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NO-ITERATION PSEUDOMONADS

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Abstract. We present the no-iteration version of the coherence conditions necessary to define a pseudomonad, and a description of the algebras for it in a similar fashion. We show that every no-iteration pseudomonad induces a pseudomonad, and that the corresponding algebras are equivalent. We also show that every pseudomonad induces a no-iteration pseudomonad, and again, that the corresponding algebras are equivalent. We conclude with an analysis of the algebras for the 2-monad $(-)^{\mathbf{2}}$ on **Cat** in the light of the no-iteration description of the algebras.

1. Introduction

In this paper we extend the results from [Marmolejo & Wood, 2010] to higher dimensional monads. We recall that that paper built on the idea of [Manes, 1976] (Exercise 1.3, page 32) that a monad (T, η, μ) can be presented in a way that avoids iteration of the endofunctor T. This leads in many particular cases to a significant simplification of the calculations necessary to verify that something is indeed a monad. This kind of presentation was extended to the algebras in [Marmolejo & Wood, 2010], and applied to distributive laws to achieve an alternative presentation of these that avoids iteration of the functors in question. After the latter paper had been published, we learned from R.F.C. Walters that he had a similar no iteration presentation of the monad, and also of the algebras, in his doctoral dissertation [Walters, 1970].

With these results in mind, we moved on to higher dimensions, where the no iteration idea is even more helpful; this is clear from just a glance at the coherence conditions necessary for a distributive law of one pseudomonad over another (see [Marmolejo, 1999] or [Marmolejo & Wood, 2008]). Our first stop in the direction of higher dimensional monads was [Marmolejo & Wood, 2012], where we dealt with the simplest non-trivial examples of pseudomonads, namely, (co-)lax idempotent or (co-)KZ, whose definition was originally given by Kock and Zöberlein in [Kock, 1973] and [Zöberlein, 1976]. However, [Marmolejo & Wood, 2012] did not follow strictly the logic of [Marmolejo & Wood, 2010]. This was due to the fact that, at some point, we realized that it was much more efficient to codify all the structure involved in terms of Kan extensions.

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This paper follows its lower dimensional counterpart [Marmolejo & Wood, 2010] more closely.

We begin in Section 2 with the definition of what we will call, only for the purposes of this paper, a no-iteration pseudomonad. This consists of six items of data and eight coherence conditions. Observe on the one hand that this suffices to define the endo pseudofunctor D, and also the pseudomonad over it, and on the other that there is no instance of D^n, for $n > 1$ anywhere in the coherence conditions.

In Section 3 we show first how to extend the function on objects D from a no-iteration pseudomonad to an endo pseudofunctor on a 2-category, followed by the definition of the rest of the structure that makes up a pseudomonad, the pseudomonad induced by a no-iteration pseudomonad.

In Section 4 we define the 2-category of algebras for a no-iteration pseudomonad in a similar no-iteration fashion, and in Section 5 we give the proof that this latter is equivalent to the usual algebras for the pseudomonad induced by a no-iteration pseudomonad.

Section 6 is devoted to the opposite direction, that is, every pseudomonad induces a no-iteration pseudomonad and the corresponding algebras are equivalent.

As an application we use the new description of the algebras for a pseudomonad to analyze the proof, given in [Korostenski & Tholen, 93] and [Rosebrugh & Wood, 02], of the fact that the algebras for the 2-monad $(-)^{\mathbf{2}}$ are factorizations systems.

This leaves the task of producing the corresponding no-iteration version of distributive laws between pseudomonads, the theme of a forthcoming paper.

2. No-iteration pseudomonads

We assume the reader is familiar with the different aspects of the theory of pseudomonads as given in, for example, [Marmolejo, 1997], with distributive laws between pseudomonads as given in [Marmolejo, 1999] and [Marmolejo, 2004], with the corresponding revision given in [Marmolejo & Wood, 2008]. We begin with the alternative no-iteration presentation of a pseudomonad in the 2-category $\mathcal{A}$.

2.1. Definition. *A* no-iteration pseudomonad $\mathbb{D}$ *on* $\mathcal{A}$ *consists of*

1. *A function* $D : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{A})$.

2. *For every* $A \in \mathcal{A}$, *a 1-cell* $dA : A \to DA$.

3. *For every* $A, B \in \mathcal{A}$, *a functor* $(\ )^{\mathbb{D}} : \mathcal{A}(A, DB) \to \mathcal{A}(DA, DB)$.

4. *For every* $A \in \mathcal{A}$, *an invertible 2-cell* $\mathbb{D}_A : \mathrm{Id}_{DA} \to dA^{\mathbb{D}}$.

5. *For every* $f : A \to DB$, *an invertible 2-cell*
$$\begin{tikzcd} A \arrow[r, "dA"] \arrow[dr, "f"'] & DA \arrow[d, "f^{\mathbb{D}}"] \arrow[dl, phantom, "\Swarrow \mathbb{D}_f"] \\ & DB. \end{tikzcd}$$

6. *For every* $f : A \to DB$ *and* $h : B \to DC$*, an invertible 2-cell*
$$\begin{tikzcd} DA \arrow[r, "f^{\mathbb{D}}"] \arrow[dr, "(h^{\mathbb{D}}f)^{\mathbb{D}}"'] & DB \arrow[d, "h^{\mathbb{D}}"] \arrow[dl, phantom, "\Swarrow \mathbb{D}_{f,h}"] \\ & DC. \end{tikzcd}$$

Subject to the coherence conditions

1. *For every* $A \in \mathcal{A}$*,*
$$\begin{tikzcd} A \arrow[r, "dA"] \arrow[dr, "dA"'] & DA \arrow[d, bend right, "dA^{\mathbb{D}}"'] \arrow[d, bend left, "\mathrm{Id}_{DA}"] \arrow[dl, phantom, "\Swarrow \mathbb{D}_{dA}"] \arrow[d, phantom, "\Leftarrow \mathbb{D}_A"] \\ & DA \end{tikzcd} = id_{dA} \tag{1}$$

2. *For every* $f : A \to DB$*,*
$$\begin{tikzcd} & DA \arrow[dr, "f^{\mathbb{D}}"] & \\ DA \arrow[ur, bend left=50, "\mathrm{Id}_{DA}"] \arrow[ur, "dA^{\mathbb{D}}"'] \arrow[rr, "(f^{\mathbb{D}}dA)^{\mathbb{D}}"'] \arrow[rr, bend right=40, "f^{\mathbb{D}}"'] \arrow[ur, phantom, bend left=20, "\Searrow \mathbb{D}_A"] \arrow[rr, phantom, bend left=15, "\Swarrow \mathbb{D}_{dA,f}"] \arrow[rr, phantom, bend right=20, "\Downarrow (\mathbb{D}_f)^{\mathbb{D}}"] & & DB \end{tikzcd} = id_{f^{\mathbb{D}}} \tag{2}$$

3. *For every* $f : A \to DB$*,*
$$\begin{tikzcd} & DB \arrow[dr, bend left, "\mathrm{Id}_{DB}"] \arrow[dr, "dB^{\mathbb{D}}"'] \arrow[dr, phantom, "\Swarrow \mathbb{D}_B"] & \\ DA \arrow[ur, "f^{\mathbb{D}}"] \arrow[rr, "(dB^{\mathbb{D}}f)^{\mathbb{D}}"'] \arrow[rr, phantom, bend left=15, "\Swarrow \mathbb{D}_{f,dB}"] & & DB \end{tikzcd} = (\mathbb{D}_B f)^{\mathbb{D}}. \tag{3}$$

4. *For every 2-cell* $\varphi : f \to g : A \to DB$*,*
$$\begin{tikzcd} A \arrow[r, "dA"] \arrow[dr, "g"'] & DA \arrow[d, bend right, "g^{\mathbb{D}}"'] \arrow[d, bend left, "f^{\mathbb{D}}"] \arrow[dl, phantom, "\Swarrow \mathbb{D}_g"] \arrow[d, phantom, "\Leftarrow \varphi^{\mathbb{D}}"] \\ & DB \end{tikzcd} = \begin{tikzcd} A \arrow[r, "dA"] \arrow[dr, bend left=15, "f"] \arrow[dr, bend right=30, "g"'] \arrow[dr, phantom, "\Swarrow \varphi"] & DA \arrow[d, "f^{\mathbb{D}}"] \arrow[dl, phantom, "\Swarrow \mathbb{D}_f"] \\ & DB. \end{tikzcd} \tag{4}$$

5. *For every* $f : A \to DB$*,* $h : B \to DC$
$$\begin{tikzcd} A \arrow[r, "dA"] \arrow[dr, "f"'] & DA \arrow[d, "f^{\mathbb{D}}"] \arrow[dl, phantom, "\Swarrow \mathbb{D}_f"] \\ & DB \arrow[d, "h^{\mathbb{D}}"] \\ & DC \end{tikzcd} = \begin{tikzcd} A \arrow[r, "dA"] \arrow[ddr, "h^{\mathbb{D}}f"'] & DA \arrow[dr, "f^{\mathbb{D}}"] \arrow[dd, "(h^{\mathbb{D}}f)^{\mathbb{D}}"'] \arrow[ddl, phantom, "\Swarrow \mathbb{D}_{(h^{\mathbb{D}}f)}"] & \\ & & DB \arrow[dl, "h^{\mathbb{D}}"] \arrow[l, phantom, "\Leftarrow \mathbb{D}_{f,h}"] \\ & DC. & \end{tikzcd} \tag{5}$$

6. *For every* $\varphi : f \to g : A \to DB$ *and every* $h : B \to DC$,

$$\begin{tikzcd}[column sep=large] & DB \arrow[dr, "h^{\mathbb{D}}"] & \\ DA \arrow[ur, bend left=45, "f^{\mathbb{D}}"] \arrow[ur, "g^{\mathbb{D}}"'] \arrow[ur, phantom, bend left=20, "\Downarrow \varphi^{\mathbb{D}}"] \arrow[rr, bend right=10, "(h^{\mathbb{D}}g)^{\mathbb{D}}"'] \arrow[rr, phantom, bend left=20, "\Swarrow \mathbb{D}_{g,h}"] & & DC \end{tikzcd} = \begin{tikzcd}[column sep=large] & DB \arrow[dr, "h^{\mathbb{D}}"] \arrow[d, phantom, "\Downarrow \mathbb{D}_{f,h}"] & \\ DA \arrow[ur, "f^{\mathbb{D}}"] \arrow[rr, bend left=10, "(h^{\mathbb{D}}f)^{\mathbb{D}}"] \arrow[rr, phantom, "(h^{\mathbb{D}}\varphi)^{\mathbb{D}} \Downarrow"] \arrow[rr, bend right=10, "(h^{\mathbb{D}}g)^{\mathbb{D}}"'] & {} & DC. \end{tikzcd} \tag{6}$$

7. *For every* $f : A \to DB$ *and every* $\psi : h \to k : B \to DC$,

$$\begin{tikzcd}[column sep=large] & DB \arrow[dr, bend left=30, "h^{\mathbb{D}}"] \arrow[dr, "k^{\mathbb{D}}"'] \arrow[dr, phantom, bend left=15, "\Swarrow \psi^{\mathbb{D}}"] & \\ DA \arrow[ur, "f^{\mathbb{D}}"] \arrow[rr, bend right=10, "(h^{\mathbb{D}}g)^{\mathbb{D}}"'] \arrow[rr, phantom, bend left=20, "\mathbb{D}_{f,k} \Swarrow"] & & DC \end{tikzcd} = \begin{tikzcd}[column sep=large] & DB \arrow[dr, "h^{\mathbb{D}}"] \arrow[d, phantom, "\Downarrow \mathbb{D}_{f,h}"] & \\ DA \arrow[ur, "f^{\mathbb{D}}"] \arrow[rr, bend left=10, "(h^{\mathbb{D}}f)^{\mathbb{D}}"] \arrow[rr, phantom, "(\psi^{\mathbb{D}}f)^{\mathbb{D}} \Downarrow"] \arrow[rr, bend right=10, "(k^{\mathbb{D}}f)^{\mathbb{D}}"'] & {} & DC. \end{tikzcd} \tag{7}$$

8. *For every* $f : A \to DB$, $h : B \to DC$ *and* $\ell : C \to DE$,

$$\begin{tikzcd} & DB \arrow[r, "h^{\mathbb{D}}"] \arrow[d, phantom, "\Searrow \mathbb{D}_{f,h}"] & DC \arrow[dr, "\ell^{\mathbb{D}}"] \arrow[d, phantom, "\mathbb{D}_{h^{\mathbb{D}}f,\ell} \Downarrow"] & \\ DA \arrow[ur, "f^{\mathbb{D}}"] \arrow[urr, bend left=10, "(h^{\mathbb{D}}f)^{\mathbb{D}}"'] \arrow[rrr, "(\ell^{\mathbb{D}}h^{\mathbb{D}}f)^{\mathbb{D}}"'] \arrow[rrr, bend right=25, "((\ell^{\mathbb{D}}h)^{\mathbb{D}}f)^{\mathbb{D}}"'] \arrow[rrr, phantom, bend right=12, "(\mathbb{D}_{h,\ell}f)^{\mathbb{D}} \Downarrow"] & {} & {} & DE \end{tikzcd} = \begin{tikzcd} & DB \arrow[r, "h^{\mathbb{D}}"] \arrow[drr, bend right=10, "(\ell^{\mathbb{D}}h)^{\mathbb{D}}"'] \arrow[d, phantom, "\Downarrow \mathbb{D}_{f,\ell^{\mathbb{D}}h}"] & DC \arrow[dr, "\ell^{\mathbb{D}}"] \arrow[d, phantom, "\Swarrow \mathbb{D}_{h,\ell}"] & \\ DA \arrow[ur, "f^{\mathbb{D}}"] \arrow[rrr, bend right=15, "((\ell^{\mathbb{D}}h)^{\mathbb{D}}f)^{\mathbb{D}}"'] & {} & {} & DE \end{tikzcd} \tag{8}$$

3. The pseudomonad induced by a no-iteration pseudomonad

Let $\mathbb{D}$ be a no-iteration pseudomonad. We extend the function on objects D to a pseudofunctor as follows.

3.1. PROPOSITION. *Given a no-iteration pseudomonad* $\mathbb{D}$ *on a 2-category* $\mathcal{A}$*, we induce a pseudofunctor* $D : \mathcal{A} \to \mathcal{A}$ *as follows:* D *as given by* $\mathbb{D}$ *on objects. For objects* A, B *we define* $D_{A,B}$ *as the composite*

$$\mathcal{A}(A,B) \xrightarrow{-\circ dB} \mathcal{A}(A,DB) \xrightarrow{(\ )^{\mathbb{D}}} \mathcal{A}(DA,DB).$$

For every A, $D_A = \mathbb{D}_A$. *For* $f : A \to B$ *and* $h : B \to C$, *define* $D^{f,h} : Dh\, Df \to D(hf)$ *as* $(\mathbb{D}_{dC\,h}\, f)^{\mathbb{D}} \cdot \mathbb{D}_{dB\,f, dC\,h}$.

PROOF. Assume we have

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \varphi"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow \psi"] & C \arrow[r, "\ell"] & E. \end{tikzcd}$$

Then $D(\psi\, f)\cdot D^{f,h}$ is the composition of the top arrow and the rightmost one in the diagram

$$\begin{tikzcd}
Dh\,Df \arrow[r, "\mathbb{D}_{dB\,f,dC\,h}"] \arrow[d, "(dC\,\psi)^{\mathbb{D}}Df"'] & (Dh\,dB\,f)^{\mathbb{D}} \arrow[r, "(\mathbb{D}_{dC\,h}f)^{\mathbb{D}}"] \arrow[d, "((dC\,\psi)^{\mathbb{D}}dB\,f)^{\mathbb{D}}"'] & Dhf \arrow[d, "(dC\,\psi f)^{\mathbb{D}}"] \\
Dk\,Df \arrow[r, "\mathbb{D}_{dB\,f,dC\,k}"'] & (Dk\,dB\,f)^{\mathbb{D}} \arrow[r, "(\mathbb{D}_{dC\,k}f)^{\mathbb{D}}"'] & Dkf.
\end{tikzcd}$$

The right rectangle commutes by the functoriality of $(\ )^{\mathbb{D}}$ and (4), the left one by (7). The other composition in the diagram is $D^{f,k}\cdot D\psi\,Df$. Similarly we have that $D(h\,\varphi)\cdot D^{f,h} = D^{g,h}\cdot(Dh\,D\varphi)$ is shown by the commutativity of

$$\begin{tikzcd}
Dh\,Df \arrow[r, "\mathbb{D}_{dB\,f,dC\,h}"] \arrow[d, "Dh\,D\varphi"'] & (Dh\,dB\,f)^{\mathbb{D}} \arrow[r, "(\mathbb{D}_{dC\,h}f)^{\mathbb{D}}"] \arrow[d, "(Dh\,dB\,\varphi)^{\mathbb{D}}"'] & Dhf \arrow[d, "(dC\,h\varphi)^{\mathbb{D}}"] \\
Dh\,Dg \arrow[r, "\mathbb{D}_{dB\,g,dC\,h}"'] & (Dh\,dB\,g)^{\mathbb{D}} \arrow[r, "(\mathbb{D}_{dC\,h}g)^{\mathbb{D}}"'] & Dhg,
\end{tikzcd}$$

using naturality and (6).

Furthermore, we have

$$D^{\mathrm{Id}_A,f}\cdot Df\,D_A = (\mathbb{D}_{dB\,f})^{\mathbb{D}}\cdot\mathbb{D}_{dA,dB\,f}\cdot(dB\,f)^{\mathbb{D}}D_A = id_{Df}$$

by (2) applied to $dB\,f$, and

$$\begin{aligned}
D^{f,\mathrm{Id}_B}\cdot D_BDf &= (\mathbb{D}_{dB}f)^{\mathbb{D}}\cdot\mathbb{D}_{dB\,f,dB}\cdot D_B(dB\,f)^{\mathbb{D}}\\
&= (\mathbb{D}_{dB}f)^{\mathbb{D}}\cdot(D_BdB\,f)^{\mathbb{D}}\\
&= (\mathbb{D}_{dB}f\cdot D_BdB\,f)^{\mathbb{D}} = (id_{dB\,f})^{\mathbb{D}} = id_{Df}
\end{aligned}$$

by (3) and (1). Finally, we show that $D^{hf,\ell}\cdot D\ell D^{f,h} = D^{f,\ell h}\cdot D^{h,\ell}Df$ by means of the following commutative diagram, where the label inside each subdiagram is the corresponding axiom needed:

$$\begin{tikzcd}
D\ell\,Dh\,Df \arrow[r, "D\ell\,\mathbb{D}_{dB\,f,dC\,h}"] \arrow[dd, "\mathbb{D}_{dC\,h,dE\,\ell}Df"] \arrow[ddr, phantom, "(8)"] & D\ell(Dh\,dB\,f)^{\mathbb{D}} \arrow[rr, "D\ell(\mathbb{D}_{dC\,h}f)^{\mathbb{D}}"] \arrow[d, "\mathbb{D}_{Dh\,dB\,f,dE\,\ell}"] \arrow[drr, phantom, "(6)"] & & D\ell\,Dhf \arrow[dd, "\mathbb{D}_{dC\,hf,dE\,\ell}"'] \\
& (D\ell\,Dh\,dB\,f)^{\mathbb{D}} \arrow[d, "(\mathbb{D}_{dC\,h,dE\,\ell}dB\,f)^{\mathbb{D}}"] \arrow[drr, "(D\ell\,\mathbb{D}_{dC\,h}f)^{\mathbb{D}}"] \arrow[dr, phantom, "(5)"] & & \\
(D\ell\,dC\,h)^{\mathbb{D}}Df \arrow[r, "\mathbb{D}_{dB\,f,D\ell\,dC\,h}"] \arrow[d, "(\mathbb{D}_{dE\,\ell}h)^{\mathbb{D}}Df"] \arrow[dr, phantom, "(7)"] & ((D\ell\,dC\,h)^{\mathbb{D}}dB\,f)^{\mathbb{D}} \arrow[rr, "(\mathbb{D}_{D\ell\,dC\,h}f)^{\mathbb{D}}"] \arrow[d, "((\mathbb{D}_{dE\,\ell}h)^{\mathbb{D}}dB\,f)^{\mathbb{D}}"] \arrow[drr, phantom, "(4)"] & & (D\ell\,dC\,hf)^{\mathbb{D}} \arrow[d, "(\mathbb{D}_{dE\,\ell}hf)^{\mathbb{D}}"'] \\
Dh\ell\,Df \arrow[r, "\mathbb{D}_{dB\,f,dE\,\ell h}"'] & (D\ell h\,dB\,f)^{\mathbb{D}} \arrow[rr, "(\mathbb{D}_{dE\,\ell h}f)^{\mathbb{D}}"'] & & D\ell hf
\end{tikzcd}$$

■

We now extend the family of 1-cells $\langle dA : A \to DA \rangle_{A \in \mathcal{A}}$ to a strong transformation.

3.2. Proposition. *If for every $f : A \to B$ we define $d_f := \mathbb{D}_{dB\, f}$:*

$$\begin{tikzcd} A \arrow[r, "dA"] \arrow[d, "f"'] & DA \arrow[d, "(dB\, f)^{\mathbb{D}} = Df"] \arrow[dl, phantom, "\overset{d_f}{\swarrow}"] \\ B \arrow[r, "dB"'] & DB. \end{tikzcd}$$

then we obtain a strong transformation $d : \mathrm{Id}_{\mathcal{A}} \to D$.

Proof. The only non-trivial condition is

$$\begin{aligned} d_h f \cdot Dh\, d_f &= \mathbb{D}_{dC\, h} f \cdot Dh\, \mathbb{D}_{dB\, f} = \mathbb{D}_{dC\, h} f \cdot \mathbb{D}_{Dh\, dB\, f} \cdot \mathbb{D}_{dB\, f, dC\, h} dA \\ &= \mathbb{D}_{dC\, h\, f} \cdot (\mathbb{D}_{dC\, h} f)^{\mathbb{D}} dA \cdot \mathbb{D}_{dB\, f, dC\, h} dA = d_{hf} \cdot D^{f,h} \end{aligned}$$

by (5) and (4). ∎

We now define a strong transformation $m : D^2 \to D$.

3.3. Proposition. *If for every A we define $mA = (\mathrm{Id}_{DA})^{\mathbb{D}}$ and for every $f : A \to B$ we define m_f as*

$$\begin{tikzcd}[column sep=huge, row sep=huge] D^2A \arrow[r, "(\mathrm{Id}_{DA})^{\mathbb{D}}"] \arrow[d, "D^2 f"'] \arrow[dr, bend left=15, "(Df)^{\mathbb{D}}" description] \arrow[dr, bend right=15, "((\mathrm{Id}_{DB})^{\mathbb{D}} dDB\, Df)^{\mathbb{D}}" description] & DA \arrow[d, "Df"] \arrow[dl, phantom, "\overset{\mathbb{D}_{\mathrm{Id}_{DA}, dB\, f}}{\swarrow}"] \\ D^2B \arrow[r, "(\mathrm{Id}_{DB})^{\mathbb{D}}"'] \arrow[ur, phantom, "\overset{\mathbb{D}^{-1}_{dDB\, Df, \mathrm{Id}_{DB}}}{\swarrow}\quad\overset{(\mathbb{D}^{-1}_{\mathrm{Id}_{DB}} Df)^{\mathbb{D}}}{\swarrow}"] & DB \end{tikzcd}$$

then we obtain a strong transformation $m : D^2 \to D$.

Proof. For $\varphi : f \to g : A \to B$ the diagram

$$\begin{tikzcd} Df\, mA \arrow[r, "m_f"] \arrow[d, "D\varphi\ mA"'] & mB\, D^2 f \arrow[d, "mB\, D^2\varphi"] \\ Dg\, mA \arrow[r, "m_g"'] & mB\, D^2 g \end{tikzcd}$$

is the exterior of the diagram

$$\begin{tikzcd} Df\, mA \arrow[r, "\mathbb{D}_{\mathrm{Id}_{DA}, dB\, f}"] \arrow[d, "D\varphi\ mA"] & Df^{\mathbb{D}} \arrow[r, "(\mathbb{D}^{-1}_{\mathrm{Id}_{DB}} Df)^{\mathbb{D}}"] \arrow[d, "D\varphi^{\mathbb{D}}"'] & (mB\, dDB\, Df)^{\mathbb{D}} \arrow[r, "\mathbb{D}^{-1}_{dDB\, Df, \mathrm{Id}_{DB}}"] \arrow[d, "(mB\, dDB\, D\varphi)^{\mathbb{D}}"'] & mB\, D^2 f \arrow[d, "mB\, D^2\varphi"'] \\ Dg\, mA \arrow[r, "\mathbb{D}_{\mathrm{Id}_{DA}, dB\, g}"'] & Dg^{\mathbb{D}} \arrow[r, "(\mathbb{D}^{-1}_{\mathrm{Id}_{DB}} Dg)^{\mathbb{D}}"'] & (mB\, dDB\, Dg)^{\mathbb{D}} \arrow[r, "\mathbb{D}^{-1}_{dDB\, Dg, \mathrm{Id}_{DB}}"'] & mB\, D^2 g. \end{tikzcd}$$

The first rectangle commutes because of (7), the second by naturality, and the third by (6).

Now take $f : A \to B$, $h : B \to C$. The equality $(D^2)^{f,g} \cdot m_h\, D^2 f \cdot Dh\, m_f = m_{hf} \cdot mA\, D^{f,h}$ is proved by the following commutative diagram:

$$
\begin{tikzcd}
{} \arrow[rr, "\mathbb{D}_{\mathrm{Id}_{DB},dC\,h}D^2f"] & & {} \arrow[rr, "(\mathbb{D}^{-1}_{\mathrm{Id}_{DC}}Dh)^{\mathbb{D}}D^2f"] \arrow[drr, phantom, "(7)"] & & {} \arrow[r, "\mathbb{D}^{-1}_{dDC\,Dh,\mathrm{Id}_{DC}}D^2f"] \arrow[d, "\mathbb{D}_{dDB\,Df,mC\,dDC\,Dh}"'] & {} \arrow[dd, "mC\mathbb{D}_{dDB\,Df,dDC\,Dh}"'] \\
& & {} \arrow[u, "\mathbb{D}^{-1}_{dDB\,Df,Dh}"'] \arrow[rr, "((\mathbb{D}^{-1}_{\mathrm{Id}_{DC}}Dh)^{\mathbb{D}}dDB\,Df)^{\mathbb{D}}"'] & & {} \arrow[d, "(\mathbb{D}^{-1}_{dDC\,Dh,\mathrm{Id}_{DC}}dDB\,Df)^{\mathbb{D}}"'] & \\
{} \arrow[uu, "Dh\mathbb{D}_{dDB\,Df,\mathrm{Id}_{DB}}"'] \arrow[rr, "\mathbb{D}_{mB\,dDB\,Df,dC\,h}"'] \arrow[uurr, phantom, "(8)"] & & {} \arrow[u, "(\mathbb{D}_{\mathrm{Id}_{DB},dC\,h}dDB\,Df)^{\mathbb{D}}"] \arrow[ddrr, phantom, "(*)"] & & {} \arrow[r, "\mathbb{D}^{-1}_{D^2h\,dDB\,Df,\mathrm{Id}_{DC}}"] \arrow[d, "(mC\mathbb{D}_{dDC\,Dh}Df)^{\mathbb{D}}"'] \arrow[uur, phantom, "(8)"] \arrow[ddr, phantom, "(6)"] & {} \arrow[d, "mC(\mathbb{D}_{dDC\,Dh}Df)^{\mathbb{D}}"'] \\
{} \arrow[u, "Dh(\mathbb{D}^{-1}_{\mathrm{Id}_{DB}}Df)^{\mathbb{D}}"'] \arrow[rr, "\mathbb{D}_{Df,dC\,h}"'] \arrow[urr, phantom, "(6)"] & & {} \arrow[u, "(Dh\mathbb{D}^{-1}_{\mathrm{Id}_{DB}}Df)^{\mathbb{D}}"] & & {} \arrow[d, "(mC\,dDC\mathbb{D}_{dB\,f,dC\,h})^{\mathbb{D}}"'] & {} \arrow[d, "mC(dDC\mathbb{D}_{dB\,f,dC\,h})^{\mathbb{D}}"'] \\
& & {} \arrow[u, "(\mathbb{D}^{-1}_{dB\,f,dC\,h})^{\mathbb{D}}"] \arrow[dr, "(\mathbb{D}_{dC\,h}f)^{\mathbb{D}\mathbb{D}}"] & & {} \arrow[d, "(mC\,dDC(\mathbb{D}_{dC\,h}f)^{\mathbb{D}})^{\mathbb{D}}"'] & {} \arrow[d, "mC\,D((\mathbb{D}_{dC\,h}f)^{\mathbb{D}})"'] \\
{} \arrow[uu, "Dh\mathbb{D}_{\mathrm{Id}_{DA},dB\,f}"'] \arrow[r, "\mathbb{D}_{dB\,f,dC\,h}mA"'] \arrow[uurr, phantom, "(8)"] & {} \arrow[ur, "\mathbb{D}_{\mathrm{Id}_{DA},Dh\,dB\,f}"] \arrow[r, "(\mathbb{D}_{dC\,h}f)^{\mathbb{D}}mA"'] & {} \arrow[r, "\mathbb{D}_{\mathrm{Id}_{DA},dC\,hf}"'] \arrow[u, phantom, "(7)"] & {} \arrow[r, "(\mathbb{D}^{-1}_{\mathrm{Id}_{DC}}Dh\,f)^{\mathbb{D}}"'] & {} \arrow[r, "\mathbb{D}^{-1}_{dDC\,Dhf,\mathrm{Id}_{DC}}"'] & {}
\end{tikzcd}
$$

where the labels inside the subdiagrams indicate the equation needed for its commutativity, and $(*)$ is $(\ )^{\mathbb{D}}$ of the following commutative diagram:

$$
\begin{tikzcd}
{} \arrow[r, "Dh\mathbb{D}^{-1}_{\mathrm{Id}_{DB}}Df"] & {} \arrow[r, "\mathbb{D}_{\mathrm{Id}_{DB},dC\,h}dDB\,Df"] & {} \arrow[r, "(\mathbb{D}^{-1}_{\mathrm{Id}_{DC}}Dh)^{\mathbb{D}}dDB\,Df"] \arrow[d, "\mathbb{D}_{Dh}Df"] \arrow[drr, phantom, "(4)"] & {} \arrow[r, "\mathbb{D}^{-1}_{dDC\,Dh,\mathrm{Id}_{DC}}dDB\,Df"] \arrow[dr, "\mathbb{D}_{mC\,dDC\,Dh}Df"'] \arrow[dr, phantom, "(5)", near end, shift left=3] & {} \arrow[d, "mC\mathbb{D}_{dDC\,Dh}Df"] \\
{} \arrow[u, "\mathbb{D}^{-1}_{dB\,f,dC\,h}"'] \arrow[drr, "id_{(Dh\,dB\,f)^{\mathbb{D}}}"'] \arrow[ur, phantom, "(5)"] & & {} \arrow[rr, "\mathbb{D}^{-1}_{\mathrm{Id}_{DC}}Dh\,Df"'] \arrow[d, "\mathbb{D}_{dB\,f,dC\,h}"'] & & {} \arrow[d, "mC\,dDC\mathbb{D}_{dB\,f,dC\,h}"] \\
& & {} \arrow[d, "(\mathbb{D}_{dC\,h}f)^{\mathbb{D}}"'] & & {} \arrow[d, "mC\,dDC(\mathbb{D}_{dC\,h}f)^{\mathbb{D}}"] \\
& & {} \arrow[rr, "\mathbb{D}^{-1}_{\mathrm{Id}_{DC}}Dhf"'] & & {}
\end{tikzcd}
$$

where the unlabeled subdiagram commutes by naturality.

Finally,

$$
\begin{aligned}
m_{\mathrm{Id}_{DA}} \cdot D_A\, mA &= \mathbb{D}^{-1}_{dDA\,D\mathrm{Id}_{DA},\mathrm{Id}_{DA}} \cdot (\mathbb{D}^{-1}_{\mathrm{Id}_{DA}} D\mathrm{Id}_A)^{\mathbb{D}} \cdot \mathbb{D}_{\mathrm{Id}_{DA},dA} \cdot D_A\, mA \\
&= \mathbb{D}^{-1}_{dDA\,D\mathrm{Id}_{DA},\mathrm{Id}_{DA}} \cdot (\mathbb{D}^{-1}_{\mathrm{Id}_{DA}} D\mathrm{Id}_A)^{\mathbb{D}} \cdot (D_A)^{\mathbb{D}} \\
&= \mathbb{D}^{-1}_{dDA\,D\mathrm{Id}_{DA},\mathrm{Id}_{DA}} \cdot (mA\, dDA\, D_A)^{\mathbb{D}} \cdot (\mathbb{D}^{-1}_{\mathrm{Id}_{DA}})^{\mathbb{D}} \\
&= mA\, D(D_A) \cdot \mathbb{D}^{-1}_{dDA,\mathrm{Id}_{DA}} \cdot (\mathbb{D}^{-1}_{\mathrm{Id}_{DA}})^{\mathbb{D}} \\
&= mA\, D(D_A) \cdot mA\, D_{\mathrm{Id}_{DA}} = mA\, (D^2)_A
\end{aligned}
$$

by (3), naturality, (6) and (2). ∎

The following two propositions give the 2-dimensional data for the construction of a pseudomonad.

3.4. Proposition. *If for every A in $\mathcal{A}$ we define $\beta A = \mathbb{D}_{\mathrm{Id}_{DA}}$ and we define ηA as*

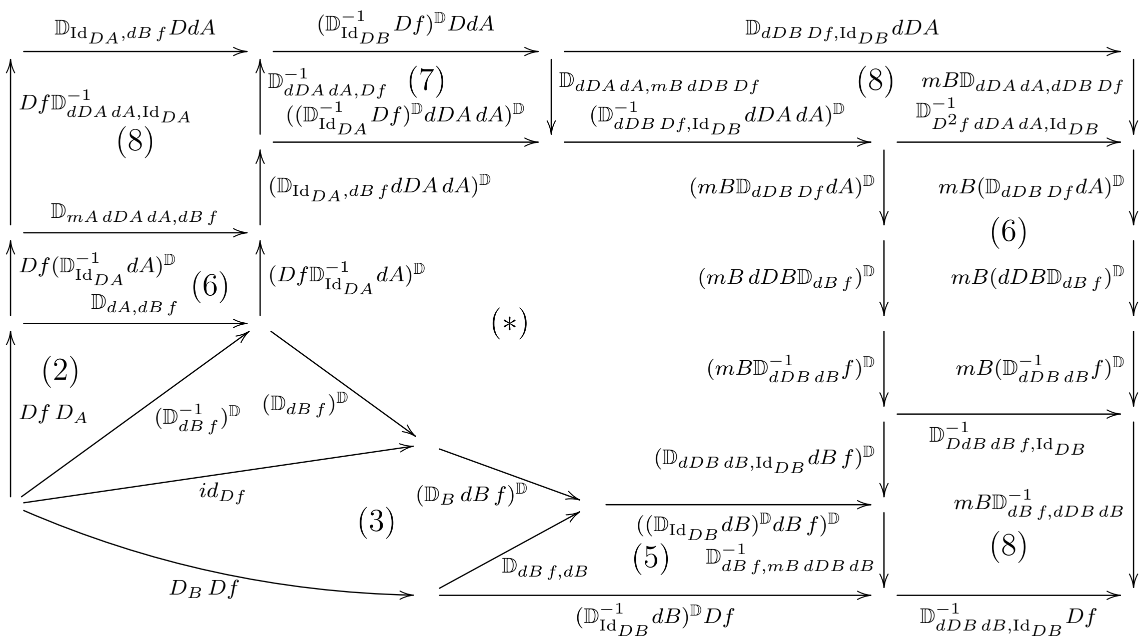
$$
\begin{tikzcd}
DA \arrow[rr, bend left=30, "\mathrm{Id}_{DA}"] \arrow[rr, "dA^{\mathbb{D}}"] \arrow[rr, bend right=20] \arrow[dr, bend right=20, "DdA"'] & \arrow[d, phantom, "\Downarrow D_A"] & DA \\
& D^2A \arrow[ur, bend right=20, "mA"'] \arrow[u, phantom, "(\mathbb{D}^{-1}_{\mathrm{Id}_{DA}} dA)^{\mathbb{D}} \Downarrow \quad \mathbb{D}^{-1}_{dDA\, dA, \mathrm{Id}_{DA}} (mA\, dDA\, dA)^{\mathbb{D}} \Downarrow"] &
\end{tikzcd}
$$

then we obtain modifications $\beta : mA\, dD \to \mathrm{Id}_D$ and $\eta : \mathrm{Id}_D \to mA\, DdA$.

Proof. The statement for β follows from the following commutative diagram for every $f : A \to B$:

$$
\begin{tikzcd}
Df^{\mathbb{D}} dDA \arrow[r, "(\mathbb{D}^{-1}_{\mathrm{Id}_{DB}} Df)^{\mathbb{D}} dDA"] \arrow[ddrr, "\mathbb{D}_{Df}"] & (mB\, dDB\, Df)^{\mathbb{D}} dDA \arrow[r, "\mathbb{D}^{-1}_{dDB\, Df, \mathrm{Id}_{DB}} dDA"] \arrow[dr, "\mathbb{D}_{mB\, dDB\, Df}"'] & mB\, D^2 f\, dDA \arrow[d, "mB \mathbb{D}_{dDA\, Df}"] \arrow[dl, phantom, "(5)"] \\
\arrow[r, phantom, "(5)"] & \arrow[r, phantom, "(4)"] & mB\, dDB\, Df \arrow[d, "\mathbb{D}_{\mathrm{Id}_{DB}} Df"] \\
Df\, mA\, dDA \arrow[uu, "\mathbb{D}_{\mathrm{Id}_{DA}, dB\, f} dDA"] \arrow[rr, "Df \mathbb{D}_{\mathrm{Id}_{DA}}"] & & Df
\end{tikzcd}
$$

The one for η follows from the commutative diagram



where $(*)$ is $(\ )^{\mathbb{D}}$ applied to the following commutative diagram

$$
\begin{tikzcd}[column sep=small]
Df^{\mathbb{D}}\, dDA\, dA \arrow[rr, "(\mathbb{D}^{-1}_{\mathrm{Id}_{DB}} Df)^{\mathbb{D}} dDA\, dA"] \arrow[ddrr, "\mathbb{D}_{Df} dA"] & & (mB\, dDB\, Df)^{\mathbb{D}} dDA\, dA \arrow[rr, "\mathbb{D}^{-1}_{dDB\, Df, \mathrm{Id}_{DB}} dDA\, dA"] \arrow[drr, "\mathbb{D}_{mB\, dDB\, Df} dA"'] \arrow[d, phantom, "(4)"] & \arrow[dr, phantom, "(5)"] & mB\, D^2 f\, dDA\, dA \arrow[d, "mB\, \mathbb{D}_{dDB\, Df} dA"] & \\
 & & {} & & mB\, dDB\, Df\, dA \arrow[dd, "mB\, dDB\, \mathbb{D}_{dB\, f}"] & \\
Df\, mA\, dDA\, dA \arrow[uu, "\mathbb{D}_{\mathrm{Id}_{DA}, dB\, f} dDA\, dA"] \arrow[dr, phantom, "(5)"] & & Df\, dA \arrow[urr, "\mathbb{D}^{-1}_{\mathrm{Id}_{DB}} Df\, dA"'] & & & \\
 & {} & & & mB\, dDB\, dB\, f \arrow[dr, "mB \mathbb{D}^{-1}_{dDB\, dB} f"] \arrow[ddd, "\mathbb{D}^{-1}_{mB\, dDB\, dB} f"'] & \\
Df\, dA \arrow[uu, "Df \mathbb{D}^{-1}_{\mathrm{Id}_{DA}} dA"] \arrow[uurr, "id_{Df\, dA}"'] \arrow[rr, "\mathbb{D}_{dB\, f}"'] & & dB\, f \arrow[urr, "\mathbb{D}^{-1}_{\mathrm{Id}_{DB}} dB\, f"'] \arrow[dd, bend right=40, "\mathbb{D}_B\, dB\, f"'] \arrow[dd, bend left=40, "\mathbb{D}^{-1}_{dB} f"] \arrow[dd, phantom, "(1)"] \arrow[dr, phantom, "(4)"] & & & mB\, DdB\, dB\, f \arrow[ddl, "\mathbb{D}_{dDB\, dB, \mathrm{Id}_{DB}} dB\, f"] \arrow[dl, phantom, "(5)"] \\
 & & & {} & {} & \\
 & & dB^{\mathbb{D}} dB\, f \arrow[rr, "(\mathbb{D}^{-1}_{\mathrm{Id}_{DB}} dB)^{\mathbb{D}} dB\, f"'] & & (mB\, dDB\, dB)^{\mathbb{D}} dB\, f &
\end{tikzcd}
$$

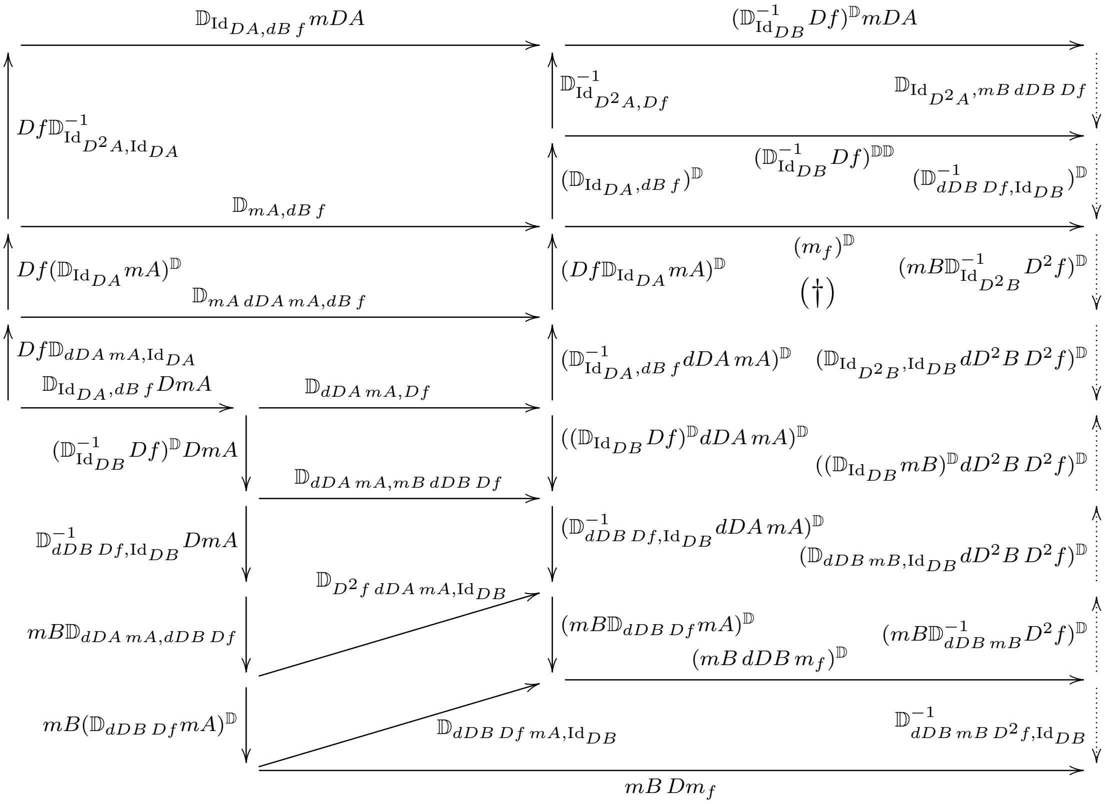
■

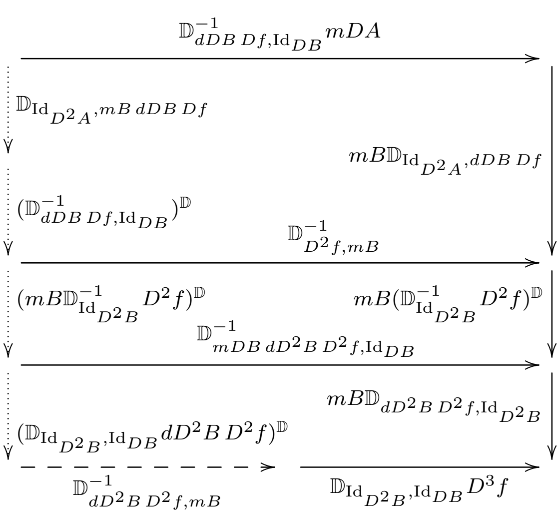
3.5. Proposition. *If for every A we define μA as*

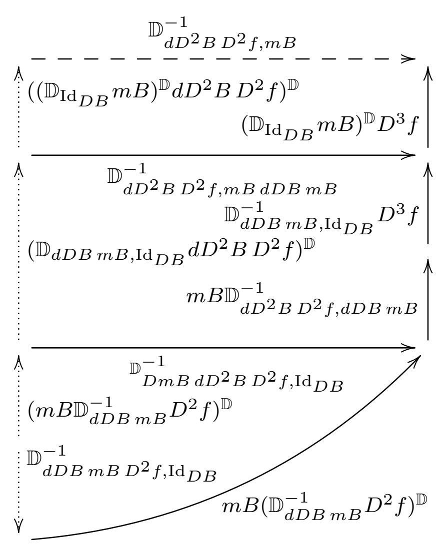
$$
\begin{tikzcd}[column sep=huge, row sep=huge]
D^3A \arrow[rr, "DmA"] \arrow[d, "mDA"'] \arrow[drr, bend left=15, "(mA\, dDA\, mA)^{\mathbb{D}}"] \arrow[drr, bend right=15, "mA^{\mathbb{D}}"'] & & D^2A \arrow[d, "mA"] \arrow[dl, phantom, shift left=4ex, "\Swarrow \mathbb{D}_{dDA\, mA, \mathrm{Id}_{DA}}"] \\
D^2A \arrow[rr, "mA"'] \arrow[ur, phantom, shift right=4ex, "\Swarrow \mathbb{D}^{-1}_{\mathrm{Id}_{D^2A}, \mathrm{Id}_{DA}}"] & {} \arrow[u, phantom, "\Swarrow (\mathbb{D}_{\mathrm{Id}_{DA}} mA)^{\mathbb{D}}"] & DA
\end{tikzcd}
$$

then we obtain a modification $\mu : mA\, DmA \to mA\, mDA$.

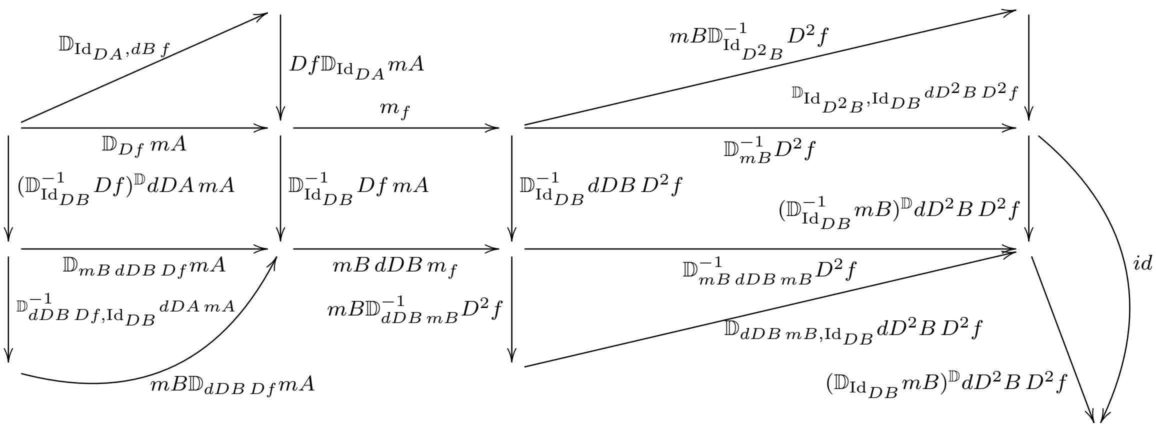
Proof. The proof is obtained by glueing the following three commutative diagrams, first the last two along the dashed arrow, and then the resulting diagram with the first along the dotted arrows:







where $(\dagger)$ is $(\ )^{\mathbb{D}}$ applied to the commutative diagram

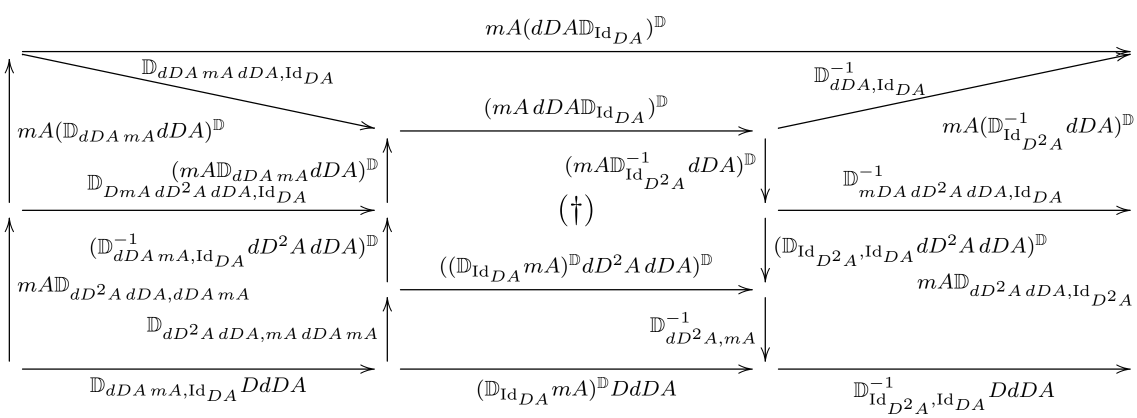


■

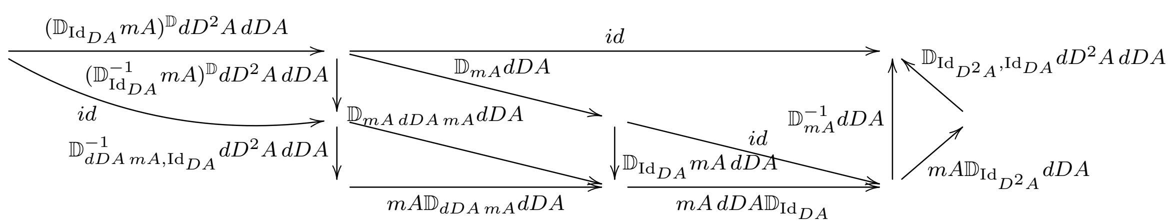
We now show that we have indeed produced a pseudomonad.

3.6. Theorem. *Let $\mathbb{D}$ be a no-iteration monad. The above constructions of the pseudofunctor D, the strong transformations d and m, and the modifications β, η, μ constitute a pseudomonad $(D, d, m, \beta, \eta, \mu)$.*

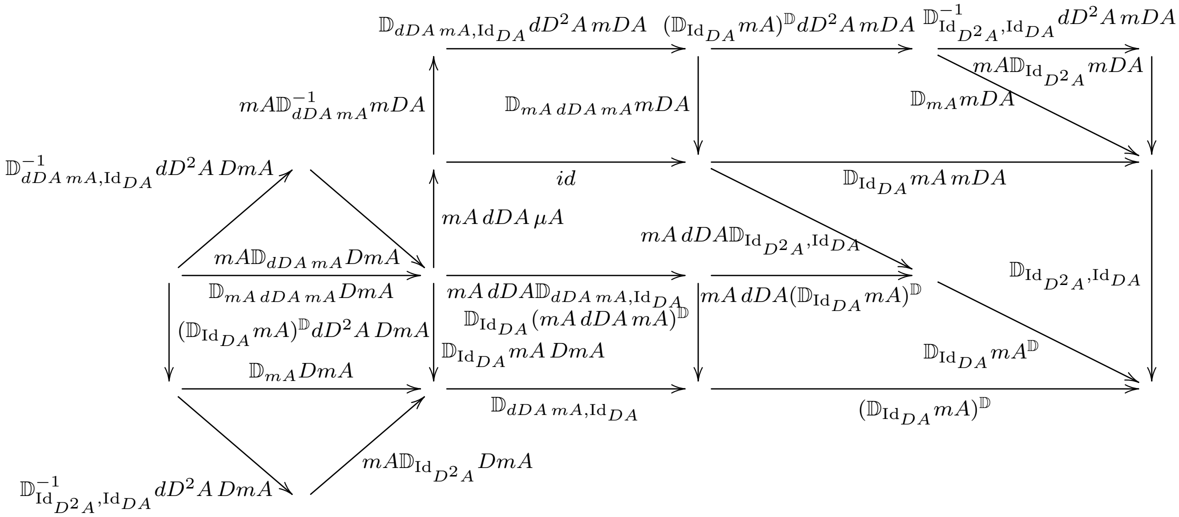
Proof. We must show the usual two coherence conditions. The equation $\mu A\,DdDA = (mA\,D\beta A)(mA\,\eta DA)$ is shown by the commutative diagram

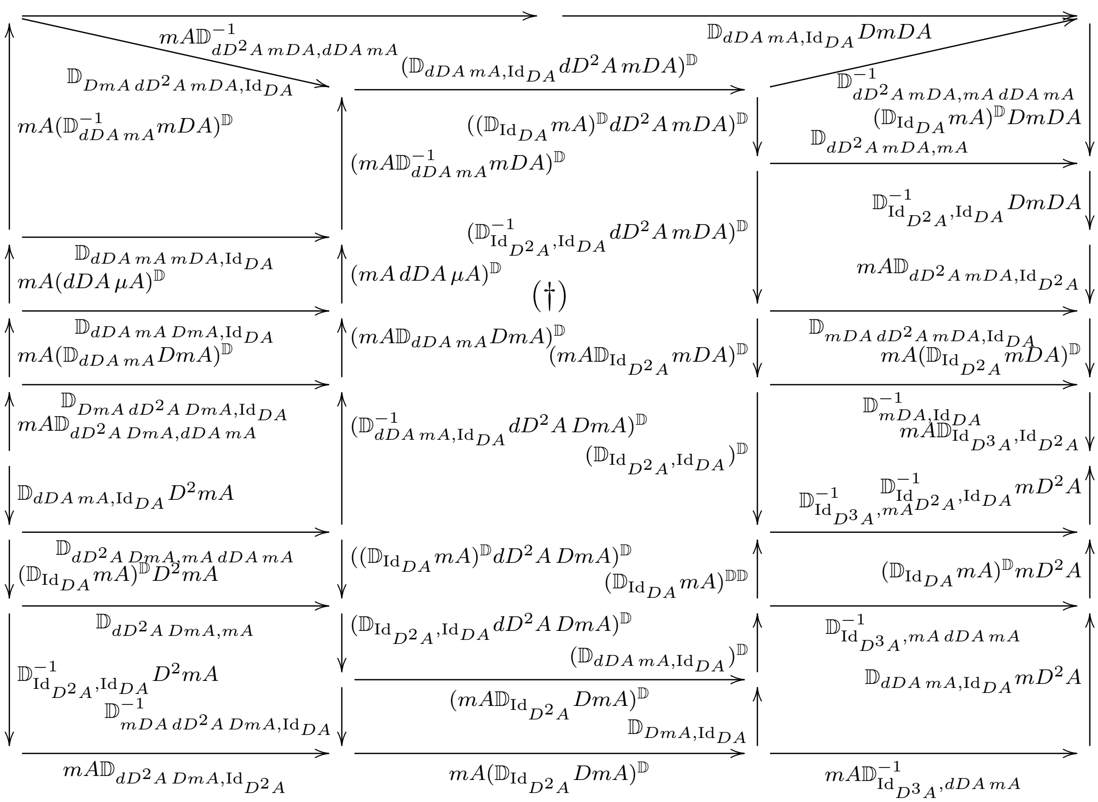


where $(\dagger)$ is $(\ )^{\mathbb{D}}$ applied to the commutative diagram



The proof of the other coherence condition is given by the commutative diagram after the following one, where (†) is $(\ )^{\mathbb{D}}$ applied to the commutative diagram





■

4. The algebras for a no-iteration pseudomonad

Let $\mathbb{D}$ be a no-iteration pseudomonad as in Definition 2.1. Following the same no-iteration framework, we define the 2-category $\widehat{\mathrm{Alg}}\text{-}\mathbb{D}$ of algebras for such a presentation as follows.

An object of $\widehat{\mathrm{Alg}}\text{-}\mathbb{D}$ consists of

1. An object A in $\mathcal{A}$.
2. For every X in $\mathcal{A}$ a functor $(\ )^{\mathbb{A}} : \mathcal{A}(X, A) \to \mathcal{A}(DX, A)$.
3. For every $h : X \to A$ an invertible 2-cell

$$\begin{tikzcd}
X \arrow[r, "dX"] \arrow[dr, "h"'] & DX \arrow[d, "h^{\mathbb{A}}"] \arrow[dl, phantom, "\Leftarrow \mathbb{A}_h"] \\
& A
\end{tikzcd}$$

4. For every $h : X \to A$ and every $g : Y \to DX$, an invertible 2-cell

$$\begin{tikzcd}
& DX \arrow[dr, "h^{\mathbb{A}}"] \arrow[d, phantom, "\mathbb{A}_{g,h}\Downarrow"] & \\
DY \arrow[ur, "g^{\mathbb{D}}"] \arrow[rr, "(h^{\mathbb{A}}g)^{\mathbb{A}}"'] & {} & A
\end{tikzcd}$$

subject to the axioms

1. For every $\varphi : h \to k : X \to A$

$$\begin{tikzcd}
X \arrow[r, "dX"] \arrow[dr, "k"'] & DX \arrow[d, bend right, "k^{\mathbb{A}}"'] \arrow[d, bend left, "h^{\mathbb{A}}"] \arrow[d, phantom, "\overset{\varphi^{\mathbb{A}}}{\Leftarrow}"] \arrow[dl, phantom, "\Leftarrow \mathbb{A}_k"] \\
& A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "dX"] \arrow[dr, bend left, "h"] \arrow[dr, bend right, "k"'] \arrow[dr, phantom, "\Leftarrow \varphi"] & DX \arrow[d, "h^{\mathbb{A}}"] \arrow[dl, phantom, "\Leftarrow \mathbb{A}_h"] \\
& A
\end{tikzcd} \tag{9}$$

2. For every $h : X \to A$

$$\begin{tikzcd}
& DX \arrow[dr, "h^{\mathbb{A}}"] \arrow[d, phantom, "\Leftarrow \mathbb{A}_{dX,h}"] & \\
DX \arrow[ur, bend left=50, "\mathrm{Id}_{DX}"] \arrow[ur, "dX^{\mathbb{D}}"'] \arrow[ur, phantom, "\Downarrow D_X"] \arrow[rr, "(h^{\mathbb{A}}dX)^{\mathbb{A}}"'] \arrow[rr, bend right=40, "h^{\mathbb{A}}"'] \arrow[rr, phantom, bend right=20, "\Downarrow(\mathbb{A}_h)^{\mathbb{A}}"] & {} & A
\end{tikzcd}
= id_{h^{\mathbb{A}}} \tag{10}$$

3. For every $h : X \to A$ and every $g : Y \to DX$

$$\begin{tikzcd}
Y \arrow[r, "dY"] \arrow[dr, "g"'] & DY \arrow[d, "g^{\mathbb{D}}"] \arrow[dl, phantom, "\Leftarrow \mathbb{D}_g"] \\
& DX \arrow[d, "h^{\mathbb{A}}"] \\
& A
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, "dY"] \arrow[dr, "h^{\mathbb{A}}g"'] & DY \arrow[d, "(h^{\mathbb{A}}g)^{\mathbb{A}}"] \arrow[dl, phantom, "\Leftarrow \mathbb{A}_{(h^{\mathbb{A}}g)}"] \arrow[r, "g^{\mathbb{D}}"] & DX \arrow[dl, "h^{\mathbb{A}}"] \arrow[l, phantom, "\overset{\mathbb{A}_{g,h}}{\Leftarrow}"] \\
& A. &
\end{tikzcd} \tag{11}$$

4. For every $h\colon X \to A$ and every $\psi\colon f \to g : Y \to DX$

$$
\begin{tikzcd}
 & DX \arrow[dr, "h^{\mathbb{A}}"] \arrow[d, phantom, "\Leftarrow \mathbb{A}_{g,h}"] & \\
DY \arrow[ur, bend left=50, "f^{\mathbb{D}}"] \arrow[ur, "g^{\mathbb{D}}"'] \arrow[ur, phantom, bend left=25, "\Downarrow \psi^{\mathbb{D}}"] \arrow[rr, "(h^{\mathbb{A}}g)^{\mathbb{A}}"'] & {} & A
\end{tikzcd}
=
\begin{tikzcd}
 & DX \arrow[dr, "h^{\mathbb{A}}"] & \\
DY \arrow[ur, "f^{\mathbb{D}}"] \arrow[rr, bend left=15, "(h^{\mathbb{A}}f)^{\mathbb{A}}\ \Downarrow \mathbb{A}_{f,h}"] \arrow[rr, phantom, "(h^{\mathbb{A}}\psi)^{\mathbb{A}}\Downarrow"] \arrow[rr, bend right=15, "(h^{\mathbb{A}}g)^{\mathbb{A}}"'] & & A
\end{tikzcd}
\tag{12}
$$

5. For every $\varphi\colon h \to k\colon X \to A$ and every $g\colon Y \to DX$

$$
\begin{tikzcd}
 & DX \arrow[dr, bend left=30, "h^{\mathbb{A}}"] \arrow[dr, "k^{\mathbb{A}}"'] \arrow[dr, phantom, bend left=15, "\Leftarrow \varphi^{\mathbb{A}}"] \arrow[d, phantom, "\Leftarrow \mathbb{A}_{g,k}"] & \\
DY \arrow[ur, "g^{\mathbb{D}}"] \arrow[rr, "(k^{\mathbb{A}}g)^{\mathbb{A}}"'] & {} & A
\end{tikzcd}
=
\begin{tikzcd}
 & DX \arrow[dr, "h^{\mathbb{A}}"] & \\
DY \arrow[ur, "g^{\mathbb{D}}"] \arrow[rr, bend left=15, "(h^{\mathbb{A}}g)^{\mathbb{A}}\ \Downarrow \mathbb{A}_{g,h}"] \arrow[rr, phantom, "(\varphi^{\mathbb{A}}g)^{\mathbb{A}}\Downarrow"] \arrow[rr, bend right=15, "(k^{\mathbb{A}}g)^{\mathbb{A}}"'] & & A
\end{tikzcd}
\tag{13}
$$

6. For every $h\colon X \to A$, $g\colon Y \to DX$ and $\ell\colon Z \to DY$

$$
\begin{tikzcd}
 & DY \arrow[r, "g^{\mathbb{D}}"] \arrow[d, phantom, "\Downarrow \mathbb{D}_{\ell,g}"] & DX \arrow[dr, "h^{\mathbb{A}}"] \arrow[d, phantom, "\mathbb{A}_{g^{\mathbb{D}}\ell,h}\Downarrow"] & \\
DZ \arrow[ur, "\ell^{\mathbb{D}}"] \arrow[urr, bend left=10, "(g^{\mathbb{D}}\ell)^{\mathbb{D}}"'] \arrow[rrr, "(h^{\mathbb{A}}g^{\mathbb{D}}\ell)^{\mathbb{A}}\quad (\mathbb{A}_{h,g}\ell)^{\mathbb{A}}\Downarrow"'] \arrow[rrr, bend right=30, "((h^{\mathbb{A}}g)^{\mathbb{A}}\ell)^{\mathbb{A}}"'] & {} & {} & A
\end{tikzcd}
=
\begin{tikzcd}
 & DY \arrow[r, "g^{\mathbb{D}}"] \arrow[drr, "(h^{\mathbb{A}}g)^{\mathbb{A}}"'] \arrow[d, phantom, "\Downarrow \mathbb{A}_{\ell,h^{\mathbb{A}}g}"] & DX \arrow[dr, "h^{\mathbb{A}}"] \arrow[d, phantom, "\Leftarrow \mathbb{A}_{g,h}"] & \\
DZ \arrow[ur, "\ell^{\mathbb{D}}"] \arrow[rrr, bend right=20, "((h^{\mathbb{A}}g)^{\mathbb{A}}\ell)^{\mathbb{A}}"'] & {} & {} & A
\end{tikzcd}
\tag{14}
$$

Given objects $(A, (\ )^{\mathbb{A}})$ and $(B, (\ )^{\mathbb{B}})$ in $\widehat{\text{Alg}}$-$\mathbb{D}$ (we omit the rest of the structure), a 1-cell in $\widehat{\text{Alg}}$-$\mathbb{D}$ is a 1-cell $f\colon A \to B$ in $\mathcal{A}$ together with an invertible 2-cell

$$
\begin{tikzcd}
DX \arrow[d, "h^{\mathbb{A}}"'] \arrow[dr, "(fh)^{\mathbb{B}}"] \arrow[dr, phantom, shift right=3, "\Leftarrow f[h]"] & \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

for every $h : X \to A$ in $\mathcal{A}$, subject to the coherence conditions

$$
\begin{tikzcd}
X \arrow[r, "dX"] \arrow[dr, "h"'] \arrow[dr, phantom, shift left=3, "\Leftarrow \mathbb{A}_h"] & DX \arrow[d, "h^{\mathbb{A}}"] \arrow[dr, "(fh)^{\mathbb{B}}"] \arrow[dr, phantom, shift right=3, "f[h] \Leftarrow"] & \\
 & A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "dX"] \arrow[d, "h"'] \arrow[dr, phantom, "\Leftarrow \mathbb{B}_{fh}"] & DX \arrow[d, "(fh)^{\mathbb{B}}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\tag{15}
$$

and for any $g\colon Y \to DX$

$$
\begin{tikzcd}
DY \arrow[r, "g^{\mathbb{D}}"] \arrow[dr, "(h^{\mathbb{A}}g)^{\mathbb{A}}"'] \arrow[dr, phantom, shift left=3, "\Leftarrow \mathbb{A}_{g,h}"] & DX \arrow[d, "h^{\mathbb{A}}"] \arrow[dr, "(fh)^{\mathbb{B}}"] \arrow[dr, phantom, shift right=3, "f[h] \Leftarrow"] & \\
 & A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=huge]
DY \arrow[r, "g^{\mathbb{D}}"] \arrow[d, "(h^{\mathbb{A}}g)^{\mathbb{A}}"'] \arrow[dr, bend left=20, "((fh)^{\mathbb{B}}g)^{\mathbb{B}}"'] \arrow[dr, bend right=20, "(fh^{\mathbb{B}}g)^{\mathbb{B}}"] \arrow[dr, phantom, "\Leftarrow (f[h]\,g)^{\mathbb{B}}"] \arrow[dr, phantom, shift right=8, "\Leftarrow f[h^{\mathbb{A}}g]"] \arrow[dr, phantom, shift left=8, "\Leftarrow \mathbb{B}_{g,fh}"] & DX \arrow[d, "(fh)^{\mathbb{B}}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\tag{16}
$$

and for every $\kappa : h \to k : X \to A$,

$$
\begin{tikzcd}
DX \arrow[d, bend right, "k^{\mathbb{A}}"'] \arrow[d, bend left, "h^{\mathbb{A}}"] \arrow[d, phantom, "\overset{\kappa^{\mathbb{A}}}{\Leftarrow}"] \arrow[dr, "(fh)^{\mathbb{B}}"] & \\
A \arrow[r, "f"'] \arrow[ur, phantom, "\overset{f[h]}{\Swarrow}"] & B
\end{tikzcd}
=
\begin{tikzcd}
DX \arrow[d, "k^{\mathbb{A}}"'] \arrow[dr, "(fk)^{\mathbb{B}}"'] \arrow[dr, bend left=50, "(fh)^{\mathbb{B}}"] \arrow[dr, phantom, bend left=25, "\overset{(f\kappa)^{\mathbb{B}}}{\Swarrow}"] & \\
A \arrow[r, "f"'] \arrow[ur, phantom, "\overset{f[k]}{\Swarrow}"] & B
\end{tikzcd}
\tag{17}
$$

For 1-cells $f : A \to B$, $f' : B \to C$ in $\widehat{\text{Alg}}$-$\mathbb{D}$, composition is given by the usual composition $f'f$ in $\mathcal{A}$ and, for every $h : X \to A$, the 2-cell

$$
(f'f)[h] =
\begin{tikzcd}
 & DX \arrow[dl, "h^{\mathbb{A}}"'] \arrow[d, "(fh)^{\mathbb{B}}"] \arrow[dr, "(f'fh)^{\mathbb{C}}"] & \\
A \arrow[r, "f"'] \arrow[ur, phantom, "\overset{f[h]}{\Swarrow}"] & B \arrow[r, "f'"'] \arrow[ur, phantom, "\overset{f'[fh]}{\Swarrow}"] & C
\end{tikzcd}
$$

It is direct to show that the composition $f'f$ satisfies the conditions (15), (16) and (17). A 2-cell $\alpha : f \to g : A \to B$ in $\widehat{\text{Alg}}$-$\mathbb{D}$ is a 2-cell $\alpha : f \to g$ in $\mathcal{A}$ such that for every $h : X \to A$ we have that

$$
\begin{tikzcd}
DX \arrow[d, "h^{\mathbb{A}}"'] \arrow[dr, "(gh)^{\mathbb{B}}"'] \arrow[dr, bend left=50, "(fh)^{\mathbb{B}}"] \arrow[dr, phantom, bend left=25, "\overset{(\alpha h)^{\mathbb{B}}}{\Swarrow}"] & \\
A \arrow[r, "g"'] \arrow[ur, phantom, "\overset{g[h]}{\Swarrow}"] & B
\end{tikzcd}
=
\begin{tikzcd}
DX \arrow[d, "h^{\mathbb{A}}"'] \arrow[dr, "(fh)^{\mathbb{B}}"] & \\
A \arrow[r, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, bend right=25, "\alpha\Downarrow"] \arrow[ur, phantom, "\overset{f[h]}{\Swarrow}"] & B
\end{tikzcd}
\tag{18}
$$

Composition of 2-cells in $\widehat{\text{Alg}}$-$\mathbb{D}$ is as in $\mathcal{A}$.

For every object A in $\widehat{\text{Alg}}$-$\mathbb{D}$, Id_A is the usual $Id_A : A \to A$ in $\mathcal{A}$ together with the identity two cell

$$
\begin{tikzcd}
DX \arrow[d, "h^{\mathbb{A}}"'] \arrow[dr, "h^{\mathbb{A}}"] & \\
A \arrow[r, "Id_A"'] \arrow[ur, phantom, "\overset{Id_A[h]}{\Swarrow}"] & A
\end{tikzcd}
$$

on $h^{\mathbb{A}}$ for every $h : X \to A$ in $\mathcal{A}$.

5. The biequivalence $\mathbb{D}$-Alg $\simeq \widehat{\text{Alg}}$-$\mathbb{D}$

In this section we consider a no-iteration pseudomonad, called $\mathbb{D}$, as in Definition 2.1, and induce the pseudomonad, also called $\mathbb{D}$, as in Section 3. We denote the usual category of algebras for $\mathbb{D}$ as $\mathbb{D}$-Alg.

5.1. THEOREM. *The categories* $\mathbb{D}$-Alg *and* $\widehat{\text{Alg}}$-$\mathbb{D}$ *are biequivalent.*

PROOF. We define pseudofunctors in both directions.

Define $F : \widehat{\text{Alg}}\text{-}\mathbb{D} \to \mathbb{D}\text{-Alg}$ as follows. Take $(A, (\ )^{\mathbb{A}}) \underset{g}{\overset{f}{\underset{\varphi\Downarrow}{\rightrightarrows}}} (B, (\ )^{\mathbb{B}})$ in $\widehat{\text{Alg}}$-$\mathbb{D}$. Let $\alpha_0 = {Id_A}^{\mathbb{A}}$, $\alpha_1 = \mathbb{A}_{Id_A}$, $\beta_0 = {Id_B}^{\mathbb{B}}$, $\beta_1 = \mathbb{B}_{Id_B}$, and define FA as

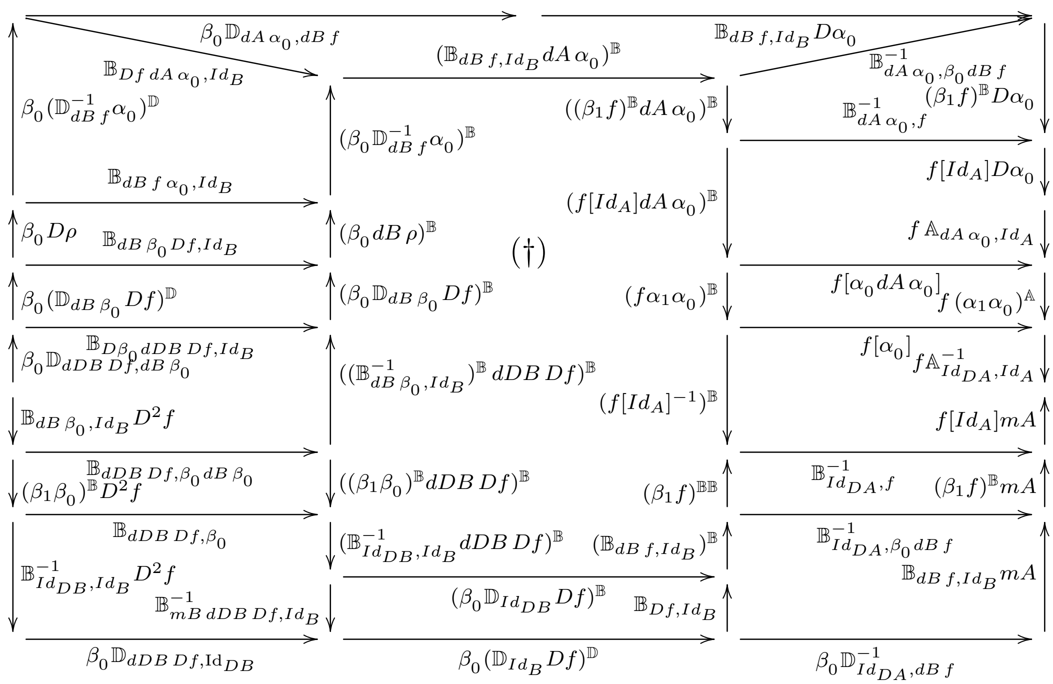
$$\left(\begin{tikzcd} A \arrow[r, "dA"] \arrow[dr, "Id_A"'] & DA \arrow[d, "\alpha_0"] \arrow[dl, phantom, "\Leftarrow\alpha_1"] \\ & A \end{tikzcd}\ ,\ \begin{tikzcd}[column sep=huge, row sep=huge] D^2A \arrow[r, "D(\alpha_0)"] \arrow[d, "mA"'] \arrow[dr, bend left, "(\alpha_0 dA\,\alpha_0)^{\mathbb{A}}"] \arrow[dr, bend right, "{\alpha_0}^{\mathbb{A}}"'] \arrow[dr, phantom, "\Leftarrow(\alpha_1\alpha_0)^{\mathbb{A}}"] & DA \arrow[d, "\alpha_0"] \arrow[dl, phantom, "\Leftarrow\mathbb{A}_{dA\,\alpha_0, Id_A}"] \\ DA \arrow[r, "\alpha_0"'] \arrow[ur, phantom, "\Leftarrow{\mathbb{A}_{Id_{DA}, Id_A}}^{-1}"] & A \end{tikzcd}\right) \tag{19}$$

For convenience, we write α_2 for the second coordinate of FA. Define Ff as

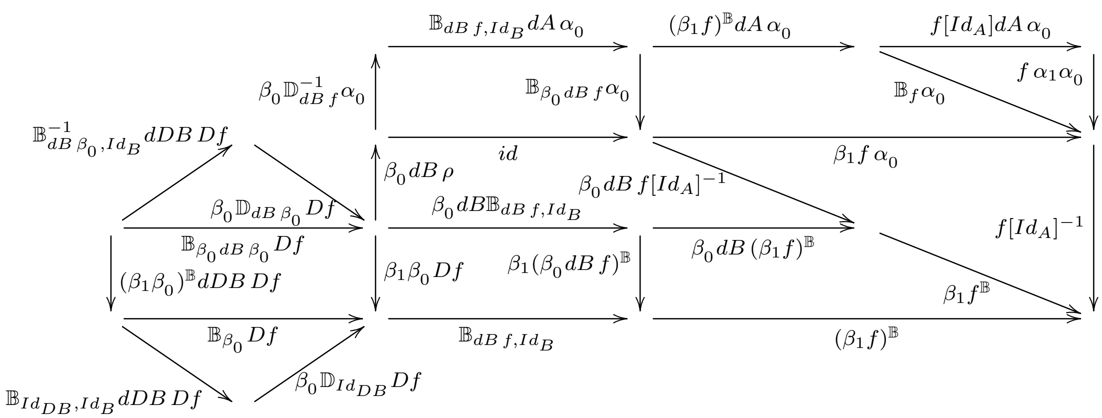
$$\begin{tikzcd}[column sep=huge, row sep=huge] DA \arrow[r, "Df"] \arrow[d, "\alpha_0"'] \arrow[dr, bend left, "(\beta_0 dB\, f)^{\mathbb{B}}"] \arrow[dr, bend right, "f^{\mathbb{B}}"'] \arrow[dr, phantom, "\Leftarrow(\beta_1 f)^{\mathbb{B}}"] & DB \arrow[d, "\beta_0"] \arrow[dl, phantom, "\Leftarrow\mathbb{B}_{dB\, f, Id_B}"] \\ A \arrow[r, "f"'] \arrow[ur, phantom, "\Leftarrow f[Id_A]"] & B \end{tikzcd}$$

For future reference, call this two cell ρ. And define $F\varphi = \varphi$.

We must verify first that FA is indeed an object in $\mathbb{D}$-Alg, but this is essentially the same proof given in Theorem 3.6 except that some $\mathbb{D}$'s should now be $\mathbb{A}$'s. We must also show that $Ff : FA \to FB$ is in $\mathbb{D}$-Alg. The diagram for this has the same shape, namely



where $(\dagger)$ is $(\ )^{\mathbb{B}}$ applied to the commutative diagram



It is not hard to see that $F\varphi$ is a 2-cell in $\mathbb{D}$-Alg. We have that $F(Id_A) = Id_{FA}$ by (10). Furthermore, if $f : A \to B$ and $f' : B \to C$ are 1-cells in $\widehat{\text{Alg}}$-$\mathbb{D}$, then the proof that $F(f'f) = F(f')F(f)$ is given by the following commutative diagram (with $\gamma_0 = (Id_C)^{\mathbb{C}}$

and $\gamma_1 = \mathbb{C}_{Id_C}$)

$$\begin{tikzcd}[column sep=huge]
{} \arrow[rr, "(\gamma_1 f')^{\mathbb{C}} Df"] \arrow[dr, "\mathbb{C}_{dB\,f,\gamma_0\,dC\,f'}"] & & {} \arrow[r, "f'[Id_B]Df"] \arrow[d, "\mathbb{C}_{dB\,f,f'}"] & {} \arrow[dd, "f'\mathbb{B}_{dB\,f,Id_B}"] & \\
{} \arrow[u, "\mathbb{C}_{dC\,f',Id_C}Df"] & {} \arrow[r, "((\gamma_1 f')^{\mathbb{C}} dB\,f)^{\mathbb{C}}"] & {} \arrow[d, "(f'[Id_B]dB\,f)^{\mathbb{C}}"] & & \\
{} \arrow[u, "\gamma_0\mathbb{D}^{-1}_{dB\,f,dC\,f'}"] \arrow[r, "\mathbb{C}_{D(f')dB\,f,Id_C}"] & {} \arrow[u, "(\mathbb{C}_{dC\,f',Id_C}dB\,f)^{\mathbb{C}}"'] & {} \arrow[r, "{f[\beta_0 dB\,f]}"'] \arrow[d, "(f'\beta_1 f)^{\mathbb{C}}"'] & {} \arrow[d, "f'(\beta_1 f)^{\mathbb{B}}"] & \\
{} \arrow[u, "\gamma_0(\mathbb{D}^{-1}_{dC\,f'})^{\mathbb{D}}"] \arrow[r, "\mathbb{C}_{dC\,f'f,Id_C}"'] & {} \arrow[u, "(\gamma_0\mathbb{D}^{-1}_{dC\,f'}f)^{\mathbb{C}}"] \arrow[r, "(\gamma_1 f' f)^{\mathbb{C}}"'] & {} \arrow[r, "{f'[f]}"'] & {} \arrow[r, "{f'\,f[Id_A]}"'] & {}
\end{tikzcd} .$$

In the other direction take

$$\begin{tikzcd}
DA \arrow[r, "Df"] \arrow[d, "\alpha_0"'] & DB \arrow[d, "\beta_0"] \arrow[dl, phantom, "\rho"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

$$\left(
\begin{tikzcd}
A \arrow[r, "dA"] \arrow[dr, "Id_A"'] & DA \arrow[d, "\alpha_0"] \arrow[dl, phantom, "\alpha_1"] \\
 & A
\end{tikzcd} ,
\begin{tikzcd}
D^2A \arrow[r, "D\alpha_0"] \arrow[d, "mA"'] & DA \arrow[d, "\alpha_0"] \arrow[dl, phantom, "\alpha_2"] \\
DA \arrow[r, "\alpha_0"'] & A
\end{tikzcd}
\right)
\begin{tikzcd}
{} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\varphi"] & {}
\end{tikzcd}
\left(
\begin{tikzcd}
B \arrow[r, "dB"] \arrow[dr, "Id_B"'] & DB \arrow[d, "\beta_0"] \arrow[dl, phantom, "\beta_1"] \\
 & B
\end{tikzcd} ,
\begin{tikzcd}
D^2B \arrow[r, "D\beta_0"] \arrow[d, "mB"'] & DB \arrow[d, "\beta_0"] \arrow[dl, phantom, "\beta_2"] \\
DB \arrow[r, "\beta_0"'] & B
\end{tikzcd}
\right)$$

$$\begin{tikzcd}
DA \arrow[r, "Df'"] \arrow[d, "\alpha_0"'] & DB \arrow[d, "\beta_0"] \arrow[dl, phantom, "\rho'"] \\
A \arrow[r, "f'"'] & B
\end{tikzcd}$$

in $\mathbb{D}$-Alg. Define $G(\alpha_1, \alpha_2)$ as

1. The same object A.
2. For every X in $\mathcal{A}$, $(\ )^{\mathbb{A}}$ is $\mathcal{A}(X,A) \xrightarrow{D} \mathcal{A}(DX,DA) \xrightarrow{\mathcal{A}(X,\alpha_0)} \mathcal{A}(DX,A)$.
3. For every $h\colon X \to A$ define $\mathbb{A}_h$ as $\begin{tikzcd}
X \arrow[r, "dX"] \arrow[d, "h"'] & DX \arrow[d, "Dh"] \arrow[dl, phantom, "d_h"] \\
A \arrow[r, "dA"] \arrow[dr, "Id_A"'] & DA \arrow[d, "\alpha_0"] \arrow[dl, phantom, "\alpha_1"] \\
 & A
\end{tikzcd}$
4. For every $g\colon Y \to DX$ and $h\colon X \to A$ define $\mathbb{A}_{g,h}$ as $\begin{tikzcd}[column sep=huge]
DY \arrow[r, bend left=40, "g^{\mathbb{D}}"] \arrow[r, bend left=20, phantom, "(\mathbb{D}^{-1}_{Id_{DX}}g)^{\mathbb{D}}\Downarrow"] \arrow[r, "(mX\,dDX\,g)^{\mathbb{D}}"'] \arrow[d, "Dg"'] & DX \arrow[ddd, "Dh"] \\
D^2X \arrow[ur, bend right, "mX"'] \arrow[ur, phantom, "\mathbb{D}^{-1}_{dDX\,g,Id_X}\Downarrow"] \arrow[dd, "D^2h"'] & \\
 & \\
D^2A \arrow[r, "mA"'] \arrow[d, "D\alpha_0"'] \arrow[ur, phantom, "m_h"] & DA \arrow[d, "\alpha_0"] \arrow[dl, phantom, "\alpha_2^{-1}"] \\
DA \arrow[r, "\alpha_0"'] & A
\end{tikzcd}$

Define $G(f,\rho)[h]$ as $\begin{tikzcd}
DX \arrow[d, "Dh"'] \arrow[dr, "D(fh)"] & \\
DA \arrow[r, "Df"'] \arrow[d, "\alpha_0"'] \arrow[ur, phantom, "(D^{h,f})^{-1}"] & DB \arrow[d, "\beta_0"] \arrow[dl, phantom, "\rho"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$. And define $G\varphi = \varphi$.

To see that $G(\alpha_1, \alpha_2)$ is an object in $\widehat{\text{Alg}}\text{-}\mathbb{D}$, observe first that condition (9) is direct. The show condition (10) observe that the pasting of D_X and $A_{dX,h}$ is $\alpha_2^{-1} D^2h\, Ddx \cdot \alpha_0 m_h\, DdX \cdot \alpha_0 Dh\, \eta X$, and then consider the following commutative diagram:

$$
\begin{tikzcd}
\cdot \arrow[rr, "\alpha_2^{-1}\, D^2h\, DdX"] \arrow[drr, "\alpha_0\, mX\, Dd_h"'] & & \cdot \arrow[rr, "\alpha_0\, D\alpha_0\, Dd_h"] & & \cdot \arrow[rr, "\alpha_0\, D^{dA,\alpha_0}\, Dh"] & & \cdot \arrow[d, "\alpha_0\, D\alpha_1\, Dh"] \\
& & \cdot \arrow[urr, "\alpha_2^{-1}\, DdA\, Dh"'] \arrow[ddrrrr, "\alpha_0\, \eta X^{-1}\, Dh"] & & & & \cdot \arrow[dd, "\alpha_0\, D_A^{-1}\, Dh"] \\
\cdot \arrow[uu, "\alpha_0\, m_h\, DdX"] \arrow[drrrrrr, "\alpha_0\, Dh\, \eta X^{-1}"] & & & & & & \\
\cdot \arrow[u, "\alpha_0\, Dh\, \eta X"] \arrow[rrrrrr, "id_{\alpha_0 Dh}"'] & & & & & & \cdot
\end{tikzcd}
$$

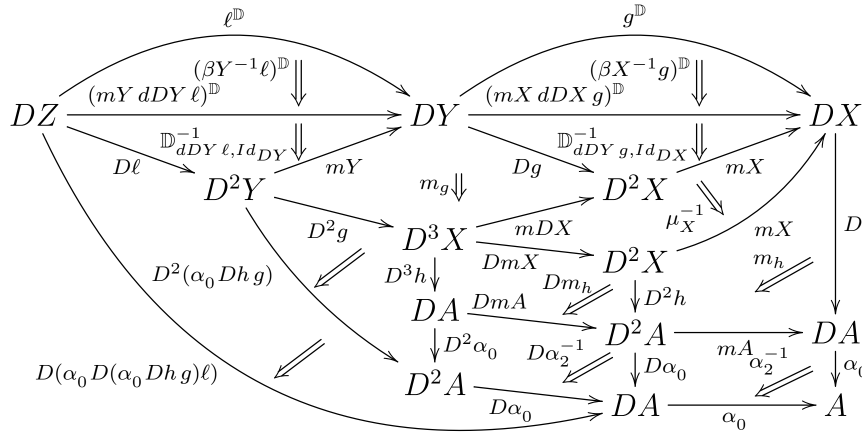
Condition (11) is the commutativity of the diagram

$$
\begin{tikzcd}
& & \cdot \arrow[r, "\alpha_2^{-1} D^2h\, Dg\, dY"] \arrow[dd, "\alpha_0 mA\, D^2h\, d_g"] & \cdot \arrow[r, "\alpha_0 D\alpha_0 D^{g,Dh} dY"] \arrow[dd, "\alpha_0 D\alpha_0 D^2h\, d_g"] & \cdot \arrow[r, "\alpha_0 D^{Dh\,g,\alpha_0} dY"] & \cdot \arrow[dd, "\alpha_0 d_{\alpha_0\, Dh\, g}"'] \\
\cdot \arrow[urr, "\alpha_0 m_h Dg\, dY"] \arrow[dr, "\alpha_0 Dh\, mX\, d_g"] & & & & & \\
& \cdot \arrow[r, "\alpha_0 m_h dDX\, g"] \arrow[dddrrrr, "\alpha_0 Dh\, \beta X\, g"'] & \cdot \arrow[r, "\alpha_2^{-1} D^2h\, dDX\, g"] \arrow[ddr, "\alpha_0 mA\, d_{Dh g}"] & \cdot \arrow[r, "\alpha_0 D\alpha_0 d_{Dh g}"] & \cdot \arrow[r, "\alpha_0 d_{\alpha_0} Dh\, g"'] & \cdot \arrow[ddd, "\alpha_1 \alpha_0 Dh\, g"'] \\
\cdot \arrow[uu, "\alpha_0\, Dh\, \mathbb{D}^{-1}_{dDX\, g, \mathrm{Id}_{DA}}"] \arrow[ur, "\alpha_0 Dh\, \mathbb{D}_{mX\, dDX\, g}"'] & & & & & \\
& & & \cdot \arrow[uur, "\alpha_2^{-1} dDA\, Dh\, g"'] \arrow[drr, "\alpha_0 \beta A\, Dh\, g"] & & \\
\cdot \arrow[uu, "\alpha_0 Dh(\beta_X^{-1} g)^{\mathbb{D}} dY"] \arrow[rrrrr, "\alpha_0 Dh \mathbb{D}_g"] & & & & & \cdot
\end{tikzcd}
$$

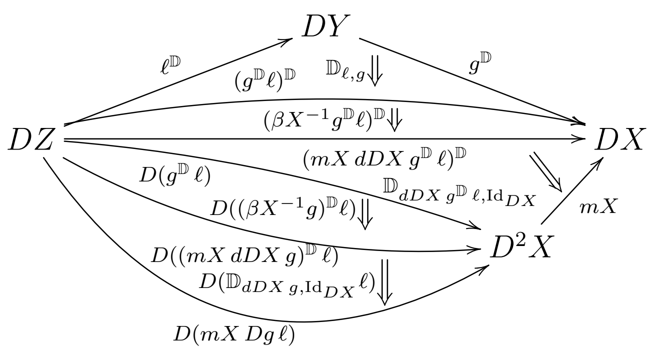
Conditions (12) and (13) are straightforward. As for (14), observe that the right hand side of (14) is

$$
\begin{tikzcd}
DZ \arrow[rr, bend left=40, "\ell^{\mathbb{D}}"] \arrow[rr, "(mY\, dDY\, \ell)^{\mathbb{D}}" description] \arrow[rr, phantom, shift left=4, "\Downarrow (\beta Y^{-1}\ell)^{\mathbb{D}}"] \arrow[rr, phantom, shift right=4, "\Downarrow \mathbb{D}^{-1}_{dDY\, \ell, Id_{DY}}"] \arrow[dr, "D\ell"'] \arrow[ddddrrr, bend right=30, "D(\alpha_0 D(\alpha_0 Dh\, g)\ell)"'] & & DY \arrow[rr, bend left=40, "g^{\mathbb{D}}"] \arrow[rr, "(mX\, dDX\, g)^{\mathbb{D}}" description] \arrow[rr, phantom, shift left=4, "\Downarrow (\beta X^{-1}g)^{\mathbb{D}}"] \arrow[rr, phantom, shift right=4, "\Downarrow \mathbb{D}^{-1}_{dDX\, g, Id_{DX}}"] \arrow[dr, "Dg"'] \arrow[dddrr, bend right=30, "D(\alpha_0\, Dh\, g)"] \arrow[ddr, phantom, "\Leftarrow m_{\alpha_0\, Dh\, g}"] & & DX \arrow[ddr, "Dh"] & \\
& D^2Y \arrow[ur, "mY"'] \arrow[ddr, "D^2(\alpha_0\, Dh\, g)"] \arrow[ddd, phantom, "\Leftarrow"] & & D^2X \arrow[ur, "mX"'] \arrow[dr, "D^2h"] \arrow[drr, phantom, "\Leftarrow m_h"] & & \\
& & & & D^2A \arrow[r, "mA"'] \arrow[d, "D\alpha_0"] \arrow[dl, phantom, "\Leftarrow"] & DA \arrow[dd, "\alpha_0"] \arrow[dl, phantom, "\Leftarrow \alpha_2^{-1}"] \\
& & D^2A \arrow[rr, "mA"'] \arrow[dr, "D\alpha_0"] & & DA \arrow[dr, "\alpha_0"] \arrow[dl, phantom, "\Leftarrow \alpha_2^{-1}"] & \\
& & & DA \arrow[rr, "\alpha_0"'] & & A
\end{tikzcd}
$$

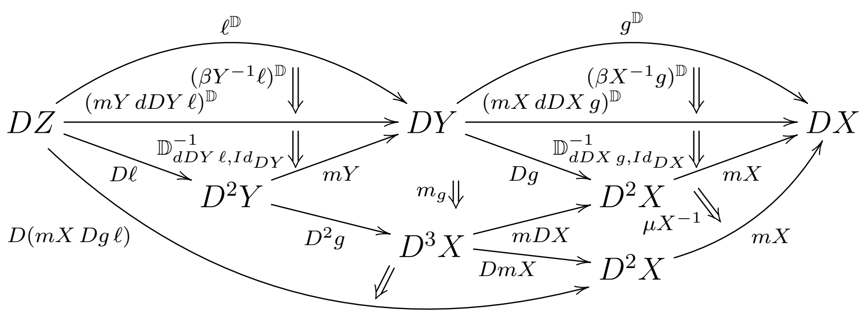
where the two unlabeled 2-cells are simply the coherence isomorphisms associated to the pseudofunctor D. In the middle replace the pasting of $m_{\alpha_0 Dh\, g}$ and the coherence isomorphism to its right by the pasting of m_g, m_{Dh}, m_{α_0} and the corresponding coherence isomorphism for D^2. Then replace the pasting of α_2^{-1}, m_{α_0} and α_2^{-1} by the pasting of μ_A^{-1}, α_2^{-1} and $D\alpha_2^{-1}$ (in this last one we omit the corresponding coherence isomorphisms associated to D). Then replace the pasting of m_h, m_{Dh} and μ_A^{-1} by the pasting of μ_X^{-1}, m_h and Dm_h. Thus we have arrived at



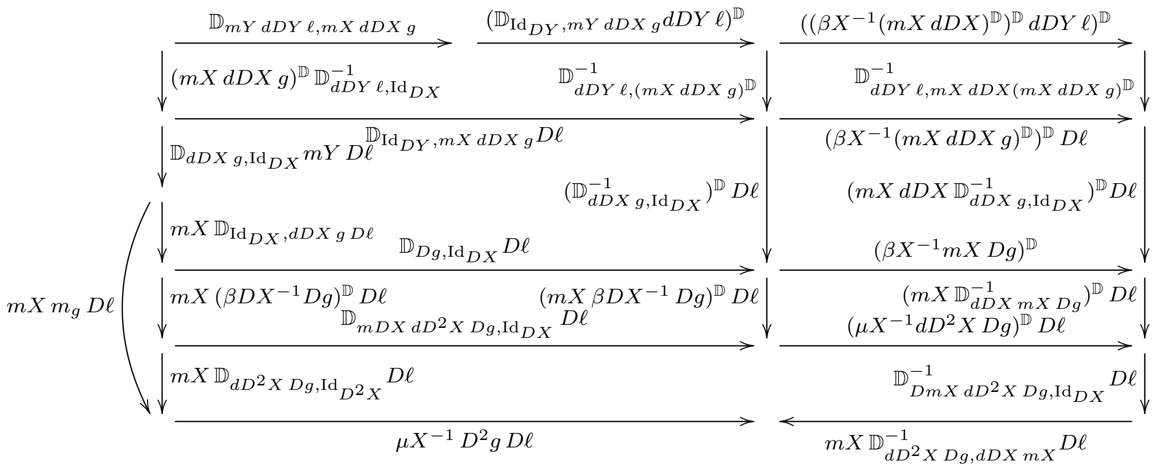
Comparing this pasting with the left hand side of (14), we see that we have to show that



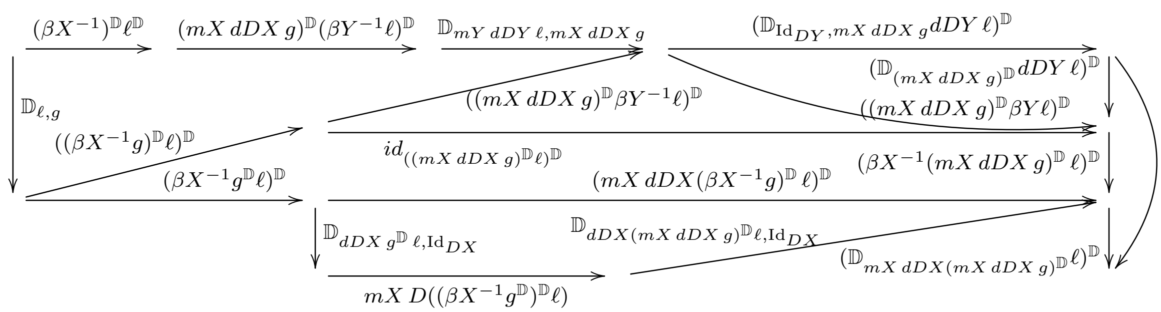
equals



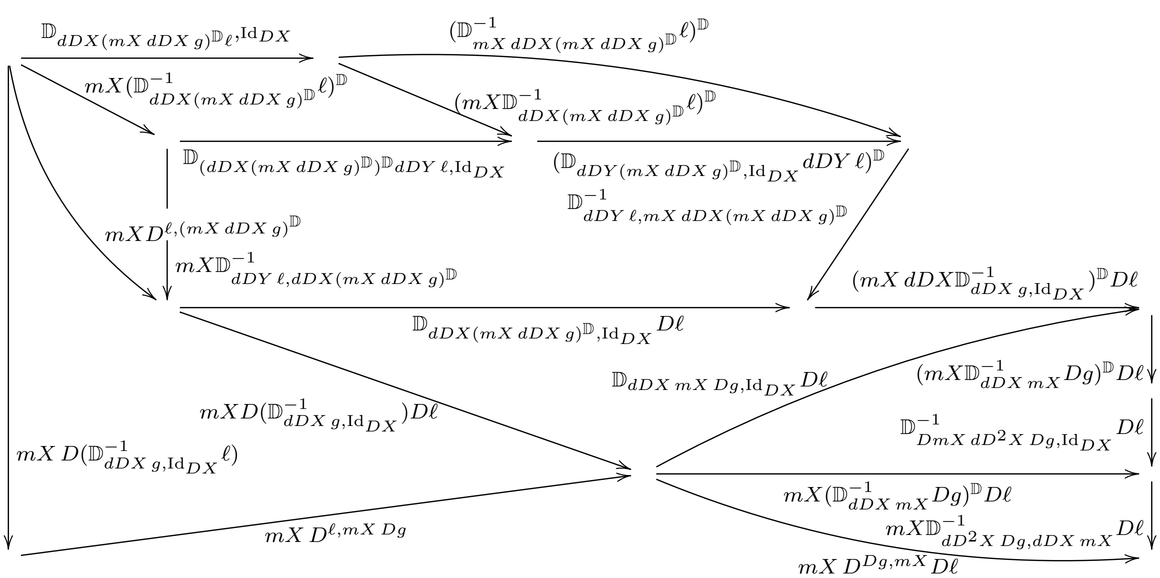
On this last pasting, substitute the pasting of $\mathbb{D}^{-1}_{dDY\,\ell,\mathrm{Id}_{DY}}$, $\mathbb{D}^{-1}_{dDX\,g,\mathrm{Id}_{DX}}$, m_g and μ_X^{-1} by the other side of the commutative diagram



Now replace the pasting of $(\beta X^{-1}g)^{\mathbb{D}}$, $(\beta Y^{-1}\ell)$ and the top row of the previous diagram by the other side of the commutative diagram



where the rightmost arrow is $((\beta X^{-1}(mX\,dDX\,g)^{\mathbb{D}})^{\mathbb{D}}dDY\,\ell)^{\mathbb{D}}$. And finish the argument with the substitution given by the commutative diagram



This completes the proof that $G(\alpha_0, \alpha_1)$ is an object in $\widehat{\text{Alg}}$-$\mathbb{D}$. Conditions (15), (16), (17) and (18) are not hard to verify, as well as the fact that $G : \mathbb{D}\text{-Alg} \to \widehat{\text{Alg}}\text{-}\mathbb{D}$ is a pseudofunctor.

Given $(A, (\ )^{\mathbb{A}}) \in \widehat{\text{Alg}}$-$\mathbb{D}$, define $t(A, (\ )^{\mathbb{A}}) : (A, (\ )^{\mathbb{A}}) \to GF(A, (\ )^{\mathbb{A}})$ as Id_A and the assignment to every $h : X \to A$ of the invertible 2-cell

$$\begin{tikzcd}
DX \arrow[r, "Dh"] \arrow[d, "h^{\mathbb{A}}"'] \arrow[dr, "(\alpha_0 dA\, h)^{\mathbb{A}}" description] \arrow[dr, phantom, shift left=4ex, "\mathbb{A}_{dA\, h, \text{Id}_A} \Swarrow"] \arrow[dr, phantom, shift right=4ex, "(\alpha_1 h)^{\mathbb{A}} \Swarrow"] & DA \arrow[d, "\alpha_0"] \\
A \arrow[r, "\text{Id}_A"'] & A
\end{tikzcd}$$

with the notation as in (19). We have that $t(A, (\ )^{\mathbb{A}})$ is a morphism in $\widehat{\text{Alg}}$-$\mathbb{D}$ (condition (16) is a bit long), and that it defines an invertible strong transformation $t : 1 \to GF$ since for any $f : (A, (\ )^{\mathbb{A}}) \to (B, (\ )^{\mathbb{B}})$ the diagram

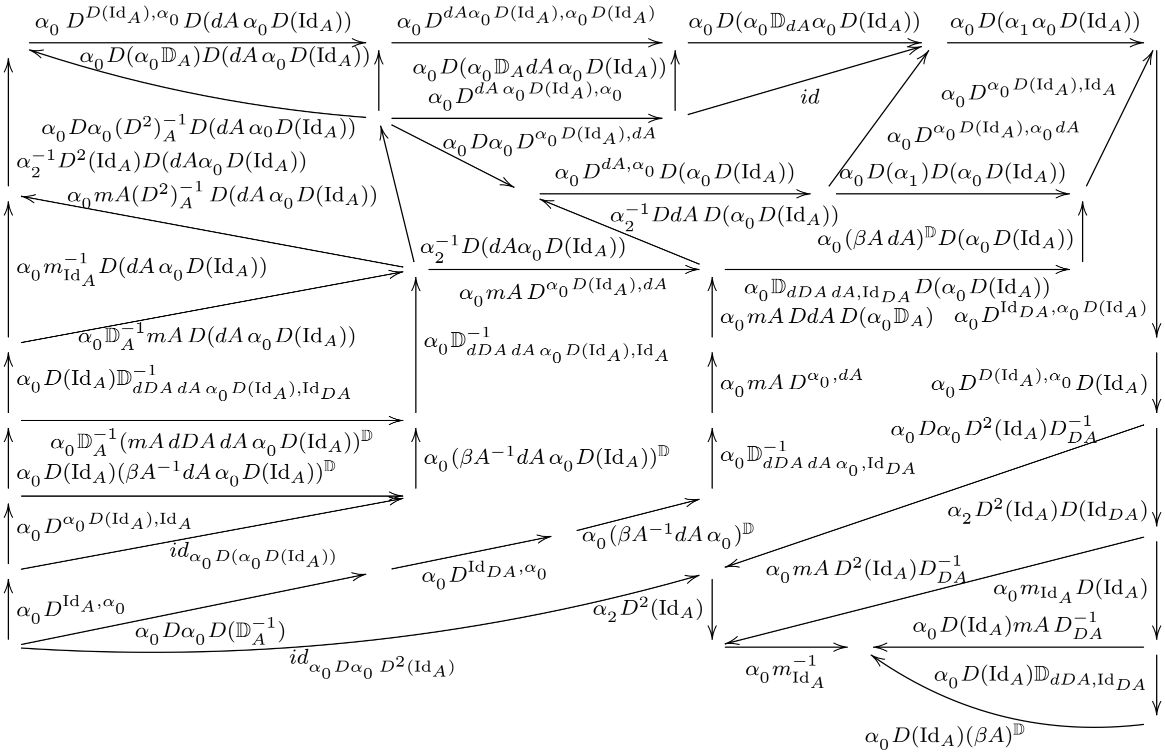
$$\begin{tikzcd}
(A, (\ )^{\mathbb{A}}) \arrow[r, "t(A, (\ )^{\mathbb{A}})"] \arrow[d, "f"'] & GF(A, (\ )^{\mathbb{A}}) \arrow[d, "GFf"] \\
(B, (\ )^{\mathbb{B}}) \arrow[r, "(B, (\ )^{\mathbb{B}})"'] & GF(B, (\ )^{\mathbb{B}})
\end{tikzcd}$$

commutes.

On the other hand, given (α_1, α_2) in $\mathbb{D}$-Alg define $s(\alpha_1, \alpha_2) : FG(\alpha_1, \alpha_2) \to (\alpha_1, \alpha_2)$ as the identity 2-cell

$$\begin{tikzcd}
DA \arrow[r, "D(\text{Id}_A)"] \arrow[d, "D(\text{Id}_A)"'] & DA \arrow[dd, "\alpha_0"] \\
DA \arrow[d, "\alpha_0"'] \arrow[r, phantom, "="] & \\
A \arrow[r, "\text{Id}_A"'] & A
\end{tikzcd}$$

One of the conditions for $s(\alpha_1, \alpha_2)$ to be a 1-cell in $\mathbb{D}$-Alg is trivial and the other one is given by the following commutative diagram:



Then the diagram

$$
\begin{tikzcd}
FG(\alpha_1,\alpha_2) \arrow[r, "s(\alpha_1,\alpha_2)"] \arrow[d, "FG(f,\rho)"'] & (\alpha_1,\alpha_2) \arrow[d, "(f,\rho)"] \\
FG(\beta_1,\beta_2) \arrow[r, "s(\beta_1,\beta_2)"'] & (\beta_1,\beta_2)
\end{tikzcd}
$$

commutes, and we obtain $s : FG \to 1$ invertible. ∎

6. Every pseudomonad induces a no-iteration pseudomonad

6.1. THEOREM. *Let $\mathbb{U} = (U, u, n, \beta, \eta, \mu)$ be a pseudomonad on $\mathcal{A}$. Define $D : Ob(\mathcal{A}) \to Ob(\mathcal{A})$ as $DA = UA$. For every A define $dA = uA$. For every A, B in $\mathcal{A}$ define $(\ )^{\mathbb{D}}$ as the composite*

$$
\mathcal{A}(A, DB) \xrightarrow{\ U\ } \mathcal{A}(DA, D^2B) \xrightarrow{\mathcal{A}(DA,nB)} \mathcal{A}(DA, DB).
$$

Define $\mathbb{D}_A = \eta A\colon Id_{DA} \to nA\ UdA$. *For* $f : A \to DB$ *in* $\mathcal{A}$, *define* $\mathbb{D}_f$ *as the pasting*

$$\begin{tikzcd}
A \arrow[r, "uA"] \arrow[d, "f"'] & DA \arrow[d, "Uf"] \arrow[dl, phantom, "\swarrow u_f"] \\
DB \arrow[r, "uDB"'] \arrow[dr, "id_{DB}"'] & D^2B \arrow[d, "nA"] \arrow[dl, phantom, "\swarrow \beta B"] \\
& DB.
\end{tikzcd}$$

And for $h : B \to DC$, *define* $\mathbb{D}_{f,h}$ *as the pasting*

$$\begin{tikzcd}
DA \arrow[r, "Uf"] & D^2B \arrow[r, "nB"] \arrow[d, "U^2h"'] & DB \arrow[d, "Uh"] \arrow[dl, phantom, "\swarrow n_h"] \\
& D^3C \arrow[r, "nDC"'] \arrow[d, "UnC"'] & D^2C \arrow[d, "nC"] \arrow[dl, phantom, "\swarrow \mu C^{-1}"] \\
& D^2C \arrow[r, "nC"'] & DC.
\end{tikzcd}$$

The data defined above is a no-iteration pseudomonad.

Proof. The proof is direct. ∎

Starting with a pseudomonad $\mathbb{U}$ on $\mathcal{A}$, produce the no-iteration pseudomonad as in the theorem above. Now produce the pseudomonad $\mathbb{D}$ induced by this presentation. We describe in detail the pseudomonad $\mathbb{D} = (D, d, m, \beta, \eta, \mu)$.

The action of D on the objects is U. For $\varphi\colon f \to g\colon A \to B$ in $\mathcal{A}$, the action of D on it is given by

$$\begin{tikzcd}
DA \arrow[r, bend left, "Uf"] \arrow[r, bend right, "Ug"'] \arrow[r, phantom, "U\varphi\Downarrow"] & DB \arrow[r, "UuB"] & D^2B \arrow[r, "nB"] & DB
\end{tikzcd}$$

$D_A = \eta A$. If $h\colon B \to C$ in $\mathcal{A}$, a brief calculation tells us that $D^{f,g}$ is

$$\begin{tikzcd}
& & D^2B \arrow[dr, "nB"] \arrow[d, phantom, "\eta B^{-1}\Downarrow"] & & & & \\
DA \arrow[r, "Uf"] & DB \arrow[ur, "UuB"] \arrow[rr, "Id_{DB}"'] & {} & DB \arrow[r, "Uh"] & DC \arrow[r, "UuC"] & D^2C \arrow[r, "nC"] & DC
\end{tikzcd}$$

Now $d\colon 1_{\mathcal{A}} \to D$ is given by $dA = uA$ for every A, and for every $f\colon A \to B$ we have that

d_f is

$$\begin{tikzcd}
A \arrow[r, "dA"] \arrow[d, "f"'] & DA \arrow[d, "Uf"] \arrow[dl, phantom, "\Leftarrow u_f"] \\
B \arrow[r, "dB"'] \arrow[d, "dB"'] & DB \arrow[d, "UdB"] \arrow[dl, phantom, "\Leftarrow u_{uB}"] \\
DB \arrow[r, "dDB"'] \arrow[dr, "Id_{DB}"'] & D^2B \arrow[d, "nB"] \arrow[dl, phantom, "\Leftarrow \beta B"] \\
& DB
\end{tikzcd}$$

For every A in $\mathcal{A}$, $mA = nA$. And for $f : A \to B$ in $\mathcal{A}$, m_f is (after one substitution)

$$\begin{tikzcd}
D^2A \arrow[rrr, "nA"] \arrow[d, "U^2f"'] & & & DA \arrow[d, "Uf"] \arrow[dl, phantom, "\Leftarrow n_f"] \\
D^2B \arrow[rrr, "nB"'] \arrow[d, "U^2uB"'] & & & DB \arrow[d, "UuB"] \arrow[dl, phantom, "\Leftarrow n_{uB}"] \\
D^3B \arrow[rrr, "nDB"'] \arrow[d, "UnB"'] & & & D^2B \arrow[d, "nB"] \arrow[dl, phantom, "\Leftarrow \mu B^{-1}"] \\
D^2B \arrow[rr, "Id_{D^2B}"] \arrow[dr, "UuDB"'] & \arrow[d, phantom, "\eta DB \Downarrow"] & D^2B \arrow[r, "nB"'] & DB \\
& D^3B \arrow[ur, "nDB"'] & &
\end{tikzcd}$$

We also have that $\beta_{\mathbb{D}}A = \beta_{\mathbb{U}}A$ whereas $\eta_{\mathbb{D}}A$ is the pasting

$$\begin{tikzcd}
DA \arrow[rrrr, "Id_{DA}"] \arrow[dr, "UuA"'] & & & \arrow[d, phantom, "\eta A \Downarrow"] & DA \\
& D^2A \arrow[rr, "Id_{D^2A}"] \arrow[dr, "UuDA"'] & \arrow[d, phantom, "\eta DA \Downarrow"] & D^2A \arrow[ur, "nA"'] & \\
& & D^3A \arrow[ur, "nDA"'] & &
\end{tikzcd}$$

Finally $\mu_{\mathbb{D}}A$ is (after one substitution) the pasting

$$\begin{tikzcd}
& & D^3A \arrow[dr, "nDA"] \arrow[d, phantom, "\eta DA^{-1} \Downarrow"] & \\
D^3A \arrow[r, "UmA"] \arrow[d, "mDA"'] & D^2A \arrow[ur, "UdDA"] \arrow[rr, "Id_{D^2A}"'] & & D^2A \arrow[d, "nA"] \arrow[dl, phantom, "\Leftarrow \mu A"] \\
D^2A \arrow[rrr, "nA"'] & & & DA
\end{tikzcd}$$

6.2. Theorem. *The bicategories* $\mathbb{U}$-Alg *and* $\mathbb{D}$-Alg, *for the pseudomonads* $\mathbb{U}$ *and* $\mathbb{D}$ *considered above, are isomorphic.*

PROOF. Define $r : U \to D$ such that for any $A \in \mathcal{A}$, $rA = \mathrm{Id}_{UA} : UA \to DA$ and for $f : A \to B$, $r_f = \eta B^{-1}\, Uf$. It is not hard to see that r is an invertible strong transformation, and that the diagrams

$$\begin{tikzcd} 1 \arrow[r, "u"] \arrow[dr, "d"'] & U \arrow[d, "r"] \\ & D \end{tikzcd} \qquad \begin{tikzcd} U^2 \arrow[r, "Ur"] \arrow[d, "n"'] & UD \arrow[r, "rD"] & D^2 \arrow[d, "m"] \\ U \arrow[rr, "r"'] & & D \end{tikzcd}$$

commute. It is not hard to see that r, together with the two commutative diagrams given above is a transition from $\mathbb{D}$ to $\mathbb{U}$ along $1 : \mathcal{A} \to \mathcal{A}$ in the notation of [Marmolejo & Wood, 2008], and thus induces a lifting $\mathbb{D}$-Alg $\to$ $\mathbb{U}$-Alg of $1_{\mathcal{A}}$. The inverse is the lifting given by the transition $s : D \to U$ from $\mathbb{U}$ to $\mathbb{D}$ along $1_{\mathcal{A}}$ given as follows. $sA = \mathrm{Id}_{UA}$ for every A in $\mathcal{A}$, for $f : A \to B$, $s_f = \eta B\; Uf$. Now the diagram

$$\begin{tikzcd} 1 \arrow[r, "d"] \arrow[dr, "u"'] & D \arrow[d, "s"] \\ & U \end{tikzcd}$$

commutes, and we define

$$\begin{tikzcd} D^2 \arrow[r, "Ds"] \arrow[d, "m"'] & DU \arrow[r, "sU"] & U^2 \arrow[d, "n"] \arrow[dl, phantom, "\swarrow\omega_2"] \\ D \arrow[rr, "s"'] & & U \end{tikzcd}$$

such that $\omega_2 A = nA\, \eta UA^{-1}$. ∎

7. Factorization systems

We take from [Korostenski & Tholen, 93] and [Rosebrugh & Wood, 02] the example of the monad $(-)^{\mathbf{2}} : \mathbf{Cat} \to \mathbf{Cat}$, where $\mathbf{2} = 0 \xrightarrow{\alpha} 1$. It is given on every category $\mathbf{A}$ by $D\mathbf{A} = \mathbf{A}^{\mathbf{2}}$, $\eta\mathbf{A} : \mathbf{A} \to \mathbf{A}^{\mathbf{2}}$ is given on $a : A \to A'$ as $(a, a) : 1_A \to 1_{A'}$. If we denote by $d_0, d_1 : \mathbf{A}^{\mathbf{2}} \to \mathbf{A}$ the domain and codomain functors, then for $F : \mathbf{X} \to \mathbf{A}^{\mathbf{2}}$ we define $F^{\mathbb{D}} : \mathbf{X}^{\mathbf{2}} \to \mathbf{A}^{\mathbf{2}}$ such that, for $f : Y \to X$ in $\mathbf{X}$, $F^{\mathbb{D}}(f)$ is

$$d_0FY \xrightarrow{FY} d_1FY \xrightarrow{d_1Ff} d_1FX,$$

and for $(h_0, h_1) : f \to f'$, with $f' : Y' \to X'$,

$$F^{\mathbb{D}}(h_0, h_1) = (d_0Fh_0, d_1Fh_1).$$

Given $\varphi : F \to G : \mathbf{X} \to \mathbf{A}^{\mathbf{2}}$, define, for $f : Y \to X$, $\varphi^{\mathbb{D}} f = (d_0\varphi Y, d_1\varphi X)$.

Then $\eta\mathbf{A}^{\mathbb{D}} = 1_{\mathbf{A}^{\mathbf{2}}}$, $F^{\mathbb{D}} \circ \eta\mathbf{A} = F$ and for $G : \mathbf{Z} \to \mathbf{X}$ we have that $F^{\mathbb{D}} \circ G^{\mathbb{D}} = (F^{\mathbb{D}} \circ G)^{\mathbb{D}}$ and thus we take $\mathbb{D}_{\mathbf{A}} = id$, $\mathbb{D}_F = id$, and $\mathbb{D}_{G,F} = id$. Then the coherence conditions are trivial.

It is well known that every category with a factorization system is an algebra for this monad. We prove this explicitly with the new formulation for the algebras. So take $\mathbf{A}$ with a factorization system $(\mathcal{E}, \mathcal{M})$. Given $F : \mathbf{X} \to \mathbf{A}$ we define $F^{\mathbb{A}} : \mathbf{X}^{\mathbf{2}} \to \mathbf{A}$ as follows. Given $f : Y \to X$ in $\mathbf{X}^{\mathbf{2}}$, we define $F^{\mathbb{A}}(f)$ as in the diagram

$$\begin{tikzcd} FY \arrow[rr, "Ff"] \arrow[dr, "e_{Ff}"'] & & FX \\ & F^{\mathbb{A}}(f) \arrow[ur, "m_{Ff}"'] & \end{tikzcd}$$

where $m_{Ff} \circ e_{Ff}$ is the $\mathcal{E}$-$\mathcal{M}$ factorization of Ff. Given $f' : Y' \to X'$ in $\mathbf{X}^{\mathbf{2}}$ and a morphism $(h, k) : f \to f'$ in $\mathbf{X}^{\mathbf{2}}$, $F^{\mathbb{A}}(h, k)$ is the unique arrow in $\mathbf{A}$ such that the diagram

$$\begin{tikzcd} FY \arrow[rr, "Ff"] \arrow[dr, "e_{Ff}"'] \arrow[ddd, "Fh"'] & & FX \arrow[ddd, "Fk"] \\ & F^{\mathbb{A}}(f) \arrow[ur, "m_{Ff}"'] \arrow[d, "F^{\mathbb{A}}(h,k)"'] & \\ & F^{\mathbb{A}}(f') \arrow[dr, "m_{Ff'}"] & \\ FY' \arrow[ur, "e_{Ff'}"] \arrow[rr, "Ff'"'] & & FX' \end{tikzcd}$$

commutes (given by the diagonal fill in property). It is clear that $F^{\mathbb{A}}$ is a functor. Furthermore, given $\varphi : F \to F' : \mathbf{X} \to \mathbf{A}$, we define $\varphi^{\mathbb{A}} : F^{\mathbb{A}} \to F'^{\mathbb{A}} : \mathbf{X}^{\mathbf{2}} \to \mathbf{A}$ such that $\varphi^{\mathbb{A}} f : F^{\mathbb{A}}(f) \to F'^{\mathbb{A}}(f)$ is the unique arrow that makes the diagram

$$\begin{tikzcd} FY \arrow[rr, "Ff"] \arrow[dr, "e_{Ff}"'] \arrow[ddd, "\varphi Y"'] & & FX \arrow[ddd, "\varphi X"] \\ & F^{\mathbb{A}}(f) \arrow[ur, "m_{Ff}"'] \arrow[d, "\varphi^{\mathbb{A}} f"'] & \\ & F'^{\mathbb{A}}(f) \arrow[dr, "m_{F'f}"] & \\ F'Y \arrow[ur, "e_{F'f}"] \arrow[rr, "F'f"'] & & F'X. \end{tikzcd}$$

commutes. It is direct to see that $\varphi^{\mathbb{A}}$ is natural, and that in fact we have defined a functor

$$(\ )^{\mathbb{A}} : \mathbf{Cat}(\mathbf{X}, \mathbf{A}) \to \mathbf{Cat}(\mathbf{X}^{\mathbf{2}}, \mathbf{A}).$$

We also have that the diagram

$$\begin{tikzcd} \mathbf{X} \arrow[r, "d\mathbf{X}"] \arrow[dr, "F"'] & \mathbf{X}^{\mathbf{2}} \arrow[d, "F^{\mathbb{A}}"] \\ & \mathbf{A} \end{tikzcd}$$

commutes, provided that the $\mathcal{E}$-$\mathcal{M}$ factorization of 1_A is $1_A \circ 1_A$ for every $A \in \mathbf{A}$ (which we assume). We thus take $\mathbb{A}_F = id$.

Given $G\colon \mathbf{Z} \to \mathbf{X^2}$, $F\colon \mathbf{X} \to \mathbf{A}$ and $h\colon W \to Z$ in $\mathbf{Z^2}$, we have that $F^{\mathbb{A}}G^{\mathbb{D}}(f)$ is given by the $\mathcal{E}$-$\mathcal{M}$ factorization of $Fd_1Gh \cdot FGW$:

$$\begin{tikzcd}
Fd_0GW \arrow[r, "FGW"] \arrow[dr, "e_{Fd_1Gh\cdot FGW}"'] & Fd_1GW \arrow[r, "Fd_1Gh"] & Fd_1GZ, \\
& F^{\mathbb{A}}G^{\mathbb{D}}(h) \arrow[ur, "m_{Fd_1Gh\cdot FGW}"'] &
\end{tikzcd}$$

whereas the calculation of $(F^{\mathbb{A}}G)^{\mathbb{A}}(h)$ can be read from the following diagram

$$\begin{tikzcd}
d_0GW \arrow[rrr, "GW"] \arrow[dr, "e_{GW}"'] \arrow[dddd, "d_0Gh"'] & & & d_1GW \arrow[dddd, "d_1Gh"] \\
& F^{\mathbb{A}}GW \arrow[urr, "m_{GW}"'] \arrow[dr, "e_{F^{\mathbb{A}}Gh}"] \arrow[dd, "F^{\mathbb{A}}Gh"'] & & \\
& & (F^{\mathbb{A}}G)^{\mathbb{A}}(h) \arrow[dl, "m_{F^{\mathbb{A}}Gh}"] & \\
& F^{\mathbb{A}}GZ \arrow[drr, "m_{GZ}"] & & \\
d_0GZ \arrow[ur, "e_{GZ}"] \arrow[rrr, "GZ"'] & & & d_1GZ
\end{tikzcd}$$

We then define $\mathbb{A}_{G,F}h$ as the unique isomorphism such that the diagram

$$\begin{tikzcd}
& & F^{\mathbb{A}}G^{\mathbb{D}}(h) \arrow[drr, "m_{Fd_1Gh\cdot FGW}"] \arrow[d, "\mathbb{A}_{G,F}h"] & & \\
Fd_0GW \arrow[urr, "e_{Fd_1Gh\cdot FGW}"] \arrow[r, "e_{FGW}"'] & F^{\mathbb{A}}GW \arrow[r, "e_{F^{\mathbb{A}}Gh}"'] & (F^{\mathbb{A}}G)^{\mathbb{A}}(h) \arrow[r, "m_{F^{\mathbb{A}}Gh}"'] & F^{\mathbb{A}}GZ \arrow[r, "m_{FGZ}"'] & Fd_1GZ
\end{tikzcd} \tag{20}$$

commutes. It is not hard to see that $\mathbb{A}_{G,F}\colon F^{\mathbb{A}}G^{\mathbb{D}} \to (F^{\mathbb{A}}G)^{\mathbb{A}}$ is a natural transformation. As for the coherence conditions, (9) is immediate, for (10) we must show that $\mathbb{A}_{d\mathbf{X},F} = id$, but this follows from the fact that (20) is transformed into

$$\begin{tikzcd}
& & F^{\mathbb{A}}(h) \arrow[drr, "m_{Fh}"] \arrow[d, "\mathbb{A}_{d\mathbf{X},F}h"] & & \\
FW \arrow[urr, "e_{Fh}"] \arrow[r, "1_{FW}"'] & FW \arrow[r, "e_{Fh}"'] & F^{\mathbb{A}}(h) \arrow[r, "m_{Fh}"'] & FZ \arrow[r, "1_{FZ}"'] & FZ
\end{tikzcd}$$

for the particular case $G = d\mathbf{X}$. (11) is the fact that $\mathbb{A}_{G,F}(1_Z) = id$, which is not hard to see. (12) is given by the fact that both composites involved, namely $\mathbb{A}_{H,F}h \cdot F^{\mathbb{A}}\psi^{\mathbb{A}}h$ and $(F^{\mathbb{A}}\psi)^{\mathbb{A}}h \cdot \mathbb{A}_{G,F}h$ make commutative the diagram

$$\begin{tikzcd}
Fd_0GW \arrow[rr, "e_{Fd_1Gh\cdot FGW}"] \arrow[d, "Fd_0\psi W"'] & & F^{\mathbb{A}}G^{\mathbb{D}}h \arrow[rr, "m_{Fd_1Gh\cdot FGW}"] \arrow[d, "\mathbb{A}_{H,F}h\cdot F^{\mathbb{A}}\psi^{\mathbb{A}}h"', "(F^{\mathbb{A}}\psi)^{\mathbb{A}}h\cdot\mathbb{A}_{G,F}h"] & & Fd_1GZ \arrow[d, "Fd_1\psi Z"] \\
Fd_0HW \arrow[r, "e_{FHW}"'] & F^{\mathbb{A}}HW \arrow[r, "e_{F^{\mathbb{A}}Hh}"'] & (F^{\mathbb{A}}H)^{\mathbb{A}}h \arrow[r, "m_{F^{\mathbb{A}}Hh}"'] & F^{\mathbb{A}}HZ \arrow[r, "m_{FHZ}"'] & Fd_1HZ.
\end{tikzcd}$$

(13) is given by the fact that both composites involved, namely $\mathbb{A}_{G,F'}h \cdot \varphi^{\mathbb{A}}G^{\mathbb{D}}h$ and $(\varphi^{\mathbb{A}}G)^{\mathbb{A}}h \cdot \mathbb{A}_{G,F}h$, make commutative the diagram

$$\begin{tikzcd}
Fd_0GW \arrow[rr, "e_{Fd_1Gh\cdot FGW}"] \arrow[d, "\varphi d_0GW"'] & & F^{\mathbb{A}}G^{\mathbb{D}}h \arrow[rr, "m_{Fd_1Gh\cdot FGW}"] \arrow[d, "\mathbb{A}_{G,F'}h\cdot\varphi^{\mathbb{A}}G^{\mathbb{D}}h"', "(\varphi^{\mathbb{A}}G)^{\mathbb{A}}h\cdot\mathbb{A}_{G,F}h"] & & Fd_1GZ \arrow[d, "\varphi d_1GZ"] \\
F'd_0GW \arrow[r, "e_{F'GW}"'] & F'^{\mathbb{A}}GW \arrow[r, "e_{F'^{\mathbb{A}}Gh}"'] & (F'^{\mathbb{A}}G)^{\mathbb{A}}h \arrow[r, "m_{F'^{\mathbb{A}}Gh}"'] & F'^{\mathbb{A}}GZ \arrow[r, "m_{F'GZ}"'] & F'd_1GZ.
\end{tikzcd}$$

Finally, (14) is given by the fact that for a $k: S \to U$ in $\mathbf{U}$ and $K: \mathbf{U} \to \mathbf{Z}^{\mathbf{2}}$, both compositions involved, namely $\mathbb{A}_{K,F^{\mathbb{A}}G}k \cdot \mathbb{A}_{F,G}K^{\mathbb{D}}k$ and $(\mathbb{A}_{G,F}K)^{\mathbb{A}}k \cdot \mathbb{A}_{G^{\mathbb{D}}K,F}k$, make commutative the diagram

$$\begin{tikzcd}
Fd_0G^{\mathbb{D}}KS \arrow[rr, "e_{Fd_1G^{\mathbb{D}}Kk\cdot FG^{\mathbb{D}}KS}"] \arrow[d, "e_{FGd_0KS}"] & & F^{\mathbb{A}}(G^{\mathbb{D}}K)^{\mathbb{D}} \arrow[rr, "m_{Fd_1G^{\mathbb{D}}Kk\cdot FG^{\mathbb{D}}KS}"] \arrow[d, "\mathbb{A}_{K,F^{\mathbb{A}}G}k\cdot\mathbb{A}_{F,G}K^{\mathbb{D}}k"', "(\mathbb{A}_{G,F}K)^{\mathbb{A}}k\cdot\mathbb{A}_{G^{\mathbb{D}}K,F}k"] & & Fd_1G^{\mathbb{D}}KU \\
F^{\mathbb{A}}Gd_0KS \arrow[r, "e_{F^{\mathbb{A}}GKS}"'] & (F^{\mathbb{A}}G)^{\mathbb{A}}KS \arrow[r, "e_{(F^{\mathbb{A}}G)^{\mathbb{A}}Kk}"'] & ((F^{\mathbb{A}}G)^{\mathbb{A}}K)^{\mathbb{A}}k \arrow[r, "m_{(F^{\mathbb{A}}G)^{\mathbb{A}}Kk}"'] & (F^{\mathbb{A}}G)^{\mathbb{A}}KU \arrow[r, "m_{F^{\mathbb{A}}KU}"'] & F^{\mathbb{A}}Gd_1KU. \arrow[u, "m_{FGd_1KU}"']
\end{tikzcd}$$

In the opposite direction, assume we have an algebra $(\mathbf{A}, (\ )^{\mathbb{A}})$ for the monad $(\ )^{\mathbf{2}}$. According to [Korostenski & Tholen, 93] we have to define the following:

1. for every f in $\mathbf{A}$, a factorization of f: $\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "e_f"'] & & \bullet; \\ & \bullet \arrow[ur, "m_f"'] & \end{tikzcd}$

2. for every $(u, v): f \to g$ in $\mathbf{A}^{\mathbf{2}}$, an arrow $t(u, v)$ that makes the following diagram commute

$$\begin{tikzcd}
\bullet \arrow[r, "e_f"] \arrow[d, "u"'] & \bullet \arrow[r, "m_f"] \arrow[d, "t(u,v)"'] & \bullet \arrow[d, "v"] \\
\bullet \arrow[r, "e_g"'] & \bullet \arrow[r, "m_g"'] & \bullet;
\end{tikzcd}$$

such that, with $\mathcal{E} = \{f | m_f \text{ is iso}\}$ and $\mathcal{M} = \{f | e_f \text{ is iso}\}$, the following conditions are satisfied:

a. if f is an isomorphism, then e_f and m_f are isomorphisms;

b. the assignment of $t(u, v)$ is functorial on (u, v);

c. $e_f \in \mathcal{E}$ and $m_f \in \mathcal{M}$.

Take an arrow $f: A \to B$. This defines a functor $\ulcorner f \urcorner : \mathbf{2} \to \mathbf{A}$ in the obvious way. Thus we have the isomorphism

$$\begin{tikzcd}
\mathbf{2} \arrow[r, "d\mathbf{2}"] \arrow[dr, "\ulcorner f \urcorner"'] & \mathbf{2}^{\mathbf{2}} \arrow[d, "\ulcorner f \urcorner^{\mathbf{A}}"] \arrow[dl, phantom, "\Swarrow \mathbb{A}_{\ulcorner f \urcorner}"] \\
& \mathbf{A}.
\end{tikzcd}$$

Observe that $\mathbf{2^2}$ is isomorphic to $\mathbf{3} = \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dr, "\beta"] & \\ 0 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & 2 \end{tikzcd}$ with $\beta\alpha = \gamma$, and that $d\mathbf{2}$ takes as value γ. We then obtain the following commutative diagram

$$\begin{tikzcd}
\ulcorner f \urcorner^{\mathbb{A}} 0 \arrow[r, "\ulcorner f \urcorner^{\mathbb{A}}\alpha"] \arrow[d, "\mathbb{A}_{\ulcorner f \urcorner} 0"'] & \ulcorner f \urcorner^{\mathbb{A}} 1 \arrow[r, "\ulcorner f \urcorner^{\mathbb{A}}\beta"] \arrow[dr, dashed, "m_f"] & \ulcorner f \urcorner^{\mathbb{A}} 2 \arrow[d, "\mathbb{A}_{\ulcorner f \urcorner} 2"] \\
A \arrow[ur, dashed, "e_f"'] \arrow[rr, "f"'] & & B,
\end{tikzcd}$$

where we define $e_f = \ulcorner f \urcorner^{\mathbb{A}}\alpha \cdot \mathbb{A}_{\ulcorner f \urcorner} 0^{-1}$ and $m_f = \mathbb{A}_{\ulcorner f \urcorner} 2 \cdot \ulcorner f \urcorner^{\mathbb{A}}\beta$. This gives the desired factorization for f.

If $g : C \to D$ is another arrow and $(u, v) : f \to g$ is an arrow in $\mathbf{A^2}$, then (u, v) defines a natural transformation $\varphi : \ulcorner f \urcorner \to \ulcorner g \urcorner$, and then a natural transformation $\varphi^{\mathbb{A}} : \ulcorner f \urcorner^{\mathbb{A}} \to \ulcorner g \urcorner^{\mathbb{A}}$. Using condition (9) it is not hard to see that the diagram

$$\begin{tikzcd}
A \arrow[r, "e_f"] \arrow[d, "u"'] & \ulcorner f \urcorner^{\mathbb{A}} 1 \arrow[r, "m_f"] \arrow[d, "\varphi^{\mathbb{A}} 1"'] & B \arrow[d, "v"'] \\
C \arrow[r, "e_g"'] & \ulcorner g \urcorner^{\mathbb{A}} 1 \arrow[r, "m_g"'] & D
\end{tikzcd}$$

commutes. That this assignment results functorial on (u, v) follows at once from the fact that $(\ )^{\mathbb{A}} : \mathbf{Cat}(\mathbf{2}, \mathbf{A}) \to \mathbf{Cat}(\mathbf{2^2}, \mathbf{A})$ is a functor.

We show now that m_{e_f} is an isomorphism. For this we observe that since the assignment $f \mapsto (e_f, m_f)$ is functorial on arrows of $\mathbf{A^2}$ and e_f is isomorphic in $\mathbf{A^2}$ to $\ulcorner f \urcorner^{\mathbb{A}}\alpha$, then it suffices to show that $m_{\ulcorner f \urcorner^{\mathbb{A}}\alpha}$ is an isomorphism. Define $G : \mathbf{2} \to \mathbf{2^2}$ the functor whose value is the arrow α. Then the image of $G^{\mathbb{D}} : \mathbf{2^2} \to \mathbf{2^2}$ is

$$\begin{tikzcd}
& 1 \arrow[dr, "id"] & \\
0 \arrow[ur, "\alpha"] \arrow[rr, "\alpha"'] & & 1.
\end{tikzcd}$$

We have that $\ulcorner f \urcorner^{\mathbb{A}} G : \mathbf{2} \to \mathbf{A}$ takes the value $\ulcorner f \urcorner^{\mathbb{A}}\alpha$, and thus the natural isomorphism $\mathbb{A}_{G, \ulcorner f \urcorner} : \ulcorner f \urcorner^{\mathbb{A}} G^{\mathbb{D}} \to (\ulcorner f \urcorner^{\mathbb{A}} G)^{\mathbb{A}}$ produces the following commutative diagram

$$\begin{tikzcd}
\ulcorner f \urcorner^{\mathbb{A}} 0 \arrow[r, "\ulcorner f \urcorner^{\mathbb{A}}\alpha"] \arrow[d, "\mathbb{A}_{G, \ulcorner f \urcorner^{\mathbb{A}}} 0"'] & \ulcorner f \urcorner^{\mathbb{A}} 1 \arrow[r, "id"] \arrow[d, "\mathbb{A}_{G, \ulcorner f \urcorner^{\mathbb{A}}} 1"'] & \ulcorner f \urcorner^{\mathbb{A}} 1 \arrow[d, "\mathbb{A}_{G, \ulcorner f \urcorner^{\mathbb{A}}} 2"] \\
(\ulcorner f \urcorner^{\mathbb{A}} G)^{\mathbb{A}} 0 \arrow[r, "(\ulcorner f \urcorner^{\mathbb{A}} G)^{\mathbb{A}}\alpha"] \arrow[d, "\mathbb{A}_{\ulcorner f \urcorner^{\mathbb{A}} G} 0"'] & (\ulcorner f \urcorner^{\mathbb{A}} G)^{\mathbb{A}} 1 \arrow[r, "(\ulcorner f \urcorner^{\mathbb{A}} G)^{\mathbb{A}}\alpha"] \arrow[dr, "m_{\ulcorner f \urcorner^{\mathbb{A}}\alpha}"'] & (\ulcorner f \urcorner^{\mathbb{A}} G)^{\mathbb{A}} 2 \arrow[d, "\mathbb{A}_{\ulcorner f \urcorner^{\mathbb{A}} G} 1"] \\
\ulcorner f \urcorner^{\mathbb{A}} 0 \arrow[ur, "e_{\ulcorner f \urcorner^{\mathbb{A}}\alpha}"'] & & \ulcorner f \urcorner^{\mathbb{A}} 1
\end{tikzcd}$$

The upper right square tells us that $(\ulcorner f \urcorner^{\mathbb{A}} G)^{\mathbb{A}}\alpha$ is an isomorphism, thus $m_{(\ulcorner f \urcorner^{\mathbb{A}}\alpha}$ is also an isomorphism. Therefore m_{e_f} is an isomorphism.

The proof that e_{m_f} is similar, just take G with value β instead of α.

We are left with showing that for f an isomorphism, e_f and m_f are isomorphisms. For this we take the category $\mathbf{I}$ of the "generic" isomorphism, that is $\mathbf{I}$ is the category with two objects and two no identity arrows as in

$$0 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 1.$$

Then a functor $F\colon \mathbf{I} \to \mathbf{A}$ is the same thing as an isomorphism f in $\mathbf{A}$. Now apply the same ideas as above observing that every arrow in $\mathbf{I}^\mathbf{2}$ is an isomorphism, and taking $G\colon \mathbf{2} \to \mathbf{I}^\mathbf{2}$ with value the arrow that goes from $\ulcorner 0 \urcorner\colon \mathbf{2} \to \mathbf{I}^\mathbf{2}$ to $\ulcorner 1 \urcorner\colon \mathbf{2} \to \mathbf{I}^\mathbf{2}$ in $\mathbf{I}^\mathbf{2}$.

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