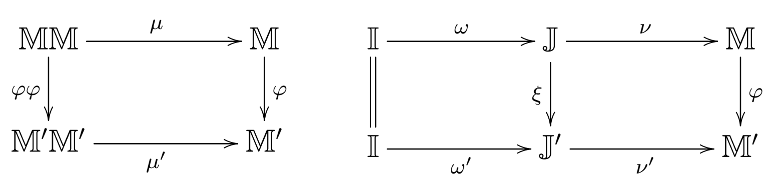
such that the following diagrams are commutative



*i.e.* for every $X \in \mathfrak{A}$ we have

$\mu_X \circ \mathbb{M}(\mu_X) = \mu_X \circ \mu_{\mathbb{M}(X)},\ \nu_{\mathbb{M}(X)} \circ \omega_{\mathbb{M}(X)} \circ \mu_X = \mathbb{I}_{\mathbb{M}\mathbb{M}(X)}$ and $\mathbb{M}(\nu_X) \circ \mathbb{M}(\omega_X) \circ \mu_X = \mathbb{I}_{\mathbb{M}\mathbb{M}(X)}$.

2.6. With $\mathbf{JMonad}_{\mathfrak{A}}$ we denote the category whose objects are $\mathbb{J}$-monads, where $\mathbb{J}$ runs over the class of endo-functors on $\mathfrak{A}$. A morphism $(\varphi; \xi) : (\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \longrightarrow (\mathbb{M}', \mu', \omega', \nu'; \mathbb{J}')$ in this category consists of natural transformations $\varphi : \mathbb{M} \longrightarrow \mathbb{M}'$ and $\xi : \mathbb{J} \longrightarrow \mathbb{J}'$ such that the following diagrams are commutative



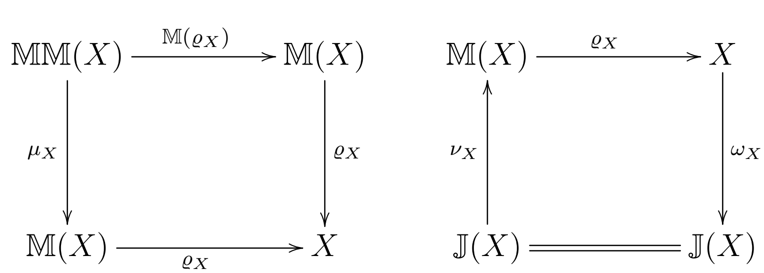
*i.e.* for every $X \in \mathfrak{A}$ we have

$$\varphi_X \circ \mu_X = \mu'_X \circ \varphi_{\mathbb{M}'(X)} \circ \mathbb{M}(\varphi_X),\ \xi_X \circ \omega_X = \omega'_X \text{ and } \varphi_X \circ \nu_X = \nu'_X \circ \xi_X.$$

For a fixed endo-functor $\mathbb{J} : \mathfrak{A} \longrightarrow \mathfrak{A}$, we denote by $\mathbb{J}$-$\mathbf{Monad}_{\mathfrak{A}}$ the subcategory of $\mathbf{JMonad}_{\mathfrak{A}}$ of $\mathbb{J}$-monads on $\mathfrak{A}$ with $\omega$ the identity natural transformation. In the special case $\mathbb{J} = \mathbb{I}_{\mathfrak{A}}$ and $\omega$ is the identity natural transformation, we drop these from our notation and recover the classical notion of *monads* on $\mathfrak{A}$.

2.7. Remark. As we saw above, a $\mathbb{J}$-monad $(\mathbb{M}, \mu, \omega, \nu; \mathbb{J})$ is a generalized notion of a monad. However, it can also be seen as just a monad $(\mathbb{M}, \mu, \eta)$ whose unit $\eta := \mathbb{I} \overset{\omega}{\longrightarrow} \mathbb{J} \overset{\nu}{\longrightarrow} \mathbb{M}$ factorizes through $\mathbb{J}$. Having this in mind, a morphism $(\varphi; \xi) : (\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \longrightarrow (\mathbb{M}', \mu', \omega', \nu'; \mathbb{J}')$ in $\mathbf{JMonad}_{\mathfrak{A}}$ is just a morphism of monads which is compatible with the factorizations of the units through $\mathbb{J}$ and $\mathbb{J}'$.

2.8. Let $(\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \in \mathbf{JMonad}_{\mathfrak{A}}$. An $(\mathbb{M}; \mathbb{J})$-*module* is an object $X \in \mathrm{Obj}(\mathfrak{A})$ with a morphism $\varrho_X : \mathbb{M}(X) \longrightarrow X$ in $\mathfrak{A}$ such that the following diagrams are commutative



The category of $(\mathbb{M};\mathbb{J})$-modules and morphisms those of $\mathbb{M}$-acts is denoted by $\mathfrak{A}_{(\mathbb{M};\mathbb{J})}$. In case $\mathbb{J} \simeq \mathbb{I}_{\mathfrak{A}}$ and $\omega$ is the identity natural transformation, we recover the category of $\mathbb{M}$-*modules* of the monad $\mathbb{M}$.

2.9. Let $(\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \in \mathbf{JMonad}_{\mathfrak{A}}$. For every $X \in \mathrm{Obj}(\mathbb{A})$, $\mathbb{M}(X)$ is an $(\mathbb{M};\mathbb{J})$-module through

$$\varrho_{\mathbb{M}(X)} : \mathbb{M}(\mathbb{M}(X)) \overset{\mu_X}{\longrightarrow} \mathbb{M}(X).$$

Such object are called *free* $(\mathbb{M};\mathbb{J})$-*modules* and we have the so called *free functor*

$$\mathcal{F}_{(\mathbb{M};\mathbb{J})} : \mathfrak{A} \longrightarrow \mathfrak{A}_{(\mathbb{M};\mathbb{J})},\ X \mapsto \mathbb{M}(X).$$

The full subcategory of free $(\mathbb{M};\mathbb{J})$-modules is called the *Kleisli category* and is denoted by $\widetilde{\mathfrak{A}}_{(\mathbb{M};\mathbb{J})}$.

2.10. Remark. Let $(\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \in \mathbf{JMonad}_{\mathfrak{A}}$ with $\mathbb{M}\mathbb{J} \simeq \mathbb{J}\mathbb{M}$. If $X$ is an $(\mathbb{M};\mathbb{J})$-module, then $\mathbb{J}(X)$ is also an $(\mathbb{M};\mathbb{J})$-module through

$$\varrho_{\mathbb{J}(X)} : \mathbb{M}\mathbb{J}(X) \simeq \mathbb{J}\mathbb{M}(X) \overset{\mathbb{J}(\varrho_X)}{\longrightarrow} \mathbb{J}(X).$$

Moreover, if $Y = \mathbb{M}(X)$ is a free $(\mathbb{M};\mathbb{J})$-module, then $\mathbb{J}(Y) = \mathbb{J}\mathbb{M}(X) \simeq \mathbb{M}\mathbb{J}(X)$ is also a free $(\mathbb{M};\mathbb{J})$-module. One can easily see that $\mathbb{J}$ can be lifted to endo-functors $\mathbb{J}' : \mathfrak{A}_{(\mathbb{M};\mathbb{J})} \longrightarrow \mathfrak{A}_{(\mathbb{M};\mathbb{J})}$ and $\widetilde{\mathbb{J}} : \widetilde{\mathfrak{A}}_{(\mathbb{M};\mathbb{J})} \longrightarrow \widetilde{\mathfrak{A}}_{(\mathbb{M};\mathbb{J})}$.

2.11. Let $(\mathbb{M}, \mu, \omega, \nu; \mathbb{J}) \in \mathbf{JMonad}_{\mathfrak{A}}$ and assume that $\mathbb{M}\mathbb{J} \simeq \mathbb{J}\mathbb{M}$. We have a natural isomorphism for every $X \in \mathfrak{A}$ and $Y \in \mathfrak{A}_{(\mathbb{M};\mathbb{J})}$ :

$$\mathfrak{A}_{(\mathbb{M};\mathbb{J})}(\mathcal{F}_{(\mathbb{M};\mathbb{J})}(X), \mathbb{J}(Y)) \simeq \mathfrak{A}(X, \mathbb{J}(Y)),\ f \mapsto f \circ (\nu \circ \omega)_X$$

with inverse $g \longmapsto \varrho_{\mathbb{J}(Y)} \circ \mathcal{F}_{(\mathbb{M};\mathbb{J})}(g)$. Consider the forgetful functor $U : \mathfrak{A}_{(\mathbb{M};\mathbb{J})} \longrightarrow \mathfrak{A}$ and the endo-functor $\mathbb{J}' : \mathfrak{A}_{(\mathbb{M};\mathbb{J})} \longrightarrow \mathfrak{A}_{(\mathbb{M};\mathbb{J})}$ (see Remark 2.10). We have a natural isomorphism

$$\mathfrak{A}_{(\mathbb{M};\mathbb{J})}(\mathcal{F}_{(\mathbb{M};\mathbb{J})}(\mathbb{J}(X)), \mathbb{J}'(Y)) \simeq \mathfrak{A}(\mathbb{J}(X), U(\mathbb{J}'(Y)));$$

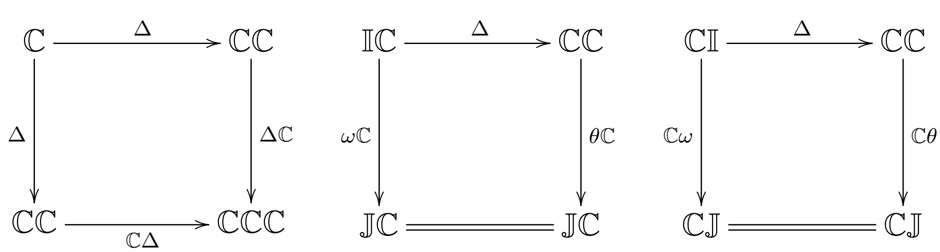
*i.e.* $(\mathcal{F}_{(\mathbb{M};\mathbb{J})}(-), U)$ is a $(\mathbb{J}, \mathbb{J}')$-adjoint pair.

$\mathbb{J}$-Comonads.

2.12. Let $\mathbb{J}$ be an endo-functor on $\mathfrak{A}$. With a $\mathbb{J}$-*comonad* on $\mathfrak{A}$ we mean a datum $(\mathbb{C}, \Delta, \omega, \theta)$ consisting of an endo-functor $\mathbb{C} : \mathfrak{A} \longrightarrow \mathfrak{A}$ associated with natural transformations

$$\Delta : \mathbb{C} \longrightarrow \mathbb{C}\mathbb{C},\ \omega : \mathbb{I} \longrightarrow \mathbb{J} \text{ and } \theta : \mathbb{C} \longrightarrow \mathbb{J}$$

such that the following diagrams are commutative



*i.e.* for every $X \in \mathfrak{A}$ we have

$$\Delta_{\mathbb{C}(X)} \circ \Delta_X = \mathbb{C}(\Delta_X) \circ \Delta_X,\ \theta_{\mathbb{C}(X)} \circ \Delta_X = \omega_{\mathbb{C}(X)} \text{ and } \mathbb{C}(\theta_X) \circ \Delta_X = \mathbb{C}(\omega_X).$$

2.13. By $\mathbf{JComonad}_{\mathfrak{A}}$ we denote the category whose objects are $\mathbb{J}$-comonads, where $\mathbb{J}$ runs over the class of endo-functors on $\mathfrak{A}$. A morphism $(\psi;\xi) : (\mathbb{C}, \Delta, \omega, \theta; \mathbb{J}) \longrightarrow (\mathbb{C}', \Delta', \omega', \theta'; \mathbb{J}')$ in this category consists of natural transformations $\psi : \mathbb{C} \longrightarrow \mathbb{C}'$ and $\xi : \mathbb{J} \longrightarrow \mathbb{J}'$ such that the following diagrams are commutative

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\ \Delta\ } & \mathbb{C}\mathbb{C} \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi\psi} \\ \mathbb{C}' & \xrightarrow[\ \Delta'\ ]{} & \mathbb{C}'\mathbb{C}' \end{array} \qquad \begin{array}{ccccc} \mathbb{C} & \xrightarrow{\ \theta\ } & \mathbb{J} & \xleftarrow{\ \omega\ } & \mathbb{I} \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \xi} & & \Vert \\ \mathbb{C}' & \xrightarrow[\ \theta'\ ]{} & \mathbb{J}' & \xleftarrow[\ \omega'\ ]{} & \mathbb{I} \end{array}$$

*i.e.* for every $X \in \mathfrak{A}$ we have

$$\psi_{\mathbb{C}'(X)} \circ \mathbb{C}(\psi_X) \circ \Delta_X = \Delta'_X \circ \psi_X,\ \xi_X \circ \theta_X = \theta'_X \circ \psi_X \text{ and } \xi_X \circ \omega_X = \omega'_X.$$

For a fixed endo-functor $\mathbb{J} : \mathfrak{A} \longrightarrow \mathfrak{A}$, we denote by $\mathbb{J}$-$\mathbf{Comonad}_{\mathfrak{A}}$ the subcategory of $\mathbb{J}$-comonads on $\mathfrak{A}$ with $\omega$ the identity transformation. In the special case $\mathbb{J} = \mathbb{I}_{\mathfrak{A}}$ and $\omega$ is the identity natural transformation, we drop these from our notation and recover the notion of *comonads* on $\mathfrak{A}$.

2.14. REMARK. $\mathbb{J}$-Comonads are *not* fully dual to $\mathbb{J}$-monads. Recall from Remark 2.7 that a $\mathbb{J}$-monad can be seen as a monad whose unit factorizes through $\mathbb{J}$. On the other hand, $\mathbb{J}$-comonads cannot be seen as a special type of comonads. The lack of duality is because not all arrows are reversed; the arrow $\omega : \mathbb{I} \longrightarrow \mathbb{J}$ is assumed for both. Notice that keeping this arrow is suggested by the concrete example in Section 5.

2.15. Let $(\mathbb{C}, \Delta, \omega, \theta; \mathbb{J}) \in \mathbf{JComonad}_{\mathfrak{A}}$. A $(\mathbb{C};\mathbb{J})$-*comodule* is an object $X \in \mathrm{Obj}(\mathfrak{A})$ along with a morphism $\varrho^X : X \longrightarrow \mathbb{C}(X)$ in $\mathfrak{A}$ such that the following diagrams are commutative

$$\begin{array}{ccc} X & \xrightarrow{\ \varrho^X\ } & \mathbb{C}(X) \\ {\scriptstyle \varrho^X}\downarrow & & \downarrow{\scriptstyle \mathbb{C}(\varrho^X)} \\ \mathbb{C}(X) & \xrightarrow[\ \Delta_X\ ]{} & \mathbb{C}\mathbb{C}(X) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\ \varrho^X\ } & \mathbb{C}(X) \\ {\scriptstyle \omega_X}\downarrow & & \downarrow{\scriptstyle \theta_X} \\ \mathbb{J}(X) & = \!= \!= & \mathbb{J}(X) \end{array}$$

The category of $(\mathbb{C};\mathbb{J})$-comodules and morphisms those of $\mathbb{C}$-coacts is denoted by $\mathfrak{A}^{(\mathbb{C};\mathbb{J})}$. In case $\mathbb{J} = \mathbb{I}_{\mathfrak{A}}$ and $\omega$ is the identity natural transformation, we recover the category of $\mathbb{C}$-*comodules* for the comonad $\mathbb{C}$.

2.16. Let $(\mathbb{C},\Delta,\varepsilon;\mathbb{J}) \in$ **JComonad**$_{\mathfrak{A}}$. For every $X \in \mathrm{Obj}(\mathfrak{A})$, $\mathbb{C}(X)$ has a canonical structure of a $(\mathbb{C};\mathbb{J})$-comodule through

$$\varrho^{\mathbb{C}(X)} : \mathbb{C}(X) \overset{\Delta_X}{\longrightarrow} \mathbb{C}\mathbb{C}(X).$$

Such object are called *cofree* $(\mathbb{C};\mathbb{J})$*-comodules* and we have the so called *cofree functor*

$$\mathcal{F}^{\mathbb{C}} : \mathfrak{A} \longrightarrow \mathfrak{A}^{(\mathbb{C};\mathbb{J})},\ X \mapsto \mathbb{C}(X).$$

The full subcategory of cofree $(\mathbb{C};\mathbb{J})$-comodules is called the *Kleisli category* of $\mathbb{C}$ and is denoted by $\widetilde{\mathfrak{A}}^{(\mathbb{C};\mathbb{J})}$.

2.17. Remark. Let $(\mathbb{C},\Delta,\omega,\theta;\mathbb{J}) \in$ **JComonad**$_{\mathfrak{A}}$ with $\mathbb{J}\mathbb{C} \simeq \mathbb{C}\mathbb{J}$. If $X$ is a $(\mathbb{C};\mathbb{J})$-comodule, then $\mathbb{J}(X)$ is also a $(\mathbb{C};\mathbb{J})$-comodule through

$$\varrho^{\mathbb{J}(X)} : \mathbb{J}(X) \overset{\mathbb{J}(\varrho^X)}{\longrightarrow} \mathbb{J}\mathbb{C}(X) \simeq \mathbb{C}(\mathbb{J}(X)).$$

If $Y = \mathbb{C}(X)$ is a cofree $(\mathbb{C};\mathbb{J})$-comodule, then $\mathbb{J}(Y) = \mathbb{J}\mathbb{C}(X) \simeq \mathbb{C}\mathbb{J}(X)$ is also a cofree $(\mathbb{C},\mathbb{J})$-comodule. One case easily see that $\mathbb{J}$ lifts to endo-functors $\mathbb{J}' : \mathfrak{A}^{(\mathbb{C};\mathbb{J})} \longrightarrow \mathfrak{A}^{(\mathbb{C};\mathbb{J})}$ and $\widetilde{\mathbb{J}} : \widetilde{\mathfrak{A}}^{(\mathbb{C};\mathbb{J})} \longrightarrow \widetilde{\mathfrak{A}}^{(\mathbb{C};\mathbb{J})}$.

2.18. Let $(\mathbb{C},\Delta,\omega,\theta;\mathbb{J}) \in$ **JComonad**$_{\mathfrak{A}}$ with $\mathbb{J}$ idempotent and $\mathbb{J}\mathbb{C} \simeq \mathbb{C}\mathbb{J}$. Consider the forgetful functor $U : \mathfrak{A}^{(\mathbb{C};\mathbb{J})} \longrightarrow \mathfrak{A}$ and the endo-functor $\mathbb{J}' : \mathfrak{A}^{(\mathbb{C};\mathbb{J})} \longrightarrow \mathfrak{A}^{(\mathbb{C};\mathbb{J})}$. We have a natural isomorphism for $X \in \mathfrak{A}$ and $Y \in \mathfrak{A}^{(\mathbb{C};\mathbb{J})}$ :

$$\mathfrak{A}^{(\mathbb{C};\mathbb{J})}(\mathbb{J}'(Y), \mathcal{F}^{\mathbb{C}}(\mathbb{J}(X))) \simeq \mathfrak{A}(U(\mathbb{J}'(Y)), \mathbb{J}(X)),\ f \mapsto \theta_{\mathbb{J}(X)} \circ f$$

with inverse $g \longmapsto \mathcal{F}^{\mathbb{C}}(g) \circ \varrho^{\mathbb{J}(Y)}$; *i.e.* $(U, \mathcal{F}^{(\mathbb{C};\mathbb{J})}(-))$ is a $(\mathbb{J}',\mathbb{J})$-adjoint pair.

2.19. Proposition. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories, $L : \mathfrak{A} \longrightarrow \mathfrak{B}$, $R : \mathfrak{B} \longrightarrow \mathfrak{A}$ be functors and $\mathbb{J} : \mathfrak{A} \longrightarrow \mathfrak{A}$, $\mathbb{K} : \mathfrak{B} \longrightarrow \mathfrak{B}$ endo-functors such that $L\mathbb{J} \simeq \mathbb{K}L$, $\mathbb{J}R \simeq R\mathbb{K}$ and $(L,R)$ is a $(\mathbb{J},\mathbb{K})$-adjoint pair.*

1. *$(\mathcal{L}, R)$ is an adjoint pair where $\mathcal{L} : \mathbb{J}(\mathfrak{A}) \overset{L}{\longrightarrow} \mathbb{K}(\mathfrak{B})$ and $\mathcal{R} : \mathbb{K}(\mathfrak{B}) \overset{R}{\longrightarrow} \mathbb{J}(\mathfrak{A})$ with unit and counit of adjunction given by*
$$\eta : \mathbb{J} \longrightarrow RL\mathbb{J} \text{ and } \varepsilon : LR\mathbb{K} \longrightarrow \mathbb{K}.$$
2. *$RL$ is a monad on $\mathbb{J}(\mathfrak{A})$ with*
$$\mu_{RL} : (RL)(RL)\mathbb{J} \simeq R(LR\mathbb{K})L \overset{R\varepsilon L}{\longrightarrow} R\mathbb{K}L \simeq (R)L\mathbb{J} \text{ and } \eta_{RL} := \eta.$$
3. *$LR$ is a comonad on $\mathbb{K}(\mathfrak{B})$ with*
$$\Delta_{LR} : (LR)\mathbb{K} \simeq L\mathbb{J}R \overset{L\eta R}{\longrightarrow} L(RL\mathbb{J})R \simeq (LR)(LR)\mathbb{K} \text{ and } \varepsilon_{LR} := \varepsilon.$$
4. *$L$ is a monad on $\mathbb{J}(\mathfrak{A})$ if and only if $R$ is a comonad on $\mathbb{K}(\mathfrak{B})$. In this case, $\mathbb{J}(\mathfrak{A})_L \simeq \mathbb{K}(\mathfrak{B})^R$.*
5. *$L$ is a comonad on $\mathbb{J}(\mathfrak{A})$ if and only if $R$ is a monad on $\mathbb{K}(\mathfrak{B})$. In this case, $\widetilde{\mathbb{J}(\mathfrak{A})}^L \simeq \widetilde{\mathbb{K}(\mathfrak{B})}_R$.*

Proof. By assumption, $L\mathbb{J} \simeq \mathbb{K}L$ whence $\mathcal{L}(\mathbb{J}(\mathfrak{A})) := L\mathbb{J}(\mathfrak{A}) = \mathbb{K}L(\mathfrak{A}) \subseteq \mathbb{K}(\mathfrak{B})$ and $\mathbb{J}R \simeq R\mathbb{K}$ whence $\mathcal{R}(\mathbb{K}(\mathfrak{B})) := R(\mathbb{K}(\mathfrak{B})) = \mathbb{J}R(\mathfrak{B}) \subseteq \mathbb{J}(\mathfrak{A})$. The assumptions imply that $(\mathcal{L}, \mathcal{R})$ is an adjoint pair. The result follows now from the classical result on right adjoint pairs (*e.g.* [EM1965, Proposition 3.1], [BW2009, 2.5, 2.6]). ■

## 3. Semiunital Semimonoidal Categories

A semimonoidal category is roughly speaking a monoidal category not necessarily with a unit object. The reader might consult the literature for the precise definitions and for the notions of (op)-semimonoidal functors between such categories. In this section, we introduce a notion of *semiunital semimonoidal categories* and *semiunital* (*op-*)*semimonoidal functors*.

Semiunits.

3.1. Let $(\mathcal{V}, \bullet)$ be a semimonoidal category with natural isomorphisms $\gamma_{X,Y,Z} : (X \bullet Y) \bullet Z \longrightarrow X \bullet (Y \bullet Z)$ for all $X, Y, Z \in \mathcal{V}$. We say that $\mathbf{I} \in \mathcal{V}$ is a *semiunit* iff

1. there is a natural transformation $\omega : \mathbb{I} \longrightarrow (\mathbf{I} \bullet -)$;
2. there exists an isomorphisms of functors $\mathbf{I} \bullet - \simeq - \bullet \mathbf{I}$, *i.e.* there is a natural isomorphism $\mathbf{I} \bullet X \overset{\ell_X}{\simeq} X \bullet \mathbf{I}$ in $\mathcal{V}$ with inverse $\wp_X$, for each object $X$ of $\mathcal{V}$, such that $\ell_{\mathbf{I}} = \wp_{\mathbf{I}}$ and the following diagram is commutative for all $X, Y \in \mathcal{V}$ :

$$\begin{array}{ccccc}
(\mathbf{I} \bullet X) \bullet Y & \xrightarrow{\gamma_{\mathbf{I},X,Y}} & \mathbf{I} \bullet (X \bullet Y) & \xrightarrow{\ell_{X \bullet Y}} & (X \bullet Y) \bullet \mathbf{I} \\
{\scriptstyle \ell_X \bullet Y}\downarrow & & & & \downarrow{\scriptstyle \gamma_{X,Y,\mathbf{I}}} \\
(X \bullet \mathbf{I}) \bullet Y & \xrightarrow[\gamma_{X,\mathbf{I},Y}]{} & X \bullet (\mathbf{I} \bullet Y) & \xrightarrow[X \bullet \ell_Y]{} & X \bullet (Y \bullet \mathbf{I})
\end{array}$$

3. the following diagram is commutative for all $X, Y \in \mathcal{V}$ :

$$\begin{array}{ccccc}
(\mathbf{I} \bullet X) \bullet Y & \xleftarrow{\omega_X \bullet Y} & X \bullet Y & \xrightarrow{X \bullet \omega_Y} & X \bullet (\mathbf{I} \bullet Y) \\
& {\scriptstyle \simeq}\searrow & \downarrow{\scriptstyle \omega_{X \bullet Y}} & \swarrow{\scriptstyle \simeq} & \\
& & \mathbf{I} \bullet (X \bullet Y) & &
\end{array}$$

3.2. If $X \overset{\omega_X}{\simeq} \mathbf{I} \bullet X$ $(\overset{\ell_X}{\simeq} X \bullet \mathbf{I})$, then we say that $X$ is *firm* and set $\lambda_X := \omega_X^{-1} : \mathbf{I} \bullet X \longrightarrow X$ and $\kappa_X : X \bullet \mathbf{I} \overset{\wp_X}{\simeq} \mathbf{I} \bullet X \overset{\omega_X^{-1}}{\longrightarrow} X$. With $\mathcal{V}^{\mathrm{firm}}$ we denote the *full* subcategory of firm objects in $\mathcal{V}$. If $\mathbf{I}$ is firm (called also *pseudo-idempotent*) and $\omega_{\mathbf{I}}^{-1} \bullet \mathbf{I} = \mathbf{I} \bullet \omega_{\mathbf{I}}^{-1}$, then one says that $\mathbf{I}$ is *idempotent* [Koc2008].

3.3. Remark. Let $(\mathcal{V}, \bullet; \gamma)$ be a semimonoidal category. One says that $\mathcal{V}$ is *monoidal* [Mac1998] iff $\mathcal{V}$ has a *unit* (or an *LR unit*), *i.e.* a distinguished object $\mathbf{I} \in \mathcal{V}$ with natural isomorphisms $\mathbf{I} \bullet X \overset{\lambda_X}{\simeq} X$ and $X \bullet \mathbf{I} \overset{\kappa_X}{\simeq} X$ such that $X \bullet \lambda_Y = \kappa_X \bullet Y$ for all $X, Y \in \mathcal{V}$ (equivalently, $\lambda_{\mathbf{I}} = \kappa_{\mathbf{I}}$, $\lambda_{X \bullet Y} = \lambda_X \bullet Y$ and $\kappa_{X \bullet Y} = X \bullet \kappa_Y$ for all $X, Y \in \mathcal{V}$). Kock [Koc2008] called an object $\mathbf{I} \in \mathcal{V}$ a *Saavedra unit* – called also a *reduced unit* – iff it is pseudo-idempotent and *cancellable* in the sense that the endo-functors $\mathbf{I} \bullet -$ and $- \bullet \mathbf{I}$ are full and faithful (equivalently, $\mathbf{I}$ is idempotent and the endo-functors $\mathbf{I} \bullet -$ and $- \bullet \mathbf{I}$ are equivalences of categories). Moreover, he showed that $\mathbf{I}$ is a unit if and only if $\mathbf{I}$ is a Saavedra unit. Indeed, every unit is a semiunit, whence our notion of semiunital semimonoidal categories generalizes the classical notion of monoidal categories.

3.4. Let $(\mathcal{V}, \bullet, \mathbf{I}_{\mathcal{V}}; \omega_{\mathcal{V}})$ and $(\mathcal{W}, \otimes, \mathbf{I}_{\mathcal{W}}; \omega_{\mathcal{W}})$ be semiunital semimonoidal categories. A semimonoidal functor $F : \mathcal{V} \longrightarrow \mathcal{W}$, with a natural transformation $\phi : F(-) \otimes F(-) \longrightarrow F(- \bullet -)$, is said to be *semiunital semimonoidal* iff there exists a *coherence morphism* $\tilde{\phi} : \mathbf{I}_{\mathcal{W}} \longrightarrow F(\mathbf{I}_{\mathcal{V}})$ in $\mathcal{W}$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathbf{I}_{\mathcal{W}} \otimes F(X) & \xrightarrow{\ \ell_{F(X)}\ } & F(X) \otimes \mathbf{I}_{\mathcal{W}} \\
{\scriptstyle \tilde{\phi} \otimes F(X)} \downarrow & & \downarrow {\scriptstyle F(X) \otimes \tilde{\phi}} \\
F(\mathbf{I}_{\mathcal{V}}) \otimes F(X) & & F(X) \otimes F(\mathbf{I}_{\mathcal{V}}) \\
{\scriptstyle \phi_{\mathbf{I}_{\mathcal{V}}, X}} \downarrow & & \downarrow {\scriptstyle \phi_{X, \mathbf{I}_{\mathcal{V}}}} \\
F(\mathbf{I}_{\mathcal{V}} \bullet X) & \xrightarrow[\ F(\ell_X)\ ]{} & F(X \bullet \mathbf{I}_{\mathcal{V}})
\end{array}$$

Moreover, we say that $F$ is a *strong* (*strict*) *semiunital semimonoidal functor* iff $F$ is strong (strict) as a semimonoidal functor and $\phi$ is an isomorphism (identity). For two semimonoidal functors $F, F' : \mathcal{V} \longrightarrow \mathcal{W}$, we say that a semimonoidal natural transformation $\varsigma : F \longrightarrow F'$ is *semiunital semimonoidal* iff the following diagram is commutative

$$\begin{array}{ccc}
 & \mathbf{I}_{\mathcal{W}} & \\
{\scriptstyle \tilde{\phi}} \swarrow & & \searrow {\scriptstyle \phi'} \\
F(\mathbf{I}_{\mathcal{V}}) & \xrightarrow[\ \varsigma_{\mathbf{I}_{\mathcal{V}}}\ ]{} & F'(\mathbf{I}_{\mathcal{V}})
\end{array}$$

One can dually define *semiunital* (*strong*, *strict*) *op-semimonoidal functors* and semiunital natural transformations between them.

3.5. Remark. Let $(\mathcal{V}, \bullet, \mathbf{I}; \omega)$ be a semiunital semimonoidal category and consider the functor

$$\mathbb{J} := \mathbf{I} \bullet - : \mathcal{V} \longrightarrow \mathcal{V}.$$

1. We have natural isomorphisms

$$\mathbb{J}(\mathbf{I}) \bullet X \simeq \mathbb{J}(\mathbb{J}(X)) \simeq X \bullet \mathbb{J}(\mathbf{I}) \text{ and } \mathbb{J}(X) \bullet Y \simeq X \bullet \mathbb{J}(Y)$$

   for all $X, Y \in \mathcal{V}$.

2. $\mathbb{J}$ is op-semimonoidal with natural transformation $\psi_{X,Y} : \mathbb{J}(- \bullet -) \longrightarrow \mathbb{J}(-) \bullet \mathbb{J}(-)$ is given by the composition of morphisms

$$\psi_{X,Y} : \mathbf{I} \bullet (X \bullet Y) \overset{\omega_{\mathbf{I}} \bullet (X \bullet Y)}{\longrightarrow} (\mathbf{I} \bullet \mathbf{I}) \bullet (X \bullet Y) \simeq (\mathbf{I} \bullet X) \bullet (\mathbf{I} \bullet Y)$$

   for all $X, Y \in \mathcal{V}$.

3. Assume that $\mathbf{I}$ is firm.

   (a) $\mathbb{J}$ is strong semiunital semimonoidal with

$$\phi_{X,Y} : (\mathbf{I} \bullet X) \bullet (\mathbf{I} \bullet Y) \simeq (\mathbf{I} \bullet \mathbf{I}) \bullet (X \bullet Y) \overset{\omega_{\mathbf{I}}^{-1} \bullet (X \bullet Y)}{\simeq} \mathbf{I} \bullet (X \bullet Y) \text{ and } \tilde{\phi} := \omega_{\mathbf{I}} : \mathbf{I} \longrightarrow \mathbf{I} \bullet \mathbf{I}$$

   for all $X, Y \in \mathcal{V}$.

   (b) $\mathbb{J}$ is strong semiunital op-semimonoidal with

$$\tilde{\psi} := \omega_{\mathbf{I}}^{-1} : \mathbf{I} \bullet \mathbf{I} \longrightarrow \mathbf{I}.$$

   (c) the full subcategory $(\mathcal{V}^{\text{firm}}, \bullet, \mathbf{I})$ is monoidal.

   (d) $(\mathbb{J}(\mathcal{V}), \bullet, \mathbf{I})$ is a monoidal full subcategory of $(\mathcal{V}^{\text{firm}}, \bullet, \mathbf{I})$ with

$$\begin{aligned} \lambda_{\mathbf{I} \bullet X} &: \mathbf{I} \bullet (\mathbf{I} \bullet X) \overset{\gamma_{\mathbf{I},\mathbf{I},X}^{-1}}{\simeq} (\mathbf{I} \bullet \mathbf{I}) \bullet X \overset{\omega_{\mathbf{I}}^{-1}}{\simeq} \mathbf{I} \bullet X; \\ \kappa_{\mathbf{I} \bullet X} &: (\mathbf{I} \bullet X) \bullet \mathbf{I} \overset{\gamma_{\mathbf{I},X,\mathbf{I}}}{\simeq} \mathbf{I} \bullet (X \bullet \mathbf{I}) \overset{\mathbf{I} \bullet \wp_X}{\simeq} \mathbf{I} \bullet (\mathbf{I} \bullet X) \overset{\lambda_{\mathbf{I} \bullet X}}{\simeq} \mathbf{I} \bullet X \end{aligned}$$

   for every $X \in \mathcal{V}$.

3.6. DEFINITION. *Let* $(\mathcal{V}, \bullet, \mathbf{I}; \omega)$ *be a semiunital semimonoidal category. We say that* $V \in \mathcal{V}$ *has a* left dual *iff there exists* $V_l^{\circledast} \in \mathcal{V}$ *along with morphisms* $f : \mathbf{I} \longrightarrow \mathbf{I} \bullet V \bullet V_l^{\circledast}$ *and* $g : \mathbf{I} \bullet V_l^{\circledast} \bullet V \longrightarrow \mathbf{I}$ *in* $\mathcal{V}$ *such that*

$$(V \bullet g) \circ (\ell_V \bullet V_l^{\circledast} \bullet V) \circ (f \bullet V) = \ell_V \text{ and } (g \bullet V_l^{\circledast}) \circ (\wp_{V^{\circledast}} \bullet V \bullet V_l^{\circledast}) \circ (V_l^{\circledast} \bullet f) = \wp_{V^{\circledast}}.$$

*A* right dual $V_r^{\circledast}$ *of* $V$ *is defined symmetrically. We say that* $\mathcal{V}$ *is* left *(*right*)* autonomous, *or* left *(*right*)* rigid *iff every object in* $\mathcal{V}$ *has a left (right) dual.*

3.7. Definition. *Let $(\mathcal{V},\bullet,\mathbf{I};\omega)$ be a semiunital semimonoidal category. We say that $\mathcal{V}$ is* right (left) closed *iff for every $V\in\mathcal{V}$, the functor $-\bullet V:\mathbb{J}(\mathcal{V})\longrightarrow\mathbb{J}(\mathcal{V})$ ($V\bullet -:\mathbb{J}(\mathcal{V})\longrightarrow\mathbb{J}(\mathcal{V})$) has a right-adjoint,* i.e. *there exists a functor $G:\mathbb{J}(\mathcal{V})\longrightarrow\mathbb{J}(\mathcal{V})$ and a natural isomorphism for every pair of objects $X,Y\in\mathcal{V}$ :*

$$\mathcal{V}(X\bullet\mathbf{I}\bullet V,Y\bullet\mathbf{I})\simeq\mathcal{V}(X\bullet\mathbf{I},G(Y\bullet\mathbf{I}))\ \textit{(resp. }\mathcal{V}(V\bullet\mathbf{I}\bullet X,Y\bullet\mathbf{I})\simeq\mathcal{V}(X\bullet\mathbf{I},G(Y\bullet\mathbf{I}))\textit{)}.$$

*Moreover, $\mathcal{V}$ is said to be* closed *iff $\mathcal{V}$ is left and right closed.*

3.8. Lemma. *Let $(\mathcal{V},\bullet,\mathbf{I};\omega)$ be a semiunital semimonoidal category. If $V\in\mathcal{V}$ has a left (right) dual $V^{\circledast}$, then $(-\bullet V,-\bullet V^{\circledast})$ ($(V\bullet -,V^{\circledast}\bullet -)$) is a $(\mathbb{J},\mathbb{J})$-adjoint pair. In particular, if $\mathcal{V}$ is left (right) autonomous, then $\mathcal{V}$ is right (left) closed.*

Proof. Assume that $V\in\mathcal{V}$ has a left dual $V^{\circledast}$. For all $X,Y\in\mathcal{V}$ we have a natural isomorphism

$$\mathcal{V}(X\bullet\mathbf{I}\bullet V,Y\bullet\mathbf{I})\simeq\mathcal{V}(X\bullet\mathbf{I},Y\bullet\mathbf{I}\bullet V^{\circledast}),\ f\longmapsto(f\bullet V^{\circledast})\circ(X\bullet\upsilon)\tag{3}$$

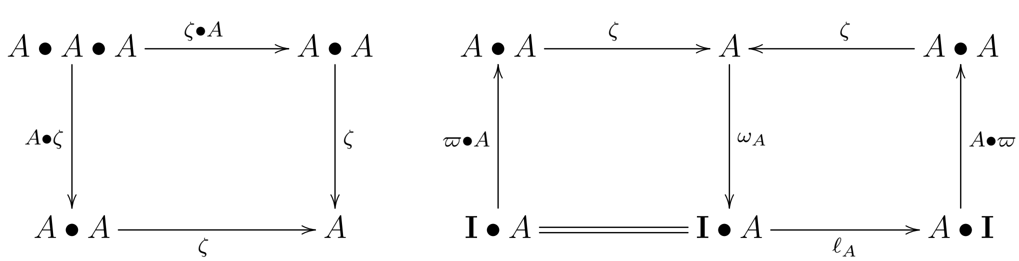
with inverse $g\longmapsto(Y\bullet\varpi)\circ(g\bullet V)$. ■

## 4. Semimonoids and Semicomonoids

In this section, we introduce notions of semimonoids and semicomonoids in semiunital semimonoidal categories. Throughout, $(\mathcal{V},\bullet,\mathbf{I};\omega)$ is a semiunital semimonoidal category, where $\mathbf{I}$ is a semiunit, $\omega:\mathbb{I}\longrightarrow\mathbb{J}$ is a natural transformation between the identity functor and the endo-functor $\mathbb{J}:=\mathbf{I}\bullet-\simeq-\bullet\mathbf{I}:\mathcal{V}\longrightarrow\mathcal{V}$ and $\gamma_{X,Y,Z}:(X\bullet Y)\bullet Z\longrightarrow X\bullet(Y\bullet Z)$ are natural isomorphisms for all $X,Y,Z\in\mathcal{V}$ (we assume the existence of natural isomorphisms $\mathbf{I}\bullet X\overset{\ell_X}{\simeq}X\bullet\mathbf{I}$ with inverse $X\bullet\mathbf{I}\overset{\wp_X:=\ell_X^{-1}}{\longrightarrow}\mathbf{I}\bullet X$ for every $X\in\mathcal{V}$).

Semimonoids.

4.1. A $\mathcal{V}$*-semimonoid* consists of a datum $(A,\zeta,\varpi)$, where $A\in\mathcal{V}$ and $\zeta:A\bullet A\longrightarrow A$, $\varpi:\mathbf{I}\longrightarrow A$ are morphisms in $\mathcal{V}$ such that the following diagrams are commutative




If $A\overset{\omega_A}{\simeq}\mathbf{I}\bullet A$, then we say that $A$ is a *unital $\mathcal{V}$-semimonoid.* A *morphism of $\mathcal{V}$-semimonoids* $f:(A,\zeta,\varpi)\longrightarrow(A',\zeta',\varpi')$ is a morphism in $\mathcal{V}$ such that the following diagrams are commutative

$$\begin{array}{ccc} A \bullet A & \xrightarrow{\zeta} & A \\ {\scriptstyle f \bullet f}\downarrow & & \downarrow{\scriptstyle f} \\ A' \bullet A' & \xrightarrow[\zeta']{} & A' \end{array} \qquad \begin{array}{ccc} \mathbf{I} & \xrightarrow{\varpi} & A \\ \| & & \downarrow{\scriptstyle f} \\ \mathbf{I} & \xrightarrow[\varpi']{} & A' \end{array}$$

The category of $\mathcal{V}$-semimonoids is denoted by $\mathbf{SMonoid}(\mathcal{V})$; the full subcategory of unital $\mathcal{V}$-semimonoids is denoted by $\mathbf{USMonoid}(\mathcal{V})$.

4.2. Let $(A, \zeta, \varpi)$ be a $\mathcal{V}$-semimonoid. A *right $A$-semimodule* is a datum $(M, \rho_M)$ where $M \in \mathcal{V}$ and $\rho_M : M \bullet A \longrightarrow M$ is a morphism in $\mathcal{V}$ such that the following diagrams are commutative

$$\begin{array}{ccc} M \bullet A \bullet A & \xrightarrow{\rho_M \bullet A} & M \bullet A \\ {\scriptstyle M \bullet \zeta}\downarrow & & \downarrow{\scriptstyle \rho_M} \\ M \bullet A & \xrightarrow[\rho_M]{} & M \end{array} \qquad \begin{array}{ccc} M \bullet A & \xrightarrow{\rho_M} & M \\ {\scriptstyle M \bullet \varpi}\uparrow & & \downarrow{\scriptstyle \omega_M} \\ M \bullet \mathbf{I} & \xleftarrow[\ell_M]{} & \mathbf{I} \bullet M \end{array}$$

If $M \overset{\omega_M}{\simeq} \mathbf{I} \bullet M$, then we say that $M$ is a *unitary right $A$-semimodule*. A morphism of right $A$-semimodules is a morphism $f : M \longrightarrow M'$ in $\mathcal{V}$ such that the following diagram is commutative

$$\begin{array}{ccc} M \bullet A & \xrightarrow{\rho_M} & M \\ {\scriptstyle f \bullet A}\downarrow & & \downarrow{\scriptstyle f} \\ M' \bullet A & \xrightarrow[\rho_{M'}]{} & M' \end{array}$$

The category of right $A$-semimodules is denoted by $\mathbf{S}_A$; the *full* subcategory of unitary right $A$-semimodules is denoted by $\mathbf{US}_A$. Analogously, one can define the category ${}_A\mathbf{S}$ of left $A$-semimodules and its *full* subcategory ${}_A\mathbf{US}$ of unitary left $A$-semimodules.

4.3. Example. *If* $\mathbf{I} \overset{\omega_{\mathbf{I}}}{\simeq} \mathbf{I} \bullet \mathbf{I}$, *then* $\mathbf{I}$ *is a unital* $\mathcal{V}$*-semimonoid with*

$$\zeta_{\mathbf{I}} : \mathbf{I} \bullet \mathbf{I} \overset{\omega_{\mathbf{I}}^{-1}}{\longrightarrow} \mathbf{I} \text{ and } \varpi_{\mathbf{I}} : \mathbf{I} \overset{\mathrm{id}}{\longrightarrow} \mathbf{I}.$$

*Moreover, every* $M \in \mathcal{V}^{\mathrm{firm}}$ *is a unitary* $(\mathbf{I}, \mathbf{I})$*-bisemimodule with* $\rho_M^l : \mathbf{I} \bullet M \overset{\omega_M^{-1}}{\longrightarrow} M$ *and* $\rho_M^r : M \bullet \mathbf{I} \overset{\omega_M^{-1} \circ \wp_M}{\longrightarrow} M$.

4.4. Let $A$ be a $\mathcal{V}$-semimonoid and $M$ a right $A$-semimodule. We have a functor

$$- \bullet M : \mathcal{V} \longrightarrow \mathbf{S}_A,$$

where for all $X \in \mathcal{V}$ we have a structure of a right $A$-semimodule on $X \bullet M$ given by

$$\rho_{X \bullet M} : (X \bullet M) \bullet A \overset{\gamma_{X,M,A}}{\longrightarrow} X \bullet (M \bullet A) \longrightarrow X \bullet M.$$

Similarly, if $M$ is a left $A$-semimodule, then we have a functor $M \bullet - : \mathcal{V} \longrightarrow {}_A\mathbf{S}$.

4.5. Let $A$ and $B$ be $\mathcal{V}$-semimonoids. Let $M$ be a left $B$-semimodule as well as a right $A$-semimodule and consider $B \bullet M \in \mathbf{S}_A$ and $M \bullet A \in {}_B\mathbf{M}$. We say that $M$ is a $(B, A)$-*bisemimodule* iff $\rho_{(M;B)} : B \bullet M \longrightarrow M$ is a morphism in $\mathbf{S}_A$, or equivalently iff $\rho_{(M;A)} : M \bullet A \longrightarrow M$ is a morphism in ${}_B\mathbf{S}$. The category of (unitary) $(B, A)$-bisemimodules with morphisms being in ${}_B\mathbf{S} \cap \mathbf{S}_A$ is denoted by ${}_B\mathbf{S}_A$ (${}_B\mathbf{US}_A$). Indeed, every (unital) $\mathcal{V}$-semimonoid $A$ is a (unitary) $(A, A)$-bisemimodule in a canonical way.

4.6. Proposition. *Every semiunital semimonoidal functor* $F : (\mathcal{V}, \bullet, \mathbf{I}_\mathcal{V}) \longrightarrow (\mathcal{W}, \otimes, \mathbf{I}_\mathcal{W})$ *lifts to a functor*

$$\widetilde{F} : \mathbf{SMonoid}(\mathcal{V}) \longrightarrow \mathbf{SMonoid}(\mathcal{W}),\ A \longmapsto F(A)$$

*that commutes with the forgetful functors*

$$U_\mathcal{V} : \mathbf{SMon}(\mathcal{V}) \longrightarrow \mathcal{V} \text{ and } U_\mathcal{W} : \mathbf{SMon}(\mathcal{W}) \longrightarrow \mathcal{W}.$$

Proof. Let $(A, \zeta_A, \varpi_A)$ be a semimonoid in $\mathcal{V}$ and consider $B := F(A)$. Define

$$\begin{aligned} \zeta_B &: F(A) \otimes F(A) \overset{\phi_{A,A}}{\longrightarrow} F(A \bullet A) \overset{F(\zeta_A)}{\longrightarrow} F(A); \\ \varpi_B &: \mathbf{I}_\mathcal{W} \overset{\tilde{\phi}}{\longrightarrow} F(\mathbf{I}_\mathcal{V}) \overset{F(\varpi_A)}{\longrightarrow} F(A). \end{aligned}$$

One checks easily that $(B, \zeta_B, \varpi_B)$ is a semimonoid in $\mathcal{W}$. If $f : A \longrightarrow A'$ is a morphism of $\mathcal{V}$-semimonoids, then examining the involved diagrams shows that $F(f) : F(A) \longrightarrow F(A')$ is a morphism of $\mathcal{W}$-semimonoids. Finally, it is clear that $U_\mathcal{W} \circ \widetilde{F} = F \circ U_\mathcal{V}$. ■

4.7. Proposition. *Let* $(A, \zeta, \varpi)$ *be a* $\mathcal{V}$*-semimonoid.*

1. *We have* $\mathbb{J}$*-monads*

$$- \bullet A : \mathcal{V} \longrightarrow \mathcal{V} \text{ and } A \bullet - : \mathcal{V} \longrightarrow \mathcal{V}$$

*and isomorphisms of categories*

$$\mathbf{S}_A \simeq \mathcal{V}_{(- \bullet A; \mathbb{J})} \text{ and } {}_A\mathbf{S} \simeq \mathcal{V}_{(A \bullet -; \mathbb{J})}.$$

2. *If* $B$ *is a* $\mathcal{V}$*-semimonoid, then we have* $\mathbb{J}$*-monads*

$$- \bullet A : {}_B\mathbf{S} \longrightarrow {}_B\mathbf{S} \text{ and } B \bullet - : \mathbf{S}_A \longrightarrow \mathbf{S}_A$$

*and isomorphisms of categories*

$$({}_B\mathbf{S})_{(- \bullet A; \mathbb{J})} \simeq {}_B\mathbf{S}_A \simeq (\mathbf{S}_A)_{(B \bullet -; \mathbb{J})}.$$

PROOF. Consider the natural transformations

$$\begin{aligned} \mu &: (- \bullet A) \bullet A \longrightarrow - \bullet A,\ \mu_X : (X \bullet A) \bullet A \overset{\gamma_{X,A,A}}{\simeq} X \bullet (A \bullet A) \overset{X \bullet \mu}{\longrightarrow} X \bullet A, \\ \nu &: \mathbb{J} \longrightarrow - \bullet A,\ \nu_X : \mathbf{I} \bullet X \overset{\ell_X}{\simeq} X \bullet \mathbf{I} \overset{X \bullet \varpi}{\longrightarrow} X \bullet A. \end{aligned}$$

One can easily check that $(- \bullet A, \mu, \omega, \nu)$ is a $\mathbb{J}$-monad. The isomorphism $\mathbf{S}_A \simeq \mathcal{V}_{(-\bullet A;\mathbb{J})}$ follows immediately from comparing the corresponding diagrams. The other assertions can also be checked easily. ■

An object $G$ in a cocomplete category $\mathfrak{A}$ is said to be a (*regular*) *generator* iff for every $X \in \mathfrak{A}$, there exists a canonical (*regular*) *epimorphism* $f_X : \coprod_{f \in \mathfrak{A}(G,X)} G \longrightarrow X$ [BW2005, p. 199] (see also [Kel2005], [Ver]); recall that an arrow in $\mathfrak{A}$ is said to be a *regular epimorphism* iff it is a coequalizer (of its *kernel pair*).

4.8. THEOREM. *Let $\mathcal{V}$ be cocomplete, $\mathbf{I}$ and $A \in \mathcal{V}$ be firm and assume that $\mathbf{I}$ is a regular generator in $\mathcal{V}$ and that both $A \bullet -$ and $- \bullet A$ preserve colimits in $\mathcal{V}$. There is a bijective correspondence between the structures of unital semimonoids on $A$, the structures of $\mathbb{J}$-monads on $- \bullet A$ and the structures of $\mathbb{J}$-monads on $A \bullet -$.*

PROOF. Assume that $(- \bullet A, \mu, \omega, \nu)$ is a $\mathbb{J}$-monad and consider

$$\begin{aligned} \mu &: A \bullet A \overset{\omega_A \bullet A}{\longrightarrow} \mathbf{I} \bullet A \bullet A \overset{\mu_{\mathbf{I}}}{\longrightarrow} \mathbf{I} \bullet A \overset{\lambda_A}{\simeq} A; \\ \varpi &: \mathbf{I} \overset{\omega_{\mathbf{I}}}{\longrightarrow} \mathbf{I} \bullet \mathbf{I} \overset{\nu_{\mathbf{I}}}{\longrightarrow} \mathbf{I} \bullet A \overset{\lambda_A}{\simeq} A. \end{aligned}$$

Clearly, $(A, \mu, \varpi)$ is a (unital) semimonoid. The converse follow by Proposition 4.7. The proof of the bijective correspondence is similar to that in the proof of [Ver, Theorem 3.9]. The statement corresponding to the endo-functor $A \bullet -$ can be proved analogously. ■

SEMICOMONOIDS.

4.9. A *$\mathcal{V}$-semicomonoid* is a datum $(C, \delta, \epsilon)$ where $C \in \mathcal{V}$, $\delta : C \longrightarrow C \bullet C$, $\epsilon : C \longrightarrow \mathbf{I}$ are morphisms in $\mathcal{V}$ such that the following diagrams are commutative

$$\begin{array}{ccc} C & \overset{\delta}{\longrightarrow} & C \bullet C \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta \bullet C} \\ C \bullet C & \underset{C \bullet \delta}{\longrightarrow} & C \bullet C \bullet C \end{array} \qquad \begin{array}{ccccc} C \bullet C & \overset{\delta}{\longleftarrow} & C & \overset{\delta}{\longrightarrow} & C \bullet C \\ {\scriptstyle \epsilon \bullet C}\downarrow & & \downarrow{\scriptstyle \omega_C} & & \downarrow{\scriptstyle C \bullet \epsilon} \\ \mathbf{I} \bullet C & = & \mathbf{I} \bullet C & \underset{\ell_C}{\longrightarrow} & C \bullet \mathbf{I} \end{array}$$

If $C \overset{\omega_C}{\simeq} I \bullet C$, then we say that $C$ is a *counital $\mathcal{V}$-semicomonoid*. A *morphism of $\mathcal{V}$-semicomonoids* $f : (C, \delta, \epsilon) \longrightarrow (C', \delta', \epsilon')$ is a morphism in $\mathcal{V}$ such that the following diagrams are commutative

$$\begin{array}{ccc} C & \xrightarrow{\delta} & C \bullet C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \bullet f} \\ C' & \xrightarrow[\delta']{} & C' \bullet C' \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\epsilon} & \mathbf{I} \\ {\scriptstyle f}\downarrow & & \Vert \\ C' & \xrightarrow[\epsilon']{} & \mathbf{I} \end{array}$$

The category of $\mathcal{V}$-semicomonoids is denoted by $\mathbf{SComonoid}(\mathcal{V})$; the *full* subcategory of counital $\mathcal{V}$-semicomonoids is denoted by $\mathbf{CSComonoid}(\mathcal{V})$.

4.10. Let $(C, \delta, \epsilon)$ be a $\mathcal{V}$-semicomonoid. A *right $C$-semicomodule* is a datum $(M, \rho^M)$ where $M \in \mathcal{V}$ and $\rho^M : M \longrightarrow M \bullet C$ is a morphisms in $\mathcal{V}$ such that the following diagrams are commutative

$$\begin{array}{ccc} M & \xrightarrow{\rho^M} & M \bullet C \\ {\scriptstyle \rho^M}\downarrow & & \downarrow{\scriptstyle \rho^M \bullet C} \\ M \bullet C & \xrightarrow[M \bullet \delta_C]{} & M \bullet C \bullet C \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho^M} & M \bullet C \\ {\scriptstyle \omega_M}\downarrow & & \downarrow{\scriptstyle M \bullet \epsilon} \\ \mathbf{I} \bullet M & \xleftarrow[\wp_M]{} & M \bullet \mathbf{I} \end{array}$$

A *morphism of right $C$-semicomodules* is a morphism $f : M \longrightarrow M'$ in $\mathcal{V}$ such that the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\rho^M} & M \bullet C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \bullet C} \\ M' & \xrightarrow[\rho^{M'}]{} & M' \bullet C \end{array}$$

The category of right $C$-semicomodules is denoted by $\mathbf{S}^C$; the category of *counitary* right $C$-semicomodules is denoted by $\mathbf{CS}^C$. Analogously, one can define the category ${}^C\mathbf{S}$ of *left $C$-semicomodules* and its *full* subcategory ${}^C\mathbf{CS}$ of *counitary left $C$-semicomodules.*

4.11. Remark. We prefer to use the terminology unital semimonoids (counital semicomonoids) to distinguish them from monoids (comonoids) which we reserve for monoidal categories. For example, the category of unital semimonoids in the monoidal category **Set** of sets is the category **Monoid** of usual monoids of the sense of Abstract Algebra. The same applies for unitary semimodules (counitary semicomodules). This is also consistent with the classical terminology of semirings and semimodules used in Section 5.

4.12. Let $C$ be a $\mathcal{V}$-semicomonoid and $M$ a right $C$-semicomodule. We have a functor

$$-\bullet M : \mathcal{V} \longrightarrow \mathbf{S}^C,$$

where for every $X \in \mathcal{V}$ we have a structure of a right $C$-semicomodule on $X \bullet M$ given by

$$\rho^{X\bullet M} : X \bullet M \overset{X\bullet\rho^M}{\longrightarrow} X \bullet (M \bullet C) \overset{\gamma^{-1}_{X,M,C}}{\longrightarrow} (X \bullet M) \bullet C.$$

Similarly, if $M$ is a left $C$-semicomodule, then we have a functor $M \bullet - : \mathcal{V} \longrightarrow {}^C\mathbf{S}$.

4.13. Let $C$ and $D$ be $\mathcal{V}$-semicomonoids. Let $M$ be a left $D$-semicomodule and a right $C$-semicomodule and consider $D \bullet M \in \mathbf{S}^C$ and $M \bullet C \in {}^D\mathbf{S}$. We say that $M$ is a $(D, C)$-*bisemicomodule* iff $\rho^{(M;D)} : M \longrightarrow D \bullet M$ is a morphism in $\mathbf{S}^C$ or equivalently iff $\rho^{(M;C)} : M \longrightarrow M \bullet C$ is a morphism in ${}^D\mathbf{S}$. The category of $(D, C)$-bisemicomodules with morphisms in ${}^D\mathbf{S} \cap \mathbf{S}^C$ is denoted by ${}^D\mathbf{S}^C$. The *full* subcategory of counitary $(D, C)$-bisemicomodules is denoted by ${}^D\mathbf{CS}^C$. Indeed, every (counital) $\mathcal{V}$-semicomonoid $C$ is a (counitary) $(C, C)$-bisemicomodule in a canonical way.

4.14. Example. $\mathbf{I}$ *is* $\mathcal{V}$*-semicomonoid with*

$$\delta_{\mathbf{I}} : \mathbf{I} \overset{\omega_{\mathbf{I}}}{\longrightarrow} \mathbf{I} \bullet \mathbf{I} \text{ and } \epsilon_{\mathbf{I}} : \mathbf{I} \overset{\mathrm{id}}{\longrightarrow} \mathbf{I}.$$

*Moreover, every (firm)* $M \in \mathcal{V}$ *is a (counitary)* $(\mathbf{I}, \mathbf{I})$*-bisemicomodule with* $(\rho^M)^l : M \overset{\omega_M}{\longrightarrow} \mathbf{I} \bullet M$ *and* $(\rho^M)^r : M \overset{\ell_M \circ \omega_M}{\longrightarrow} M \bullet \mathbf{I}$.

Dual to Proposition 4.6, we have

4.15. Proposition. *Every semiunital op-semimonoidal functor* $F : (\mathcal{V}, \bullet, \mathbf{I}_{\mathcal{V}}) \longrightarrow (\mathcal{W}, \otimes, \mathbf{I}_{\mathcal{W}})$ *lifts to a functor*

$$\widetilde{F} : \mathbf{SCMonoid}(\mathcal{V}) \longrightarrow \mathbf{SCMonoid}(\mathcal{W}),\ C \longmapsto F(C)$$

*which commutes with the forgetful functors*

$$U_{\mathcal{V}} : \mathbf{SCMon}(\mathcal{V}) \longrightarrow \mathcal{V} \text{ and } U_{\mathcal{W}} : \mathbf{SCMon}(\mathcal{W}) \longrightarrow \mathcal{W}.$$

Dual to Proposition 4.7, we obtain

4.16. Proposition. *Let* $(C, \delta, \epsilon)$ *be a* $\mathcal{V}$*-semicomonoid.*

1. *We have* $\mathbb{J}$*-comonads*

$$-\bullet C : \mathcal{V} \longrightarrow \mathcal{V} \text{ and } C \bullet - : \mathcal{V} \longrightarrow \mathcal{V}$$

*and isomorphisms of categories*

$$\mathbf{S}^C \simeq \mathcal{V}^{(-\bullet C;\mathbb{J})} \text{ and } {}^C\mathbf{S} \simeq \mathcal{V}^{(C\bullet -;\mathbb{J})}.$$

*2. If $D$ is a $\mathcal{V}$-semicomonoid, then we have $\mathbb{J}$-comonads*

$$-\bullet C: \ {}^{D}\mathbf{S} \longrightarrow \ {}^{D}\mathbf{S} \text{ and } D\bullet - : \mathbf{S}^{C} \longrightarrow \mathbf{S}^{C}$$

*and isomorphisms of categories*

$$({}^{D}\mathbf{M})^{(-\bullet C;\mathbb{J})} \simeq \ {}^{D}\mathbf{S}^{C} \simeq (\mathbf{S}^{C})^{(D\bullet -;\mathbb{J})}.$$

Our second reconstruction result is obtained in a way similar to that of Theorem 4.8:

4.17. Theorem. *Let $\mathcal{V}$ be cocomplete, $\mathbf{I}$ and $C \in \mathcal{V}$ be firm and assume that $\mathbf{I}$ is a regular generator and that both $C\bullet -$ and $-\bullet C$ respect colimits in $\mathcal{V}$. There is a bijective correspondence between the structures of counital semicomonoids on $C$, the structures of $\mathbb{J}$-comonads on $(-\bullet C, \Delta, \omega, \varepsilon; \mathbb{J})$ and the structures of $\mathbb{J}$-comonads on $(C\bullet -, \widetilde{\Delta}, \omega, \widetilde{\varepsilon}; \mathbb{J})$.*

4.18. Proposition. *If $(C, \delta, \epsilon)$ is a semicomonoid and $(A, \zeta, \varpi)$ is a unital semimonoid, then $(\mathcal{V}(C, A), *, \mathfrak{e})$ is a monoid in $\mathbf{Set}$ with multiplication and neutral element given by*

$$f * g := \zeta \circ (f\bullet g) \circ \delta \text{ and } \mathfrak{e} := \varpi \circ \epsilon.$$

Proof. For every $f, g, h \in \mathcal{V}(C, A)$, we have

$$\begin{aligned} ((f*g)*h) &= \zeta \circ (\zeta\bullet A) \circ ((f\bullet g)\bullet h) \circ (\delta \bullet C) \circ \delta \\ &= \zeta \circ (A\bullet \zeta) \circ (f\bullet (g\bullet h)) \circ (C\bullet \delta) \circ \delta \\ &= f*(g*h), \end{aligned}$$

whence $*$ is associative. On the other hand, we have for every $f \in \mathcal{V}(C, A)$

$$\begin{aligned} \ell_A \circ \omega_A \circ (f*\mathfrak{e}) &= \ell_A \circ \omega_A \circ \zeta \circ (f\bullet \varpi) \circ (C\bullet \epsilon) \circ \delta \\ &= (\ell_A \circ \omega_A \circ \zeta \circ (A\bullet \varpi)) \circ (f\bullet \mathbf{I}) \circ (C\bullet \epsilon) \circ \delta \\ &= (A\bullet \mathbf{I}) \circ (f\bullet \mathbf{I}) \circ (C\bullet \epsilon) \circ \delta \\ &= (f\bullet \mathbf{I}) \circ (C\bullet \epsilon) \circ \delta \\ &= (f\bullet \mathbf{I}) \circ (\ell_C \circ \omega_C) \\ &= (\ell_A \circ \omega_A) \circ f \end{aligned}$$

Since $A \overset{\ell_A\circ\omega_A}{\simeq} A\bullet \mathbf{I}$ is an isomorphism (in particular a monomorphism), we conclude that $f*\mathfrak{e} = f$. One can conclude similarly that $\mathfrak{e}*f$ for all $f \in \mathcal{V}(C, A)$. ■

4.19. Proposition. *If $\varphi : (C, \delta_C, \epsilon_C) \longrightarrow (D, \delta_D, \epsilon_D)$ is a morphism of semicomonoids and $\sigma : (A, \zeta_A, \varpi_A) \longrightarrow (B, \zeta_B, \varpi_B)$ is a morphism of unital semimonoids, then*

$$\mathcal{V}(D, A) \overset{<-,A>}{\longrightarrow} \mathcal{V}(C, A),\ f \longmapsto f\circ\varphi \text{ and } \mathcal{V}(C, A) \overset{<C,->}{\longrightarrow} \mathcal{V}(C, B),\ g \longmapsto \sigma\circ g$$

*are morphisms of monoids in $\mathbf{Set}$. In particular, we have functors*

$$\mathcal{V}(C, -) : \mathbf{SMonoid}_{\mathcal{V}} \longrightarrow \mathbf{Monoid} \text{ and } \mathcal{V}(-, A) : \mathbf{SCMonoid}_{\mathcal{V}}^{op} \longrightarrow \mathbf{Monoid}.$$

## 5. A concrete example

In this section we give applications to the category of bisemimodules over a semialgebra. For the convenience of the reader and to make the manuscript self-contained, we begin this section by recalling some *basic* definitions and results on semirings and their semimodules.

SEMIRINGS AND SEMIMODULES.

5.1. DEFINITION. *A* semiring *is an algebraic structure* $(S,+,\cdot,0,1)$ *consisting of a non-empty set* $S$ *with two binary operations "+" (addition) and "$\cdot$" (multiplication) satisfying the following axioms:*

1. $(S,+,0)$ *is a commutative monoid with neutral element* $0_S$;
2. $(S,\cdot,1)$ *is a monoid with neutral element* $1$;
3. $x\cdot(y+z)=x\cdot y+x\cdot z$ *and* $(y+z)\cdot x=y\cdot x+z\cdot x$ *for all* $x,y,z\in S$;
4. $0\cdot s=0=s\cdot 0$ *for every* $s\in S$ *(*i.e. $0$ *is* absorbing*).*

5.2. Let $S,S'$ be semirings. A map $f:S\longrightarrow S'$ is said to be a *morphism of semirings* iff for all $s_1,s_2\in S$ :

$$f(s_1+s_2)=f(s_1)+f(s_2),\ f(s_1s_2)=f(s_1)f(s_2),\ f(0_S)=0_{S'} \text{ and } f(1_S)=1_{S'}.$$

The category of semirings is denoted by **SRng**.

5.3. Let $(S,+,\cdot)$ be a semiring. We say that $S$ is

*cancellative* iff the additive semigroup $(S,+)$ is cancellative, *i.e.* whenever $s,s',s''\in S$ we have

$$s+s'=s+s''\Rightarrow s'=s'';$$

*commutative* iff the multiplicative semigroup $(S,\cdot)$ is commutative;
*semifield* iff $(S\backslash\{0\},\cdot,1)$ is a commutative group.

5.4. EXAMPLES. *Rings are indeed semirings. The first natural example of a (*commutative*) semiring which is* not *a ring is* $(\mathbb{N}_0,+,\cdot)$, *the set of non-negative integers. The semirings* $(\mathbb{R}_0^+,+,\cdot)$ *and* $(\mathbb{Q}_0^+,+,\cdot)$ *are indeed semifields. For every associative ring* $R$ *we have a semiring structure* $(\mathrm{Ideal}(R),+,\cdot)$ *on the set* $\mathrm{Ideal}(R)$ *of (two-sided) ideals of* $R$*. Every distributive complete lattice* $(\mathcal{L},\wedge,\vee,0,1)$ *is a semiring. For more examples, the reader may refer to* [Gol1999]*. In the sequel, we assume that* $0_S\neq 1_S$*.*

5.5. Definition. *Let $S$ be a semiring. A* right $S$-semimodule *is an algebraic structure $(M,+,0_M)$ consisting of a non-empty set $M$, a binary operation "$+$" along with a right $S$-action*

$$M \times S \longrightarrow M,\ (m,s) \mapsto ms,$$

*such that:*

1. *$(M,+,0_M)$ is a commutative monoid with neutral element $0_M$;*
2. *$(ms)s' = m(ss')$, $(m+m')s = ms + m's$ and $m(s+s') = ms + ms'$ for all $s,s' \in S$ and $m,m' \in M$;*
3. *$m1_S = m$ and $m0_S = 0_M = 0_M s$ for all $m \in M$ and $s \in S$.*

5.6. Let $M, M'$ be right $S$-semimodules. A map $f : M \longrightarrow M'$ is said to be a *morphism of $S$-semimodules* (or *$S$-linear*) iff for all $m_1, m_2 \in M$ and $s \in S$ :

$$f(m_1 + m_2) = f(m_1) + f(m_2) \text{ and } f(ms) = f(m)s.$$

The set $\mathrm{Hom}_S(M,M')$ of $S$-linear maps from $M$ to $M'$ is clearly a commutative monoid under addition. The category of right $S$-semimodules is denoted by $\mathbb{S}_S$. Analogously, one can define the category ${}_S\mathbb{S}$ of left $S$-semimodules. A right (left) $S$-semimodule is said to be *cancellative* iff the semigroup $(M,+)$ is cancellative. With $\mathbb{CS}_S \subseteq \mathbb{S}_S$ (resp. ${}_S\mathbb{CS} \subseteq {}_S\mathbb{S}$) we denote the *full* subcategory of cancellative right (left) $S$-semimodules. For two semirings $S$ and $T$, an $(S,T)$-bisemimodule $M$ has a structure of a left $S$-semimodule and a right $T$-semimodule such that $(sm)t = s(mt)$ for all $m \in M$, $s \in S$ and $t \in T$. The category of $(S,T)$-bisemimodules and $S$-linear $T$-linear maps is denoted by ${}_S\mathbb{S}_T$; the *full* subcategory of cancellative $(S,T)$-bisemimodules is denoted by ${}_S\mathbb{CS}_T$.

5.7. Let $M$ be a right $S$-semimodule. An $S$-congruence on $M$ is an equivalence relation $\equiv$ such that

$$m_1 \equiv m_2 \Rightarrow m_1 s + m \equiv m_2 s + m \text{ for all } m_1, m_2, m \in M \text{ and } s \in S.$$

In particular, we have an $S$-congruence relation $\equiv_{[0]}$ on $M$ defined by

$$m \equiv_{[0]} m' \iff m + m'' = m' + m'' \text{ for some } m'' \in M.$$

The quotient $S$-semimodule $M/\equiv_{[0]}$ is indeed cancellative and we have a canonical surjection $\mathfrak{c}_M : M \longrightarrow \mathfrak{c}(M)$, where $\mathfrak{c}(M) := M/\equiv_{[0]}$, with

$$\mathrm{Ker}(\mathfrak{c}_M) = \{m \in M \mid m + m'' = m'' \text{ for some } m'' \in M\}.$$

The class of cancellative right $S$-semimodules is a *reflective* subcategory of $\mathbb{S}_S$ in the sense that the functor $\mathfrak{c} : \mathbb{S}_S \longrightarrow \mathbb{CS}_S$ is left adjoint to the embedding functor $\mathbb{CS}_S \hookrightarrow \mathbb{S}_S$, *i.e.* for every $S$-semimodule $M$ and every *cancellative* $S$-semimodule $N$ we have a natural isomorphism of commutative monoids $\mathrm{Hom}_S(\mathfrak{c}(M), N) \simeq \mathrm{Hom}_S(M,N)$ [Tak1982, p.517].

Takahashi's Tensor-like Product.

5.8. ([Gol1999, page 187]) Let $M_S$ be a right $S$-semimodule, ${}_SN$ a left $S$-semimodule and consider the commutative monoid $U := S^{(M\times N)} \times S^{(M\times N)}$. Let $U' \subseteq S^{(M\times N)} \times S^{(M\times N)}$ be the *symmetric* $S$-subsemimodule generated by the set of elements of the form

$$\begin{array}{cc} (f_{(m_1+m_2,n)}, f_{(m_1,n)} + f_{(m_2,n)}), & (f_{(m_1,n)} + f_{(m_2,n)}, f_{(m_1+m_2,n)}), \\ (f_{(m,n_1+n_2)}, f_{(m,n_1)} + f_{(m,n_2)}), & (f_{(m,n_1)} + f_{(m,n_2)}, f_{(m,n_1+n_2)}), \\ (f_{(ms,n)}, f_{(m,sn)}), & (f_{(m,sn)}, f_{(ms,n)}), \end{array}$$

where

$$f_{(m,n)}(m',n') = \begin{cases} 1_S, & (m,n) = (m',n') \\ & \\ 0, & (m,n) \neq (m',n'). \end{cases}$$

Let $\equiv$ be the $S$-congruence relation on $S^{(M\times N)}$ defined by

$$f \equiv f' \Longleftrightarrow f + g = f' + g' \text{ for some } (g,g') \in U'.$$

*Takahashi's tensor-like product* of $M$ and $N$ is defined as $M \boxtimes_S N := U/\equiv$ . Notice that there is an $S$*-balanced map*

$$\widetilde{\tau} : M \times N \longrightarrow M \boxtimes_S N, \; (m,n) \mapsto m \boxtimes_S n := (m,n)/\equiv$$

with the following universal property [Tak1982]: for every commutative monoid $G$ and every $S$-bilinear $S$-balanced map $\beta : M \times N \longrightarrow G$ there exists a *unique* morphism of monoids $\gamma : M \boxtimes_S N \longrightarrow \mathfrak{c}(G)$ such that we have a commutative diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\beta} & G \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \mathfrak{c}_G} \\ M \boxtimes_S N & \underset{\gamma}{\dashrightarrow} & \mathfrak{c}(G) \end{array} \tag{4}$$

The following result collects some properties of $- \boxtimes_S -$ (compare with [Abu] and [Gol1999, Proposition 16.15, 16.16]):

5.9. Proposition. *Let $S$ and $T$ be semirings, $M$ be a right $S$-semimodule and $N$ a left $S$-semimodule.*

1. *$M \boxtimes_S N$ is a* cancellative *commutative monoid.*
2. *$M_S$ (${}_SN$) is cancellative if and only if $\mathfrak{c}(M) \simeq M$ ($\mathfrak{c}(N) \simeq N$).*
3. *We have natural isomorphisms of functors*

$$- \boxtimes_S S \simeq \mathfrak{c}(-) : \; \mathbb{S}_S \longrightarrow \mathbb{S}_S \text{ and } S \boxtimes_S - \simeq \mathfrak{c}(-) : \; {}_S\mathbb{S} \longrightarrow {}_S\mathbb{S}.$$

*Moreover, we have isomorphisms of functors*

$$- \boxtimes_S S \simeq \mathfrak{c}(-) \simeq S \boxtimes_S - : \ {}_S\mathbb{S}_S \longrightarrow \ {}_S\mathbb{S}_S.$$

*We set*

$$M \boxtimes_S S \overset{\vartheta^r_M}{\simeq} M \text{ and } S \boxtimes_S N \overset{\vartheta^l_M}{\simeq} N.$$

4. *We have idempotent functors*

$$\mathbb{J} : S \boxtimes_S - : \ {}_S\mathbb{S} \longrightarrow \ {}_S\mathbb{S} \text{ and } \mathbb{K} := - \boxtimes_T T : \mathbb{S}_T \longrightarrow \mathbb{S}_T. \tag{5}$$

*In particular,* $\mathfrak{c}(\mathfrak{c}(M)) \simeq \mathfrak{c}(M)$ *and* $\mathfrak{c}(\mathfrak{c}(N)) \simeq \mathfrak{c}(N)$.

5. *We have natural isomorphisms of commutative monoids*

$$\mathfrak{c}(M) \boxtimes_S N \simeq \mathfrak{c}(M) \boxtimes_S \mathfrak{c}(N) \simeq M \boxtimes_S \mathfrak{c}(N) \simeq M \boxtimes_S N \simeq \mathfrak{c}(M \boxtimes_S N). \tag{6}$$

5.10. Proposition. *Let $S$ and $T$ be semirings, $M$ a right $S$-semimodule and $N$ an $(S,T)$-bisemimodule. Consider the functors*

$$- \boxtimes_S N : \mathbb{S}_S \longrightarrow \mathbb{S}_T, \ \ N \boxtimes_T - : \ {}_T\mathbb{S} \longrightarrow \ {}_S\mathbb{S}$$

*and the endo-functors $\mathbb{J}$ and $\mathbb{K}$ in* (5).

1. $(- \boxtimes_S N, \mathrm{Hom}_{-T}(N, -))$ *is a* $(\mathbb{J}, \mathbb{K})$*-adjoint pair.*
2. $(N \boxtimes_T -, \mathrm{Hom}_{S-}(N, -))$ *is a* $(\mathbb{K}, \mathbb{J})$*-adjoint pair.*

Proof. For every right $T$-semimodule $G$ we have natural isomorphisms of commutative monoids

$$\begin{array}{rcll} \mathrm{Hom}_{-T}(\mathbb{J}(M) \boxtimes_S N), \mathbb{K}(G)) & \simeq & \mathrm{Hom}_{-T}(\mathfrak{c}(M) \boxtimes_S N), \mathfrak{c}(G)) & \\ & \simeq & \mathrm{Hom}_{-T}(M \boxtimes_S N, \mathfrak{c}(G)) & \\ & \simeq & \mathrm{Hom}_{-S}(M, \mathrm{Hom}_{-T}(N, \mathfrak{c}(G))) & ([\text{Gol1999, 16.15}]) \\ & \simeq & \mathrm{Hom}_{-S}(\mathfrak{c}(M), \mathrm{Hom}_{-T}(N, \mathfrak{c}(G))) & ([\text{Tak1982, p. 517}]) \\ & \simeq & \mathrm{Hom}_{-S}(\mathbb{J}(M), \mathrm{Hom}_{-T}(N, \mathbb{K}(G))). & \end{array}$$

The second statement can be proved symmetrically. ■

In what follows, $S$ denotes a *commutative* semiring with $1_S \neq 0_S$, $A$ is an $S$*-semialgebra* (*i.e.* a semiring with a morphism of semirings $\iota_A : S \longrightarrow A$), ${}_A\mathbb{S}_A$ is the category of $(A,A)$-bisemimodules and ${}_A\mathbb{CS}_A$ is its *full* subcategory of cancellative $(A,A)$-bisemimodules. Moreover, we fix the idempotent endo-functor $\mathbb{J}$ given by

$$\mathfrak{c}(-) \simeq A \boxtimes_A - \simeq - \boxtimes_A A : \ {}_A\mathbb{S}_A \longrightarrow \ {}_A\mathbb{S}_A.$$

5.11. Theorem.

1. $({}_A\mathbb{S}_A, \boxtimes_A, A)$ *is a closed semiunital semimonoidal category.*

2. $({}_A\mathbb{CS}_A, \boxtimes_A, \mathfrak{c}(A))$ *is a closed monoidal category.*

5.12. By a *semiunital $A$-semiring* we mean an $(A, A)$-bisemimodule $\mathcal{A}$ associated with $(A, A)$-bilinear maps $\zeta_{\mathcal{A}} : \mathcal{A} \boxtimes_A \mathcal{A} \longrightarrow \mathcal{A}$ and $\varpi_{\mathcal{A}} : A \longrightarrow \mathcal{A}$ such that the following diagrams are commutative

$$\begin{array}{ccc} \mathcal{A}\boxtimes_A\mathcal{A}\boxtimes_A\mathcal{A} & \xrightarrow{\zeta_{\mathcal{A}}\boxtimes_A\mathcal{A}} & \mathcal{A}\boxtimes_A\mathcal{A} \\ {\scriptstyle \mathcal{A}\boxtimes_A\zeta_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \zeta_{\mathcal{A}}} \\ \mathcal{A}\boxtimes_A\mathcal{A} & \xrightarrow[\zeta_{\mathcal{A}}]{} & \mathcal{A} \end{array} \qquad \begin{array}{ccccc} \mathcal{A}\boxtimes_A\mathcal{A} & \xrightarrow{\zeta_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\zeta_{\mathcal{A}}} & \mathcal{A}\boxtimes_A\mathcal{A} \\ {\scriptstyle \varpi_{\mathcal{A}}\boxtimes_A\mathcal{A}}\uparrow & & \downarrow{\scriptstyle \mathfrak{c}_A} & & \uparrow{\scriptstyle \mathcal{A}\boxtimes_A\varpi_{\mathcal{A}}} \\ A\boxtimes_A\mathcal{A} & \xrightarrow[\vartheta^l_{\mathcal{A}}]{} & \mathfrak{c}(\mathcal{A}) & \xleftarrow[\vartheta^r_{\mathcal{A}}]{} & \mathcal{A}\boxtimes_A A \end{array}$$

Let $\mathcal{A}$ and $\mathcal{A}'$ be semiunital $A$-semirings. An $(A, A)$-bilinear map $f : \mathcal{A} \longrightarrow \mathcal{A}'$ is called a *morphism of semiunital $A$-semirings* iff

$$f \circ \zeta_{\mathcal{A}} = \zeta_{\mathcal{A}'} \circ (f \boxtimes_A f) \text{ and } f \circ \varpi_{\mathcal{A}} = \varpi_{\mathcal{A}'}.$$

The set of morphisms of semiunital $A$-semirings form $\mathcal{A}$ to $\mathcal{A}'$ is denoted by $\mathrm{SSRng}_A(\mathcal{A}, \mathcal{A}')$. The category of semiunital $A$-semirings will be denoted by $\mathbf{SSRng}_A$. Indeed, we have an isomorphism of categories $\mathbf{SSRng}_A \simeq \mathbf{SMonoid}({}_A\mathbb{S}_A)$.

5.13. Let $\mathcal{A}$ be a semiunital $A$-semiring. A *semiunitary right $\mathcal{A}$-semimodule* is a right $A$-semimodule along with a right $A$-linear map $\rho_M : M \boxtimes_A \mathcal{A} \longrightarrow M$ such that the following diagrams are commutative

$$\begin{array}{ccc} M\boxtimes_A\mathcal{A}\boxtimes_A\mathcal{A} & \xrightarrow{\rho_M\boxtimes_A\mathcal{A}} & M\boxtimes_A\mathcal{A} \\ {\scriptstyle M\boxtimes_A\zeta_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \rho_M} \\ M\boxtimes_A\mathcal{A} & \xrightarrow[\rho_M]{} & M \end{array} \qquad \begin{array}{ccc} M\boxtimes_A A & \xrightarrow{M\boxtimes_A\varpi_{\mathcal{A}}} & M\boxtimes_A\mathcal{A} \\ {\scriptstyle \vartheta^r_M}\downarrow & & \downarrow{\scriptstyle \rho_M} \\ \mathfrak{c}(M) & \xleftarrow[\mathfrak{c}_M]{} & M \end{array}$$

A *morphism of semiunitary right $\mathcal{A}$-semimodules* ($\mathcal{A}$-linear) is an $A$-linear map $f : M \longrightarrow M'$ such that the following diagram is commutative

$$\begin{array}{ccc} M\boxtimes_A\mathcal{A} & \xrightarrow{\rho_M} & M \\ {\scriptstyle f\boxtimes_A\mathcal{A}}\downarrow & & \downarrow{\scriptstyle f} \\ M'\boxtimes_A\mathcal{A} & \xrightarrow[\rho_{M'}]{} & M' \end{array}$$

The category of semiunitary right $\mathcal{A}$-semimodules and $\mathcal{A}$-linear maps is denoted by $\mathbb{SS}_{\mathcal{A}}$; the *full* subcategory of unitary cancellative right $\mathcal{A}$-semimodules is denoted by $\mathbb{CS}_{\mathcal{A}}$. Analogously, one can define the category ${}_{\mathcal{A}}\mathbb{SS}$ of *semiunital left* $\mathcal{A}$*-semimodules* and its *full* subcategory ${}_{\mathcal{A}}\mathbb{CS}$ of unitary cancellative left $\mathcal{A}$-semimodules. For two semiunital $A$-semirings $\mathcal{A}$ and $\mathcal{B}$, one can define the category ${}_{\mathcal{B}}\mathbb{SS}_{\mathcal{A}}$ of *semiunitary* $(\mathcal{B},\mathcal{A})$*-bisemimodules* and its *full* subcategory ${}_{\mathcal{B}}\mathbb{CS}_{\mathcal{A}}$ of unitary cancellative $(\mathcal{B},\mathcal{A})$-bisemimodules in the obvious way. Considering any semiunital $A$-semiring as a semimonoid in $({}_A\mathbb{S}_A, \boxtimes_A, A)$, we have indeed isomorphisms of categories for any two semiunital $A$-semirings $\mathcal{A}$ and $\mathcal{B}$:

$$ {}_{\mathcal{B}}\mathbb{SS} \simeq ({}_A\mathbb{SS})_{(\mathcal{B}\boxtimes_A -;\mathfrak{c})},\ \mathbb{SS}_{\mathcal{A}} \simeq (\mathbb{S}_A)_{(-\boxtimes_A \mathcal{A};\mathfrak{c})},\ {}_{\mathcal{B}}\mathbb{SS}_{\mathcal{A}} \simeq\ {}_{\mathcal{B}}\mathbf{S}_{\mathcal{A}} $$

$$ {}_{\mathcal{B}}\mathbb{CS} \simeq ({}_A\mathbb{CS})_{\mathcal{B}\boxtimes_A -},\ \mathbb{CS}_{\mathcal{A}} \simeq (\mathbb{CS}_A)_{-\boxtimes_A \mathcal{A}},\ {}_{\mathcal{B}}\mathbb{CS}_{\mathcal{A}} \simeq\ {}_{\mathcal{B}}\mathbf{US}_{\mathcal{A}} $$

5.14. Remark. We use the terminology *semiunital* $A$*-semirings* to stress that such semimonoids are defined in the semiunital semimonoidal category $({}_A\mathbb{S}_A, \boxtimes_A, A)$ and to avoid confusion with (unital) $A$*-semirings* which can be defined as monoids in the monoidal category $({}_A\mathbb{S}_A, \otimes_A, A)$. The same applies for *semicounitary* $A$*-semicorings* below.

5.15. Being a variety, in the sense of Universal Algebra, the category ${}_A\mathbb{S}_A$ of $(A,A)$-bisemimodules is cocomplete. The class of regular epimorphism in ${}_A\mathbb{S}_A$ coincides with that of surjective $(A,A)$-bilinear maps. For every $(A,A)$-bisemimodule $M$, there is a surjective $(A,A)$-bilinear map from a free $(A,A)$-bisemimodule to $M$ (compare with [Gol1999, Proposition 17.11]); whence, $A$ is a regular generator. Moreover, for every $(A,A)$-bisemimodule $X$, both $X \boxtimes_A -, - \boxtimes_A X : {}_A\mathbb{S}_A \longrightarrow {}_A\mathbb{CS}_A$ respect colimits since they are left adjoints [Tak1982, Corollary 4.5].

Applying Theorem 4.8 to ${}_A\mathbb{S}_A$, we obtain:

5.16. Corollary. *Let $A$ be cancellative and $\mathcal{A}$ a cancellative $(A,A)$-bisemimodule. There is a bijective correspondence between the structures of unital $A$-semirings on $\mathcal{A}$, the structures of $\mathfrak{c}$-monads on $\mathcal{A}\boxtimes_A -$ and the structures of $\mathfrak{c}$-monads on $-\boxtimes_A \mathcal{A}$.*

5.17. A *semicounital* $A$*-semicoring* is an $(A,A)$-bisemimodule associated with $(A,A)$-bilinear maps $\delta_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}\boxtimes_A \mathcal{C}$ and $\epsilon_{\mathcal{C}} : \mathcal{C} \longrightarrow A$ such that the following diagrams are commutative

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\delta_{\mathcal{C}}} & \mathcal{C}\boxtimes_A\mathcal{C} \\
{\scriptstyle \delta_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \mathcal{C}\boxtimes_A\delta_{\mathcal{C}}} \\
\mathcal{C}\boxtimes_A\mathcal{C} & \xrightarrow[\delta_{\mathcal{C}}\boxtimes_A\mathcal{C}]{} & \mathcal{C}\boxtimes_A\mathcal{C}\boxtimes_A\mathcal{C}
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{C}\boxtimes_A\mathcal{C} & \xleftarrow{\delta_{\mathcal{C}}} & \mathcal{C} & \xrightarrow{\delta_{\mathcal{C}}} & \mathcal{C}\boxtimes_A\mathcal{C} \\
{\scriptstyle \epsilon_{\mathcal{C}}\boxtimes_A\mathcal{C}}\downarrow & & \downarrow{\scriptstyle \mathfrak{c}_C} & & \downarrow{\scriptstyle \mathcal{C}\boxtimes_A\epsilon_{\mathcal{C}}} \\
A\boxtimes_A\mathcal{C} & \xrightarrow[\vartheta^l_{\mathcal{C}}]{} & \mathfrak{c}(\mathcal{C}) & \xleftarrow[\vartheta^r_{\mathcal{C}}]{} & \mathcal{C}\boxtimes_A A
\end{array}
\qquad (7)
$$

The map $\delta_{\mathcal{C}}$ ($\epsilon_{\mathcal{C}}$) is called the *comultiplication* (*counity*) of $\mathcal{C}$. Using Sweedler-Heyneman's notation, we have for every $c \in \mathcal{C}$ :

$$
\begin{aligned}
\sum c_{11}\boxtimes_A c_{12}\boxtimes_A c_2 &= \sum c_1 \boxtimes_A c_{21}\boxtimes_A c_{22}; \\
\mathfrak{c}_C(\sum c_1\epsilon_{\mathcal{C}}(c_2)) &= \mathfrak{c}_C(c) = \mathfrak{c}_C(\sum \epsilon_{\mathcal{C}}(c_1)c_2).
\end{aligned}
$$

Let $(\mathcal{C},\delta,\epsilon)$ and $(\mathcal{C}',\delta',\epsilon')$ be semicounital $A$-semicorings. We call an $(A,A)$-bilinear map $f:\mathcal{C}\longrightarrow\mathcal{C}'$ a *morphism of semicounital $A$-semicorings* iff

$$(f\boxtimes_A f)\circ\delta_{\mathcal{C}}=\delta_{\mathcal{C}'}\circ f \text{ and } \epsilon_{\mathcal{C}'}\circ f=\epsilon_{\mathcal{C}}.$$

The set of morphisms of semicounital $A$-semicoring from $\mathcal{C}$ to $\mathcal{C}'$ is denoted by $\mathrm{SSCog}_A(\mathcal{C},\mathcal{C}')$. The category of semicounital $A$-semicorings is denoted by $\mathbf{SSCrng}_A$. Indeed, we have an isomorphism of categories $\mathbf{SSCrng}_A\simeq\mathbf{SCMonoid}({}_A\mathbb{S}_A)$.

5.18. Let $(\mathcal{C},\delta,\epsilon)$ be an $A$-semicoring. A *semicounitary right $\mathcal{C}$-semicomodule* is a right $A$-semimodule $M$ associated with an $A$-linear map

$$\rho^M:M\longrightarrow M\boxtimes_A\mathcal{C},\ m\mapsto\sum m_{<0>}\boxtimes_A m_{<1>},$$

such that the following diagrams are commutative

$$\begin{array}{ccc} M & \xrightarrow{\rho^M} & M\boxtimes_A\mathcal{C} \\ {\scriptstyle\rho^M}\downarrow & & \downarrow{\scriptstyle M\boxtimes_A\delta_{\mathcal{C}}} \\ M\boxtimes_A\mathcal{C} & \xrightarrow[\rho^M\boxtimes_A\mathcal{C}]{} & M\boxtimes_A\mathcal{C}\boxtimes_A\mathcal{C} \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho^M} & M\boxtimes_A\mathcal{C} \\ {\scriptstyle\mathfrak{c}_M}\downarrow & & \downarrow{\scriptstyle M\boxtimes_A\epsilon_{\mathcal{C}}} \\ \mathfrak{c}(M) & \xleftarrow[\vartheta^r_M]{} & M\boxtimes_A A \end{array}$$

Using Sweedler-Heyneman's notation, we have for every $m\in M$ :

$$\begin{array}{rcl}\sum m_{<0>}\boxtimes_A m_{<1>1}\boxtimes_A m_{<1>2} & = & \sum m_{<0><0>}\boxtimes_A m_{<0><1>}\boxtimes_A m_{<1>};\\ \mathfrak{c}(\sum m_{<0>}\epsilon_{\mathcal{C}}(m_{<1>})) & = & \mathfrak{c}_M(m).\end{array}$$

For semicounitary right $\mathcal{C}$-comodules $M,M'$, we call an $A$-linear map $f:M\longrightarrow M'$ a *morphism of semicounitary right $\mathcal{C}$-semicomodules* (or *$\mathcal{C}$-colinear*) iff the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle\rho^M}\downarrow & & \downarrow{\scriptstyle\rho^N} \\ M\boxtimes_A\mathcal{C} & \xrightarrow[f\boxtimes_A\mathcal{C}]{} & N\boxtimes_A\mathcal{C} \end{array}$$

The category of semicounitary right $\mathcal{C}$-semicomodules and $\mathcal{C}$-colinear maps is denoted by $\mathbb{SS}^{\mathcal{C}}$; the *full* subcategory of counitary right $\mathcal{C}$-semicomodules is denoted by $\mathbb{CS}^{\mathcal{C}}$. Analogously, one can define the category ${}^{\mathcal{C}}\mathbb{SS}$ of *semicounitary left $\mathcal{C}$-semicomodules* and its *full* subcategory ${}^{\mathcal{C}}\mathbb{CS}$ of *counitary left $\mathcal{C}$-semicomodules*. For two semicounital $A$-semicorings $\mathcal{C}$ and $\mathcal{D}$ one can define the category ${}^{\mathcal{D}}\mathbb{SS}^{\mathcal{C}}$ of *semicounitary $(\mathcal{D},\mathcal{C})$-bisemicomodules* and its *full* subcategory ${}^{\mathcal{D}}\mathbb{CS}^{\mathcal{C}}$ of *counitary $(\mathcal{D},\mathcal{C})$-bisemicomodules* in the obvious way. Considering any semicounital $A$-semicoring as a semicomonoid in $({}_A\mathbb{S}_A,\boxtimes_A,A)$, we have indeed isomorphisms of categories for any two semicounital $A$-semicorings $\mathcal{C}$ and $\mathcal{D}$:

$${}^{\mathcal{D}}\mathbb{SS}\simeq\ ({}_A\mathbb{S})^{(\mathcal{D}\boxtimes_A-;\mathfrak{c})},\ \mathbb{SS}^{\mathcal{C}}\simeq\ (\mathbb{S}_A)^{(-\boxtimes_A\mathcal{C};\mathfrak{c})},\ {}^{\mathcal{D}}\mathbb{SS}^{\mathcal{C}}\simeq\ {}^{\mathcal{D}}\mathbf{S}^{\mathcal{C}}$$

$${}^{\mathcal{D}}\mathbb{CS}\simeq\ {}^{\mathcal{D}\boxtimes_A-}({}_A\mathbb{CS}),\ \mathbb{CS}^{\mathcal{C}}\simeq\ (\mathbb{CS}_A)^{-\boxtimes_A\mathcal{C}},\ {}^{\mathcal{D}}\mathbb{CS}^{\mathcal{C}}\simeq\ {}^{\mathcal{D}}\mathbf{CS}^{\mathcal{C}}$$

Applying Theorem 4.17 to ${}_A\mathbb{S}_A$, we obtain:

5.19. Corollary. *Let $A$ be cancellative and $\mathcal{C}$ a cancellative $(A,A)$-bisemimodule. There is a bijective correspondence between the structures of counital $A$-semicorings on $\mathcal{C}$, the structures of $\mathfrak{c}$-comonads on $\mathcal{C} \boxtimes_A -$ and the structures of $\mathfrak{c}$-comonads on $- \boxtimes_A \mathcal{C}$.*

Almost all structures of corings over rings (*e.g.* [Abu2003], [BW2003]) can be transferred to obtain structures of semicorings over semirings.

5.20. Example. *Let $f : B \longrightarrow A$ be an extension of $S$-semialgebras and consider $A$ as a $(B,B)$-bisemimodule in the canonical way. One can define Sweedler's counital $A$-semicoring $\mathcal{C} := (A \boxtimes_B A, \delta, \epsilon)$ with*

$$\begin{aligned} \delta &: A \boxtimes_B A \longrightarrow (A \boxtimes_B A) \boxtimes_A (A \boxtimes_B A),\ a \boxtimes_B \widetilde{a} \mapsto (a \boxtimes_B 1_A) \boxtimes_A (1_A \boxtimes_B \widetilde{a}); \\ \epsilon &: A \boxtimes_B A \longrightarrow A,\ a \boxtimes_B \widetilde{a} \mapsto a\widetilde{a}. \end{aligned}$$

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## References

J. Abuhlail, *Uniformly flat semimodules*, preprint: arXiv:1201.0591.

J. Abuhlail, *Rational modules for corings*, Comm. Algebra 31 (2003), 5793–5840.

M. Barr and C. Wells, *Toposes, triples and theories*; Corrected reprint of the 1985 original, Reprints in Theory Appl. Categ. 12 (2005).

G. Böhm, T. Brzeziński and R. Wisbauer, *Monads and comonads on module categories,* J. Algebra 322 (5) (2009), 1719–1747.

T. Brzeziński and R. Wisbauer, *Corings and Comodules*, Lond. Math. Soc. Lec. Not. Ser. 309, Cambridge University Press (2003).

R. Dedekind, *Über die Theorie der ganzen algebraiscen Zahlen*. Supplement XI to P. G. Dirichlet, L.: *Vorlesung über Zahlentheorie*. 4 Aufl., Druck and Verlag, Braunschweig (1894).

N. Durov, *New Approach to Arakelov Geometry,* Ph.D. dissertation, Universität Bonn – Germany (2007); available as arXiv:0704.2030v1.

E. Eilenberg and J. C. Moore, *Adjoint functors and triples,* Illinois J. Math. 9 (1965), 381-398.

K. Głazek, *A guide to the literature on semirings and their applications in mathematics and information sciences. With complete bibliography*, Kluwer Academic Publishers, Dordrecht (2002).

J. Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, Dordrecht (1999).

H. Peter Gumm, *Elements of the general theory of coalgebras*, 1999. Lecture Notes for LUATCS'99, Rand Afrikaans University, Johannesburg.

A. Joyal and J. Kock, *Weak units and homotopy* 3*-types*, Categories in algebra, geometry and mathematical physics, 257–276, Contemp. Math. 431 (2007), Amer. Math. Soc., Providence, RI.

Y. Katsov, *Tensor products and injective envelopes of semimodules over additively regular semirings*, Algebra Colloq. 4 (2) (1997), 121–131.

G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. 10, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge].

J. Kock, *Elementary remarks on units in monoidal categories*, Math. Proc. Cambridge Philos. Soc. 144 (1) (2008), 1, 53–76.

S. Mac Lane, *Categories for the Working Mathematician.* Second edition, Graduate Texts in Mathematics 5, Springer-Verlag (1998).

B. Mesablishvili and R. Wisbauer, *Bimonads and Hopf monads on categories*, J. K-Theory 7 (2) (2011), 349–388.

A. Patchkoria, *On exactness of long sequences of homology semimodules,* J. of Homotopy and Related Structures 1 (1) (2006), 229-249.

H.-E. Porst, *On categories of monoids, comonoids, and bimonoids*, Quaest. Math. 31 (2) (2008), 127–139.

J. Rutten, *Fundamental Study: Universal coalgebra: a theory of systems*, Theoretical Computer Science 249 (2000) 3–80.

M. Sweedler, *The predual theorem to the Jacobson-Bourbaki theorem.* Trans. Amer. Math. Soc. 213 (1975), 391-406.

M. Takahashi, *On the bordism categories. II. Elementary properties of semimodules*, Math. Sem. Notes Kobe Univ. 9 (2) (1981), 495–530.

M. Takahashi, *On the bordism categories. III. Functors Hom and for semimodules*, Math. Sem. Notes Kobe Univ. 10 (2) (1982), 551–562.

J. Vercruysse, *Hopf algebras – Variant notions and reconstruction theorems*, to appear in C. Heunen, M. Sadrzadeh, E. Grefenstette (eds.): Compositional methods in Physics and Linguistics, Oxford University Press, preprint; arXiv:1202.3613.

*Department of Mathematics and Statistics*
*Box 5046, KFUPM, 31261 Dhahran, KSA*

Email: `abuhlail@kfupm.edu.sa`

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