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TANGLED CIRCUITS

R. ROSEBRUGH AND N. SABADINI AND R. F. C. WALTERS

ABSTRACT. We consider *commutative Frobenius algebras in braided strict monoidal categories* in the study of the circuits and communicating systems which occur in Computer Science, including circuits in which the wires are tangled. We indicate also some possible novel geometric interest in such algebras. For example, we show how Armstrong's description ([1, 2]) of knot colourings and knot groups fit into this context.

1. Introduction

The theme of this paper is the use of *commutative Frobenius algebras in braided strict monoidal categories* in the study of varieties of circuits and communicating systems which occur in Computer Science, including circuits in which the wires are tangled. We indicate also some possible novel geometric interest in such algebras.

The main definition of the paper is of tangled circuit diagrams. To make this definition we need the notion of monoidal graph M (see Section 3). Then a tangled circuit diagram (over M) is an arrow in the free braided strict monoidal category on M in which the objects of M are equipped with commutative Frobenius algebra structures.

In order to study tangled circuit diagrams we define two different invariants for tangled circuits, one which takes values in a tangled category of relations, and the other in a tangled version of the category of spans. The second type of invariant includes as special cases the colourings of knots, and knot groups (following work of Armstrong [1, 2]).

The authors and collaborators have previously studied similar systems using symmetric monoidal categories ([8, 9, 10, 11, 17, 18, 19, 5]), with separable algebras instead of Frobenius algebras. These earlier works did not take into consideration any tangling of the wires. Further we will see in Section 6.5 the importance of considering Frobenius algebras rather than the more special separable algebras even in the symmetric monoidal case (no tangling).

There is a huge literature now relating monoidal categories and geometry beginning with [7, 12, 16]. We mention just two further items of an expository nature useful to reading this paper (apart from our own work mentioned above): the first [20] is a survey for computer scientists and others which discusses many additional structures but strangely not Frobenius algebras, and ignores our work on separable algebras; the second

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[13] is an introductory book on the relation between Frobenius algebras and 2-dimensional cobordism.

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2. Braided monoidal categories and Frobenius algebras

We review immediately the notions which are fundamental for the paper.

2.1. DEFINITION. *A* braided strict monoidal category *([7]) is a category* $\mathbf{C}$ *with a functor, called tensor,* $\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ *and a "unit" object* I *together with a natural family of isomorphisms* $\tau_{A,B} : A \otimes B \longrightarrow B \otimes A$ *called twist satisfying*

1) $\otimes$ *is associative and unitary on objects and arrows,*

2) the following diagrams commute for objects A, B, C*:*

$$B1: \quad \begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{\ \tau\ } & B \otimes C \otimes A \\ & {\scriptstyle \tau\otimes 1}\searrow \quad \nearrow{\scriptstyle 1\otimes\tau} & \\ & B \otimes A \otimes C & \end{array}$$

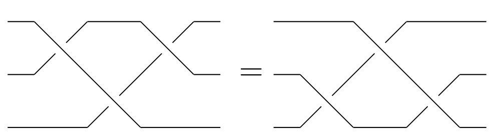
and

$$B2: \quad \begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{\ \tau\ } & C \otimes A \otimes B \\ & {\scriptstyle 1\otimes \tau}\searrow \quad \nearrow{\scriptstyle \tau\otimes 1} & \\ & A \otimes C \otimes B & \end{array}$$

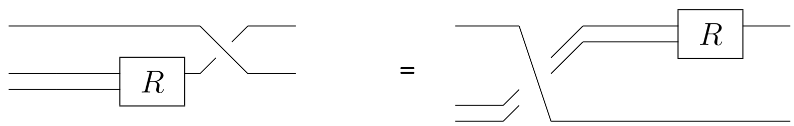
Among the consequences of the definition is the Yang-Baxter equation which reads:

$$(1 \otimes \tau)(\tau \otimes 1)(1 \otimes \tau) = (\tau \otimes 1)(1 \otimes \tau)(\tau \otimes 1) : A \otimes B \otimes C \longrightarrow C \otimes B \otimes A$$

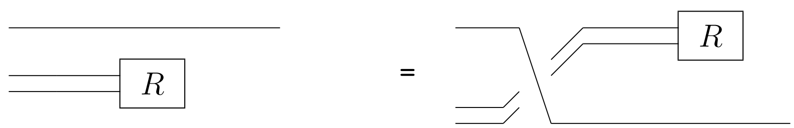
A compact and comprehensible formulation of such properties is provided by circuit or"wire" diagrams like the one below for the Yang-Baxter equation. Composition is read from left to right and $\otimes$ is vertical juxtaposition. The twist is expressed by the "positive crossing" (top wire over bottom) and its inverse by the negative crossing.



Another consequence of the axioms above is that $\tau_{A,I} = \tau_{I,A} = 1_A : A \longrightarrow A$.
The naturality of the twist τ leads to the following kind of equality of diagrams:



In the case when the codomain of the component is I, naturality is drawn, for example, as:



The following structures on an object in a braided monoidal category are fundamental to describing circuits.

2.2. Definition. *A* Frobenius algebra *in a braided monoidal category consists of an object G and four arrows $\nabla : G \otimes G \longrightarrow G$, $\Delta : G \longrightarrow G \otimes G$, $n : I \longrightarrow G$ and $e : G \longrightarrow I$ making (G, ∇, e) a monoid, (G, Δ, n) a comonoid and satisfying the equations*

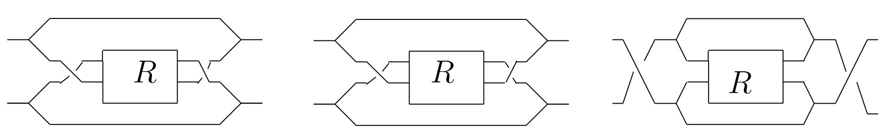
$$(1_G \otimes \nabla)(\Delta \otimes 1_G) = \Delta\nabla = (\nabla \otimes 1_G)(1_G \otimes \Delta) : G \otimes G \longrightarrow G \otimes G.$$

The Frobenius algebra is said to be commutative *if in addition the following equations involving the twist hold:*

$$\nabla\tau = \nabla : G \otimes G \longrightarrow G$$
$$\tau\Delta = \Delta : G \longrightarrow G \otimes G$$

3. Tangled circuit diagrams

We propose a definition for a category of *tangled circuit diagrams*, in which it is possible to distinguish, for example, the first and second of the following circuit diagrams, while the second and third are equal.



The main definitions are:

3.1. Definition. *A* monoidal graph M *consists of two sets* M_0 *(objects or wires) and* M_1 *(arrows or components) and two functions dom* : $M_1 \longrightarrow M_0^*$ *and cod* : $M_1 \longrightarrow M_0^*$ *where* M_0^* *is the free monoid on* M_0.

3.2. Definition. *Given a monoidal graph* M *the free braided strict monoidal category in which the objects of* M *are equipped with commutative Frobenius algebra structures is called* $\mathbf{TCirc}_M$. *Its arrows are called* tangled circuit diagrams, *or more briefly circuit diagrams. In the case that* M *has one object and no arrows we will denote* $\mathbf{TCirc}_M$ *simply by* $\mathbf{TCirc}$.

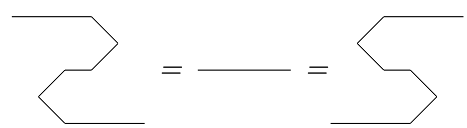
The basic structure we require on an object is the following.

3.3. Definition. *An object* X *in a braided strict monoidal category (with twist* τ*) is called a* tangle algebra *when it is equipped with arrows* $\eta : I \longrightarrow X \otimes X$ *and* $\epsilon : X \otimes X \longrightarrow I$ *that satisfy the equations (where we write* 1 *for all identities):*

(i) $(\epsilon \otimes 1)(1 \otimes \eta) = 1 = (1 \otimes \epsilon)(\eta \otimes 1)$

(ii) $\epsilon\tau = \epsilon$ *and* $\tau\eta = \eta$.

Axiom (i) says that X *is a* self-dual object. *The reader can translate these into wire diagrams. For example the wire diagram for (i) is:*



3.4. Theorem. *If* G *is a commutative Frobenius algebra in a braided strict monoidal category, then the object* G *together with the arrows* $\epsilon = e\nabla$, $\eta = \Delta n$ *form a tangle algebra.*

Proof. Let G be a commutative Frobenius algebra in a braided monoidal category. It is straightforward to give algebraic proofs for the tangle algebra axioms, but we remind the reader that these can be more easily found using wire diagrams.

To see that $(\epsilon \otimes 1)(1 \otimes \eta) = 1$ notice that

$$(e \otimes 1)(\nabla \otimes 1)(1 \otimes \Delta)(1 \otimes n) = (e \otimes 1)\Delta\nabla(1 \otimes n) = 1 \cdot 1 = 1.$$

■

3.5. Definition. *(Freyd-Yetter [6]) The category* **Tangle** *is the free braided strict monoidal category generated by one object X, equipped with a tangle algebra structure.*

The category **Tangle** has a geometric description [21] consonant with its name. In that description the arrows from I to I are unoriented knots and links.

Given any object A of M it is straightforward to see that there is an appropriate functor from Freyd and Yetter's category **Tangle** to $\mathbf{TCirc}_M$ since a symmetric Frobenius structure on A induces a tangle algebra structure on A. As a result any invariants of tangled circuit diagrams provide also invariants for tangles and knots. We conjecture that such functors $\mathbf{Tangle} \longrightarrow \mathbf{TCirc}_M$ are faithful. We also conjecture that there is a topological description of $\mathbf{TCirc}_M$ related to Freyd and Yetter's description of **Tangle** and to cobordisms.

3.6. Corollary. *Given an object A of a monoidal graph M there is a unique braided strict monoidal functor* $\mathbf{Tangle} \longrightarrow \mathbf{TCirc}_M$ *taking the generating object to A and the structure maps of* **Tangle** *to the corresponding structure maps of A in* $\mathbf{TCirc}_M$.

3.7. Remark. Notice that in $\mathbf{TCirc}_M$ the commutative Frobenius algebra structures on the objects of M induce Frobenius algebra structures on each object (that is, on multiple wires) of $\mathbf{TCirc}_M$. However these Frobenius algebra structures are *not commutative* and hence the corresponding η and ϵ do not satisfy the commutativity property (ii) of tangle algebras.

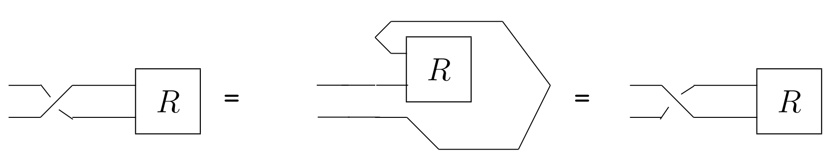
3.8. Example equations. We now give some examples of equations between circuit diagrams.

3.9. Proposition. *If $R : X \times Y \longrightarrow I$ is an arrow in the monoidal graph M then*

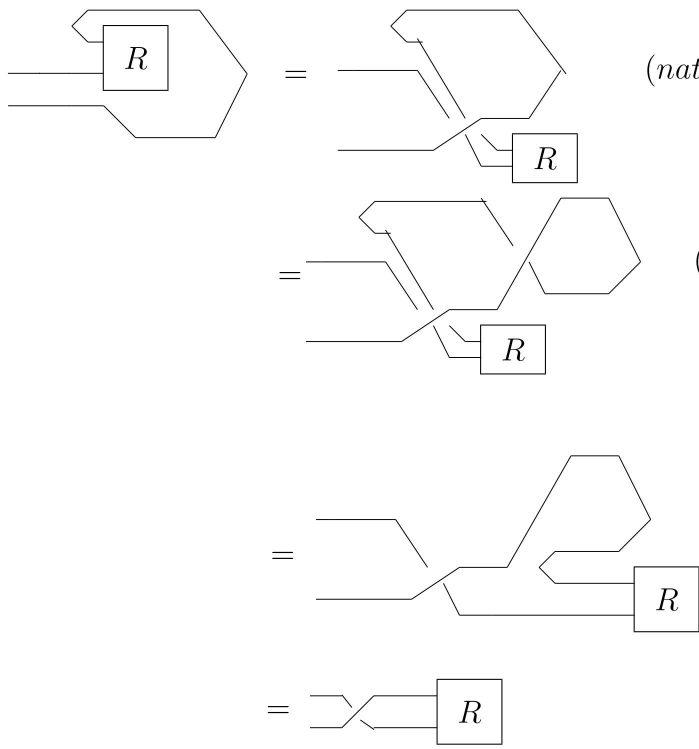
$$R\tau_{Y,X}^{-1} = \epsilon_X(1_X \otimes R \otimes 1_X)(\eta \otimes 1_Y \otimes 1_X) = R\tau_{X,Y}$$

.

Proof. First a picture of the equations:



It is clearly sufficient to prove the first equation. Consider:



(*naturality*)

(*commutativity*)

(*naturality*)

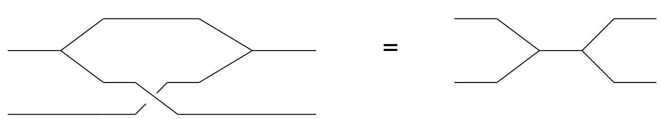
(*duality*)

■

3.10. Proposition. *If* $R : I \longrightarrow X \otimes X$ *and* $S : X \otimes X \longrightarrow I$ *then* $S\tau^{2n}R = SR$*; that is, R and S joined by an even number of twists is equal to R and S joined directly.*

Several more equations provable in **TCirc** follow:

3.11. Example.



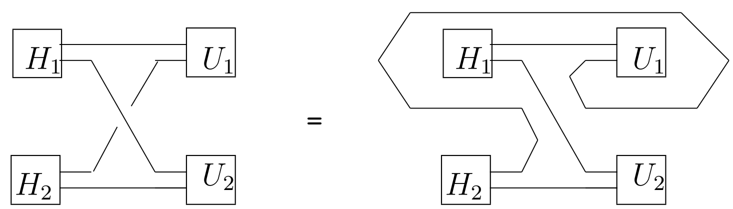
Proof.

$$(\nabla_X \otimes 1_X)(1_X \otimes \tau_X)(\Delta_X \otimes 1_X) = (\nabla_X \otimes 1_X)(\tau_X \otimes 1_X)(1_X \otimes \tau_X)(\Delta_X \otimes 1_X) \quad \text{(commutativity)}$$
$$= (\nabla_X \otimes 1_X)(1_X \otimes \Delta_X)\tau_X \quad \text{(naturality)}$$
$$= \Delta_X \nabla_X \tau_X \quad \text{(Frobenius)}$$
$$= \Delta_X \nabla_X \quad \text{(commutativity)}$$

■

3.12. Remark. The geometric intuition is that single wires are *thick* (but not ribbons) and so can be deformed contracting segments. Notice however that it is not true in general in **TCirc** that the separable axiom $\nabla\Delta = 1$ holds. That is, cycles of wires cannot be contracted to a point.

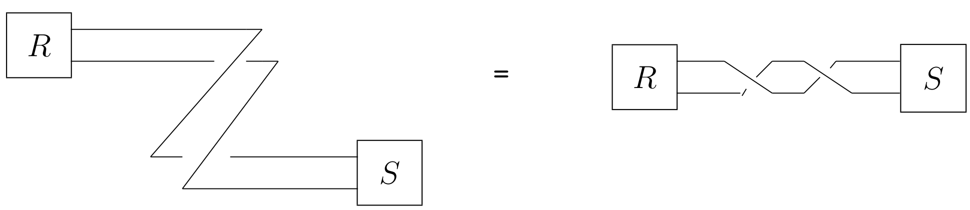
3.13. Example.



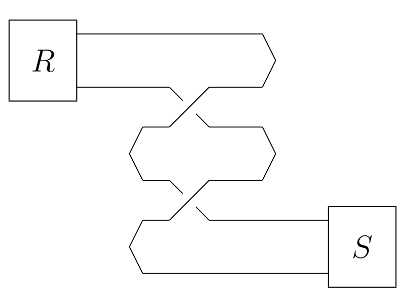
3.14. Example. If $R : I \longrightarrow X \otimes X$ and $S : X \otimes X \longrightarrow I$ then

$$S(\epsilon \otimes \epsilon \otimes 1 \otimes 1)(1 \otimes \tau^{-1} \otimes \tau^{-1} \otimes 1)(1 \otimes 1 \otimes \eta \otimes \eta)R = S\tau\tau R = SR.$$

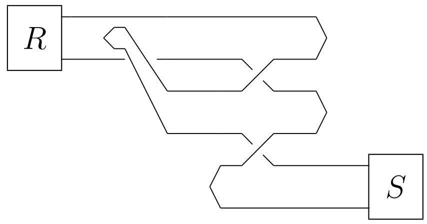
Diagrammatically:



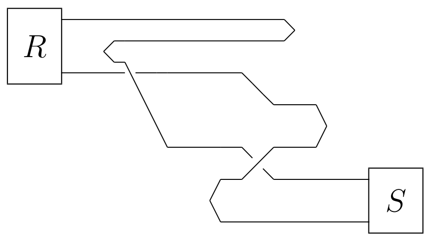
Proof. We will give a diagrammatic proof. A more explicit picture of the left hand expression is



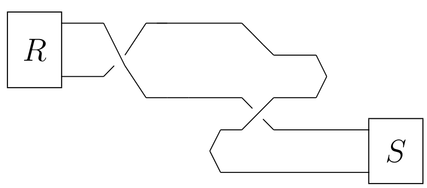
By naturality this is equal to



and hence to

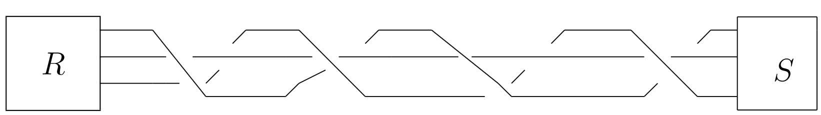


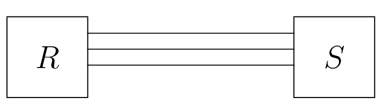
This simplifies by duality to



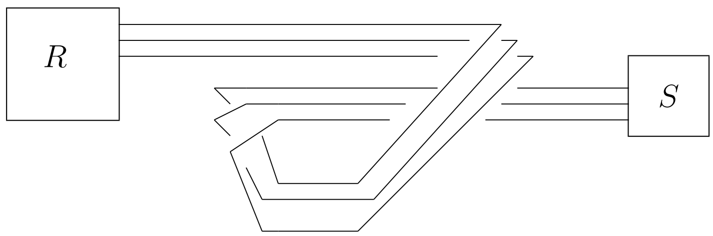
It is now clear that repeating the argument using naturality and duality we obtain the result. ■

3.15. Dirac's belt trick. We claim that the following two circuits are equal in **TCirc**, that is that a rotation through 4π of a component $I \longrightarrow X^3$ is equal to the identity. We suspect, but are unable to prove, that a rotation through 2π is not the identity.

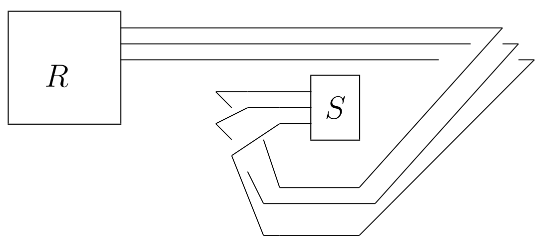




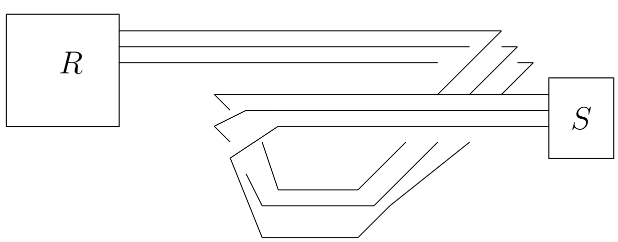
We give a sketch of a proof only. Using arguments similar to that of example 3.14 we may prove that the first (twisted) circuit is equal to



Naturality gives equality to



and then to



which is equal to the untwisted circuit.

3.16. Remark. In a separate article [15] we investigate a group structure on the set of *blocked-braids on n-strings*, that is, on circuits of the form SBR where $R : I \longrightarrow X^n$, $S : X^n \longrightarrow I$ and B is a braid on n strings.

4. A braided category of relations

The category **Rel** whose objects are sets, and whose arrows are relations is symmetric monoidal with the tensor of sets being the cartesian product, and each object has a commutative Frobenius (even separable) algebra structure provided by the diagonal functions and their reverse relations. In fact this was the motivating example for the introduction in [3] of the Frobenius equations Equivalent axioms had been given earlier by Lawvere in [14]. We describe here a modification of **Rel** which we call $\mathbf{TRel}_G$, which depends on a group G, and which is braided rather than symmetric. We further describe a commutative Frobenius algebra in $\mathbf{TRel}_G$ which hence yields a representation of $\mathbf{TCirc}_M$, and this representation enables us to distinguish various circuits. We discuss distinguishing closed circuits, a problem analogous to classifying knots, using $\mathbf{TRel}_G$.

4.1. The definition of $\mathbf{TRel}_G$. We will describe a braided modification of the category **Rel** with a commutative Frobenius object.

4.2. Definition. *Let G be a group. The objects of $\mathbf{TRel}_G$ are the formal powers of G, and the arrows from G^m to G^n are relations R from the set G^m to the set G^n satisfying:*

1) *if $(x_1,...,x_m)R(y_1,...y_n)$ then also for all g in G* $(gx_1g^{-1},...,gx_mg^{-1})R(gy_1g^{-1},...,gy_mg^{-1})$,

2) *if $(x_1,...,x_m)R(y_1,...y_n)$ then $x_1...x_m(y_1...y_n)^{-1} \in Z(G)$ (the center of G).*

Composition and identities are defined to be composition and identity of relations.

It is straightforward to verify that $\mathbf{TRel}_G$ is a category; that is, that identities and composites of relations satisfying 1) and 2) also satisfy 1) and 2). Notice also that if the group G is abelian the conditions 1) and 2) of Definition 4.2 are trivially true. We introduce some useful notation. Write $x = (x_1,...,x_m)$, $y = (y_1,...,y_n)$, and so on. Write $\overline{x} = x_1x_2...x_m$ and for g,h in G, as $g^h = hgh^{-1}$. For g in G write $x^g = (x_1^g, x_2^g, ..., x_m^g)$. Thus, $(\overline{x})^g = \overline{x^g}$, and of course for any x,y in $G^m \times G^n$, $x^gy^g = (xy)^g$ where we write xy for $(x_1,...,x_m,y_1,...,y_n)$.

4.3. Theorem. $\mathbf{TRel}_G$ *is a braided strict monoidal category with tensor defined on objects by $G^m \otimes G^n = G^{m+n}$ and on arrows by product of relations. The twist*

$$\tau_{m,n} : G^m \otimes G^n \longrightarrow G^n \otimes G^m$$

is the functional relation

$$(x,y) \sim (y^{\overline{x}}, x)$$

Proof. The monoidal structure of **Rel** restricts to $\mathbf{TRel}_G$ since if $R : G^m \longrightarrow G^t$ and $S : G^n \longrightarrow G^u$ satisfy 1) and 2) then so also does $R \times S$. To see that $R \times S$ satisfies 1) notice that if xRy and zSw then for any $g \in G$, x^gRy^g and z^gSw^g and hence $(xz)^g(R \times S)(yw)^g$. To see that $R \times S$ satisfies 2) notice that, if $\overline{x}(\overline{y})^{-1} \in Z(G)$ and $\overline{z}(\overline{w})^{-1} \in Z(G)$, then $\overline{xz}(\overline{yw})^{-1} = (\overline{x})(\overline{z})((\overline{y})(\overline{w}))^{-1} = (\overline{x})(\overline{z})(\overline{w})^{-1}(\overline{y})^{-1}$. But $\overline{z}(\overline{w})^{-1} \in Z(G)$, so $(\overline{x})(\overline{z})(\overline{w})^{-1}(\overline{y})^{-1} = (\overline{x})(\overline{y})^{-1}(\overline{z})(\overline{w})^{-1}$ and the latter is in $Z(G)$.

It is straightforward that τ satisfies properties 1) and 2). We show that $B1$ holds for τ as defined. $B2$ is similar.

First note that $\tau_{m,n+p}(xyz) = (yz)^{\overline{x}}x$. Further $(\tau_{m,n} \otimes 1_{G^p})(xyz) = y^{\overline{x}}xz$ while $(1_{G^n} \otimes \tau_{n,p})(xyz) = xz^{\overline{y}}y$. Thus

$$(1_{G^n} \otimes \tau_{n,p})(\tau_{m,n} \otimes 1_{G^p})(xyz) = (1_{G^n} \otimes \tau_{n,p})((y^{\overline{x}})xz) = (y^{\overline{x}})(z^{\overline{x}})x = \tau_{m,n+p}(xyz).$$

Lastly we need to show that $\tau_{m,n} : G^m \times G^n \longrightarrow G^n \times G^m$ is natural. This amounts to two conditions. Consider $R : G^p \longrightarrow G^m$ and $S : G^q \longrightarrow G^n$ in $\mathbf{TRel}_G$. The first condition for naturality is that

$$\tau_{m,n}(R \otimes 1_{G^n}) = (1_{G^n} \otimes R)\tau_{p,n} : G^{p+n} \longrightarrow G^{n+m}.$$

But $xyzw$ ($x \in G^p$, $y \in G^n$, $z \in G^n$, $w \in G^m$) belongs to the left-hand side iff xRw and $z = y^{\overline{w}}$, whereas $xyzw$ belongs to the right-hand side iff xRw and $z = y^{\overline{x}}$. But condition 2) implies that if xRw then for any y it follows that $y^{\overline{x}} = y^{\overline{w}}$, and hence the result.

The second condition for naturality is that

$$\tau_{m,n}(1_{G^m} \otimes S) = (S \otimes 1_{G^m})\tau_{m,q} : G^{m+q} \longrightarrow G^{n+m}.$$

But $xyzw$ ($x \in G^m$, $y \in G^q$, $z \in G^n$, $w \in G^m$) belongs to the left-hand side iff $x = w$ and $yS(z^{\overline{x}^{-1}})$, whereas $xyzw$ belongs to the right-hand side iff $x = w$ and $y^{\overline{x}}Sz$. Condition 1) implies the result. ∎

4.4. Remark. Notice that a relation in $\mathbf{TRel}_G$ from I to G^n is just a subset of G^n closed under conjugation by elements of G and whose elements x satisfy $\overline{x} \in Z(g)$. Further a relation from I to I is either the empty set or the one-point set.

4.5. The commutative Frobenius structure on G. The commutative Frobenius structure on the object G of $\mathbf{TRel}_G$ mentioned above is as follows: ∇ is a function, namely the multiplication of the group G, $n : I \longrightarrow G$ is also a function, the identity of the group; Δ is the opposite relation of ∇, e is the opposite relation of n.

For the resulting tangle algebra structure on G, notice that η is the relation $* \sim (x, x^{-1})$, and ϵ is the opposite relation of η.

It is straightforward to check that these relations belong to $\mathbf{TRel}_G$. We will just check one of the Frobenius equations, namely that

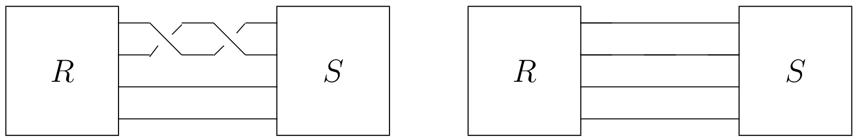
$$(1_G \otimes \nabla)(\Delta \otimes 1_G) = \Delta\nabla : G \times G \longrightarrow G \times G.$$

If g, h, p, q are in G then (g, h, p, q) belongs to the left-hand relation if there is a $r \in G$ such that $g = pr$ and $rh = q$. But this is the same as saying that $p^{-1}g = qh^{-1}$ or $gh = pq$ which is exactly the condition for (g, h, p, q) to be in the right-hand relation.

The Frobenius algebra structure on G in $\mathbf{TRel}_G$ actually satisfies the additional separable axiom $\nabla\Delta = 1$.

4.6. Proving circuits distinct in $\mathbf{TRel}_G$. In this section we discuss the possibility of distinguishing various tangled circuits by looking in $\mathbf{TRel}_G$. These include the analogue of knots, or closed circuits, which are circuits from the one-point set I to I.

4.7. Example. First an example where two circuits may be distinguished in $\mathbf{TRel}_{S_3}$, where S_3 is the symmetric group on three letters. The circuits are:



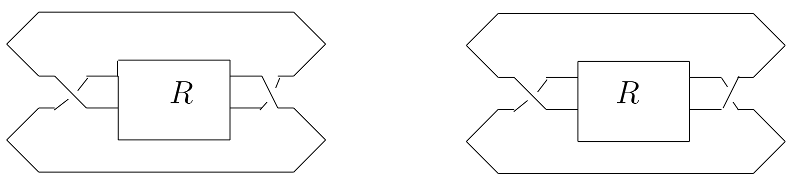
Proof. Let each of R and S be the set of conjugates of $u = ((1,2),(1,3),(2,3),(1,3))$ under the action of G (not $G \times G \times G \times G$). Notice that $(1,2)(1,3)(2,3)(1,3)$ is the identity and so in the centre of G. The second circuit evaluates as the one point set.

The first circuit evaluates instead as the empty set since the braid $\tau\tau \times 1_G \times 1_G$ in the first circuit relates $((1,2),(1,3),(2,3),(1,3))$ in R to $((1,3),(2,3),(2,3),(1,3))$ which is not in the conjugacy class of u since the second and third elements are equated by the braid.

■

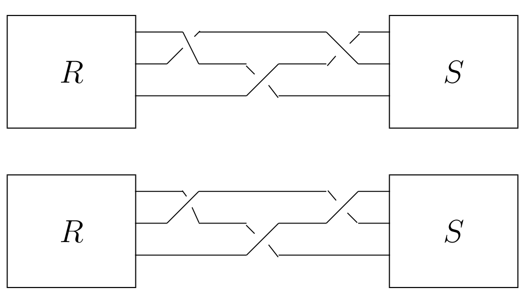
Notice that a similar argument using the symmetric group S_3 works for two components joined by $n > 3$ wires, the first two of which are tangled.

4.8. Example. We will see that the first two circuits in Section 3 can also be shown distinct in $\mathbf{TRel}_{S_3}$. It is clearly sufficient to show the following circuits distinct:



Take R to be the following subset of $(S_3)^2 \times (S_3)^2$: the conjugacy class of the element $(((1,2),(1,3)),((1,2),(1,3)))$. Then the first circuit evaluates as $\emptyset$ and the second as the one-point set.

4.9. Example. Next an example of two circuits which we believe are distinct in $\mathbf{TCirc}_M$ but are always equal in $\mathbf{TRel}_G$. For any group G, $\mathbf{TRel}_G$ cannot distinguish them.



Proof. Suppose (x, y, z) is an element of component R. Notice that since xyz is in the centre $xyz = yzx = zxy$. The braid between the two components in the first circuit relates (x, y, z) to

$$u = (xyx^{-1}zxy^{-1}x^{-1}, xyx^{-1}, z^{-1}xz) = (xyx^{-1}zxy^{-1}x^{-1}, z^{-1}yz, z^{-1}xz)$$

since $yzx = zxy$. Instead the braid in the second circuit relates (x, y, z) to

$$v = (z, z^{-1}yz, z^{-1}y^{-1}xyz) = (z, xyx^{-1}, x)$$

since $z^{-1}y^{-1}xyz = z^{-1}y^{-1}yzx = x$ and $zxy = yzx$. But $xzuz^{-1}x^{-1} = v$ since

$$xzxyx^{-1}zxy^{-1}x^{-1}z^{-1}x^{-1} = xyzxx^{-1}zxx^{-1}z^{-1}y^{-1}x^{-1} = xyzy^{-1}x^{-1} = z$$

and hence u and v are conjugate. Since S is closed under conjugacy, the element (x, y, z) gives rise to an element of the first circuit if and only if it does for the second circuit. Since this is true for any (x, y, z) the two circuits are equal in $\mathbf{TRel}_G$. ∎

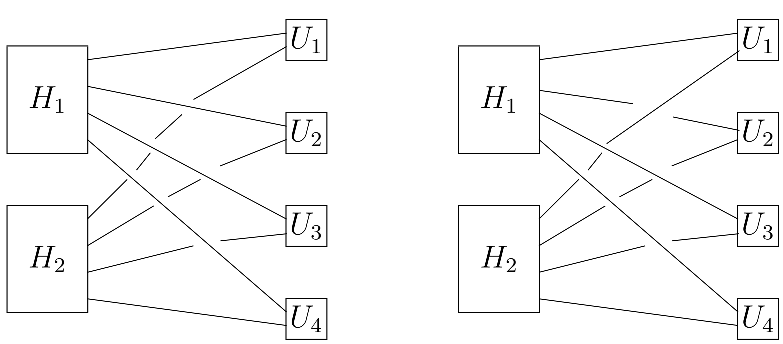
4.10. Example. In fact the last example is general for three wires. *The circuit obtained by composing in* $\mathbf{TRel}_G$ *any two components* $R : I \longrightarrow G^3$ *and* $S : G^3 \longrightarrow I$ *with a braiding in between depends only on the permutation, not the braiding.*

Proof. Suppose $(x, y, z) \in R$ then $xyz \in Z(G)$ and hence $xyx^{-1} = z^{-1}yz$, $yzy^{-1} = x^{-1}zx$ and $zxz^{-1} = y^{-1}xy$. Consider two composites R composed with $\tau \otimes 1$ and R composed with $\tau^{-1} \otimes 1$. Consider $(x, y, z) \in R$. We will show that these two composites associate (x, y, z) with conjugate triples. Repeating this we see that the argument given in the above example can be applied, showing that in a composite τ and τ^{-1} are interchangeable.

In the first composite (x, y, z) is related to $u = (xyx^{-1}, x, z) = (z^{-1}yz, y, z)$. In the second composite (x, y, z) is related to $(y, y^{-1}xy, z)$. It is immediate that $zuz^{-1} = v$. ∎

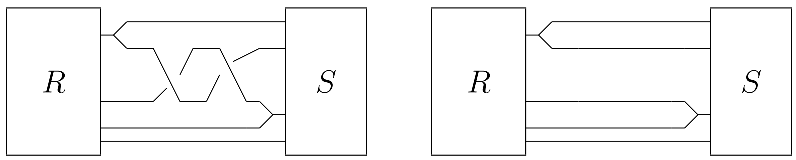
Of course such circuits with different permutations can be distinguished even in $\mathbf{TRel}_{\mathbb{Z}_2}$.

4.11. Example. Here are another two circuits we can distinguish in $\mathbf{TRel}_{S_3}$:



PROOF. Replace each of the four components U_1,U_2,U_3,U_4 by ϵ. Let H_1 be the conjugacy class of $((1,2),(1,3),(2,3),(1,3))$. The wires of the first circuit relate this element to $u = ((1,2),(2,3),(1,2),(1,3))$, and of the second circuit to $v = ((1,3),(1,2),(1,2),(1,3))$. Clearly u and v are not conjugate, and hence we can choose H_2 so that the two circuits evaluate differently in $\mathbf{TRel}_{S_3}$. ■

4.12. EXAMPLE. The following two circuits can be distinguished in $\mathbf{TRel}_{S_3}$.



PROOF. Take R to be the conjugacy class of $((1,2),(1,3),(2,3),(1,3))$ and S the conjugacy class of $((),(1,3),(),(1,3))$. The first circuit evaluates as the one-point set and the second as $\emptyset$. ■

5. A braided category of spans

The principal category we have been using in the earlier work on circuits and communicating-parallel algebras of processes is the category of spans of graphs **Span**(**Graph**). For sequential systems we have used the category of cospans of graphs **Cospan**(**Graph**). Already in the paper [8] the separable algebra structure on each object played a crucial role. The relation between another model of circuits, namely Mealy automata and **Span**(**Graph**) was discussed in [9]. One of the motivations of the present work is to produce an semantic algebra in which the twisting (or crossing) of wires is also (at least partially) expressible. To this end we introduce first a simple braided modification of the category of spans of sets **Span**(**Set**) called $\mathbf{TSpan}_G$, depending on a group G and with a commutative Frobenius algebra. It is clear that a similar construction $\mathbf{TSpan}_G(\mathbf{C})$ could be made for a group object G in a category **C** with limits in the place of **Set**.

Again, there is a representation of **Tangle** (via a representation of **TCirc**) which takes a tangle to the span of colourings of the tangle (introduced by John Armstrong in [2]). Applied to knots, the set of colourings is one of the simplest invariants for distinguishing knots. As a first example it allows one to show that a trefoil is not an unknot. The extended notion of colourings of tangled circuit diagrams gives further aid in distinguishing circuit diagrams.

The dual of the category of groups $\mathbf{Group}^{\mathbf{op}}$ has finite limits. Further the free group on one generator F is a group object in $\mathbf{Group}^{\mathbf{op}}$. The category $\mathbf{TSpan}_F(\mathbf{Group}^{\mathbf{op}})$ is braided monoidal with F equipped with a commutative Frobenius structure. The induced representation

$$\mathbf{Tangle} \longrightarrow \mathbf{TSpan}_F(\mathbf{Group}^{\mathbf{op}})$$

associates the cospan of groups introduced by John Armstrong in [1] to a tangle, and the knot group to a knot.

Strictly speaking, because we are interested in *categories* of spans, we always consider isomorphism classes of spans (in the usual sense of isomorphism of spans) but we will often describe them in terms of representatives.

5.1. Definition. *Let G be a group. The objects of $\mathbf{TSpan}_G$ are the formal powers of G, and an arrow from G^m to G^n is an isomorphism class of spans of sets, $G^m \xleftarrow{\delta_0} S \xrightarrow{\delta_1} G^n$, from the set G^m to the set G^n such that there exists a function $G \times S \longrightarrow S$ of G written $(g, s) \mapsto gs$ yielding a bijection for each $g \in G$, and satisfying:*

1) if $\delta_0(s) = (x_1, ..., x_m)$ and $\delta_1(s) = (y_1, ..., y_n)$ then $\delta_0(gs) = (x_1^g, ..., x_m^g)$ and $\delta_1(gs) = (y_1^g, ..., y_m^g)$ for all g in G,

2) if $\delta_0(s) = (x_1, ..., x_m)$ and $\delta_1(s) = (y_1, ..., y_n)$ then $x_1...x_m(y_1...y_n)^{-1} \in Z(G)$.

Composition and identities are composition and identity of spans.

It is straightforward that $\mathbf{TSpan}_G$ is a category. Like $\mathbf{TRel}_G$ it has the structure of a braided strict monoidal category. Note again that if G is abelian then 1) and 2) in the definition are satisfied.

5.2. Theorem. *$\mathbf{TSpan}_G$ is braided strict monoidal with tensor defined by $G^m \otimes G^n = G^{m+n}$ and twist $\tau_{m,n} : G^m \otimes G^n \longrightarrow G^n \otimes G^m$ is the span with $\delta_0 = 1_{G^m \otimes G^n}$ and*

$$\delta_1(x_1, ..., x_m, y_1, ..., y_n) = (y_1^{\overline{x}}, ..., y_n^{\overline{x}}, x_1, ..., x_m).$$

Proof. This is similar to Theorem 4.3. As noted, it is easy to show that identities and composites of spans satisfying conditions 1) and 2) also satisfy 1) and 2), so $\mathbf{TSpan}_G$ is a category.

To see that $\otimes$ is a functor recall that product of spans defines a tensor functor on the category $\mathbf{Span}(\mathbf{Set})$. It remains to show that $\mathbf{TSpan}_G$ is closed under $\otimes$. Suppose $R : G^m \longrightarrow G^t$ and $S : G^n \longrightarrow G^u$. If $x = \delta_0(r), y = \delta_1(r)$ and $z = \delta_0(s), w\delta_1(s)$, then for any g, $x^g = \delta_0(gr), y^g = \delta_1(gr)$ and $z^g = \delta_0(s), w^g = \delta_1(s)$, whence $(xz)^g = \delta_0(gr, gs), (yw)^g = \delta_1(gr, gs)$, so taking $g(r, s)$ to be (gr, gs) condition 1) is satisfied. For x, y, z, w as defined, condition 2) follows exactly as in Theorem 4.3.

The associative and unitary properties for $\otimes$ in $\mathbf{TSpan}_G$ are immediate from the same properties in $\mathbf{Span}(\mathbf{Set})$.

The properties $B1$ and $B2$ for τ are identical to the case of $\mathbf{TRel}_G$.

As in the case of $\mathbf{TRel}_G$ the conditions 1) and 2) assure the naturality of τ. ∎

5.3. A commutative Frobenius structure on G. As for $\mathbf{TRel}_G$ and using the same functions viewed as spans, G has the structure of a commutative Frobenius algebra in $\mathbf{TSpan}_G$. Consequently:

5.4. Corollary. *There is a unique braided strict monoidal functor*

$$\mathbf{Tangle} \longrightarrow \mathbf{TSpan}_G$$

taking the generating object to G and the structure maps of **Tangle** *to the corresponding arrows in* $\mathbf{TSpan}_G$.

While G is separable in $\mathbf{TRel}_G$, it is not so in $\mathbf{TSpan}_G$ because, unlike the relational composite, the *span* composite $\nabla\Delta$ is not the identity.

5.5. Knot colourings. The description of $\mathbf{TSpan}_G$ makes it clear that there is a faithful monoidal functor

$$\mathbf{TSpan}_G \longrightarrow \mathbf{Span}(\mathbf{Set}).$$

Consider the following composite of monoidal functors we have described above:

$$\text{COLOURINGS}_G : \mathbf{Tangle} \longrightarrow \mathbf{TCirc} \longrightarrow \mathbf{TSpan}_G \longrightarrow \mathbf{Span}(\mathbf{Set}).$$

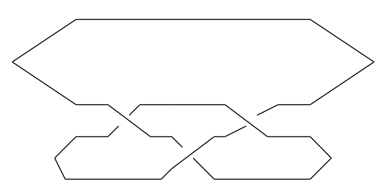
COLOURINGS_G takes the generating object X of **Tangle** to the underlying set of G, and takes ϵ_X to the span $G\times G \leftarrow \{(x,y) : xy = 1\} \rightarrow I$, η_X to $I \leftarrow \{(x,y) : xy = 1\} \rightarrow G\times G$ and τ_X to $(x,y) \leftarrow (x,y) \mapsto (xyx^{-1}, x)$.

5.6. Theorem. *(J. Armstrong [2]) If K is a knot then* COLOURINGS_G *is the set of colourings of K in the group G.*

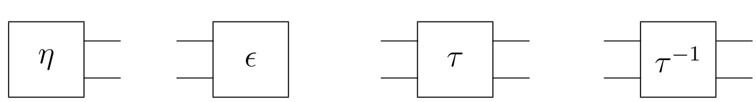
5.7. Remark. Because of the faithfulness of the functor $\mathbf{TSpan}_G \longrightarrow \mathbf{Span}(\mathbf{Set})$ the calculation of the set of colourings of a knot may be done equally in $\mathbf{TSpan}_G$ or $\mathbf{Span}(\mathbf{Set})$. The advantage of introducing $\mathbf{TSpan}_G$ as we do is that $\mathbf{TSpan}_G$ has the same structure as **Tangle** (braided monoidal with a tangle algebra) whereas $\mathbf{Span}(\mathbf{Set})$ does not.

5.8. Colourings of a trefoil. We will calculate the colourings of a trefoil knot in the dihedral group D_3 to allow us to introduce notation and indicate relations with other work. One expression in **Tangle** for a trefoil is

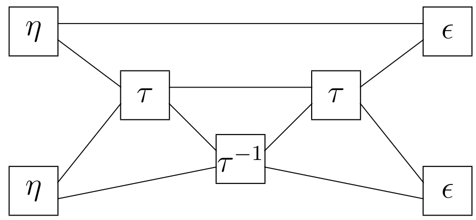
$$(\epsilon \otimes \epsilon)(1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \tau^{-1})(1 \otimes \tau \otimes 1)(\eta \otimes \eta).$$



It is convenient to represent the arrows in this expression as components as follows:

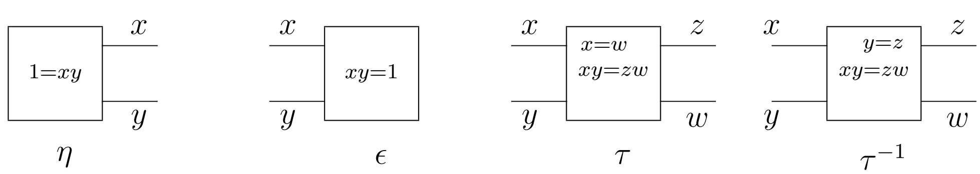


Then the trefoil may be displayed as the circuit diagram:

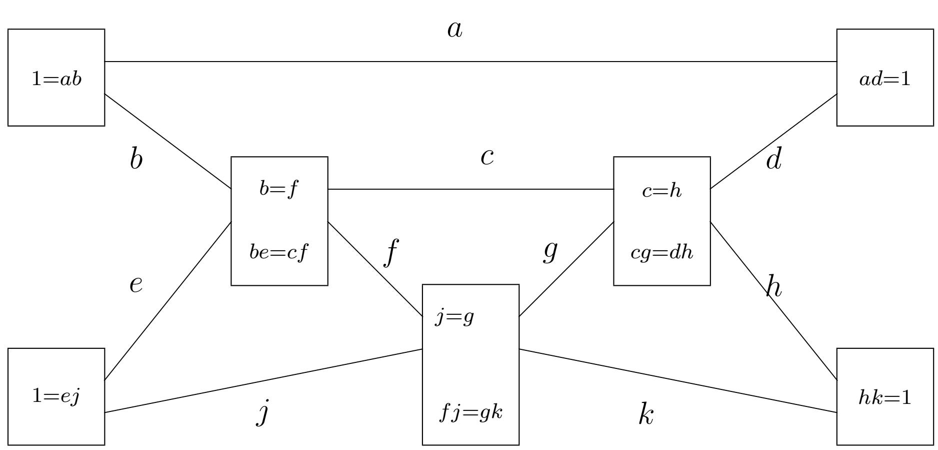


The evaluation of the expression for the trefoil in **Span**(**Set**) is a limit of the diagram in **Set** formed as follows: for each wire in the diagram take the set G and for each component take the pair of arrows constituting its span of sets. See [19] for the relation between limits in **C** and expressions in **Span**(**C**). An element of this limit is a tuple of elements of G one for each wire, satisfying the conditions of the components. Each of the components η, ϵ, τ, τ^{-1} is actually a relation from its domain to codomain, that is a subset of products of groups given by equational conditions.

It is convenient to refine the pictures of the components to include the conditions as follows:



Then a colouring of the trefoil is an element of the limit. Thus it is a tuple of elements of G on the wires satisfying the conditions of the components:



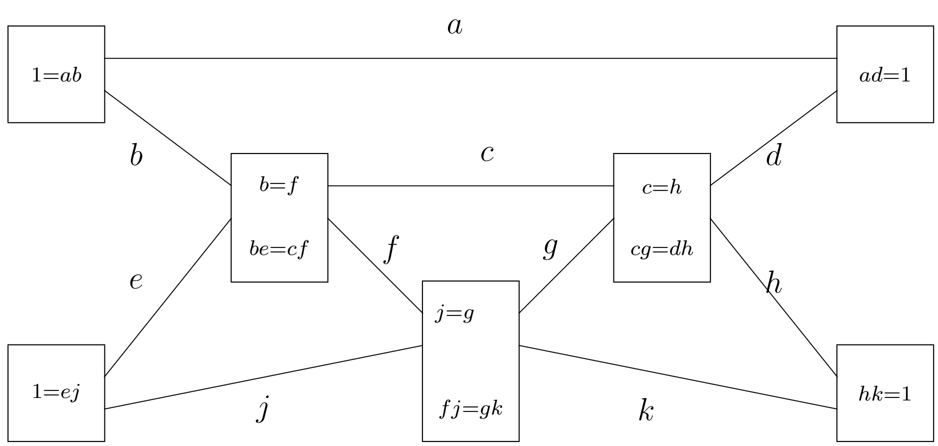
Clearly a and c determine the other letters. When the group is D_3 there are twelve colourings: one for each of $(a,c) = ((),()), ((1,2,3),(1,2,3)), ((1,3,2),(1,3,2)),$ $((1,2),(1,2)), ((1,3),(1,3)), ((2,3),(2,3)), ((1,2),(1,3)), ((1,2),(2,3)), ((1,3),(1,2)),$ $((1,3),(2,3)), ((2,3),(1,2)), ((2,3),(1,2))$, whereas the unknot has only six colourings.

5.9. Knot groups. Consider now the free group on one generator F as a group object in the category $\mathbf{Group}^{\mathbf{op}}$. As we have mentioned, the construction **TSpan** works for any category with finite limits, not just **Set**, and hence there is a braided monoidal category $\mathbf{TSpan}_F(\mathbf{Group}^{\mathbf{op}})$, and a corresponding representation

$$Gp : \mathbf{Tangle} \longrightarrow \mathbf{TSpan}_F(\mathbf{Group}^{\mathbf{op}}) \longrightarrow \mathbf{Span}(\mathbf{Group}^{\mathbf{op}}) = \mathbf{Cospan}(\mathbf{Group})$$

5.10. Theorem. *(J. Armstrong [1]) If K is a knot then $Gp(K)$ is the knot group of K.*

5.11. The knot group of a trefoil. Limits in $\mathbf{Group}^{\mathbf{op}}$ are colimits in **Group**. We can calculate the knot group from the same picture we used to calculate the knot colouring. In the diagram



a letter represents the free group F on that generator, letters on a pair of wires represent the free group on two generators $F \times F$ in $\mathbf{Group}^{\mathbf{op}}$. The components are quotients of the free group on the boundary wires by the equations. The evaluation of the circuit in $\mathbf{TSpan}_F(\mathbf{Group}^{\mathbf{op}})$ is a colimit, namely the free group on all the wires quotiented by all the equations.

In the case of the trefoil the knot group is

$$\begin{aligned} < a,b,c,d,e,f,g,h,j,k \; ; \; & ab = 1, b = f, be = cf, c = h, \\ & cg = dh, ad = 1, ej = 1, j = g, fj = gk, hk = 1 > . \end{aligned}$$

Eliminating the variables b,d,e,f,g,h,j,k this presentation reduces to

$$< a, c \; ; \; aca = cac > .$$

6. Extending $\mathbf{TRel}_G$ and $\mathbf{TSpan}_G$

We now describe an extension of $\mathbf{TRel}_G$ which depends not only on the group G but also on a set X, and we denote it $\mathbf{TRel}_{X,G}$, and a similar extension of $\mathbf{TSpan}_G$ denoted $\mathbf{TSpan}_{X,G}$. These will enable us to model circuits with state. The idea is that in $\mathbf{TRel}_{X,G}$ the object $X \times G$ is a single wire carrying data X.

6.1. Definition. *The category* $\mathbf{TRel}_{X,G}$ *has objects* $(X \times G)^n$. *An arrow of* $\mathbf{TRel}_{X,G}$ *is a relation* S *in* **Set** *from* $(X \times G)^m$ *to* $(Y \times G)^n$ *such that 1) if* $(x,h)S(y,k)$ *then for any* $g \in G$, $(x,h^g)S(y,k^g)$, *and 2) if* $(x,h)S(y,k)$ *then* $(\overline{h})(\overline{k})^{-1} \in Z(G)$. *Composition and identities are defined as in* **Rel**.

In $\mathbf{TRel}_{X,G}$ we define a tensor product by $(X \times G)^m \otimes (X \times G)^n = (X \times G)^{m+n}$.

6.2. Proposition. $\mathbf{TRel}_{X,G}$ *is a braided strict monoidal category with*

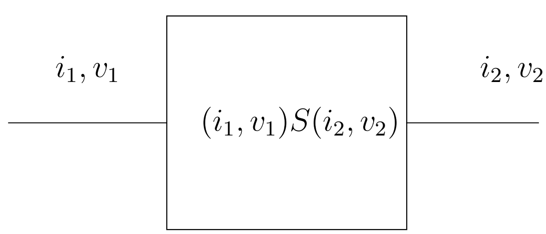
$$\tau_{(XG)^m \otimes (XG)^n} : (XG)^m \otimes (XG)^n \longrightarrow (XG)^n \otimes (XG)^m$$

defined to be the relation

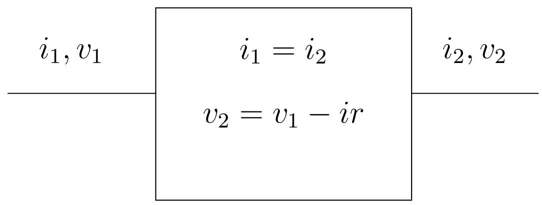
$$((x,g),(y,h)) \sim ((y,h^{\overline{g}}),(x,g)).$$

As in $\mathbf{TRel}_G$ and $\mathbf{TSpan}_G$, a "single wire" $X \times G$ in $\mathbf{TRel}_{X,G}$ admits a commutative Frobenius algebra structure, namely the multiplication is the relation $((x,g),(x,h)) \sim (x,gh)$; the comultiplication is $(x,gh) \sim ((x,g),(x,h))$, the counit is $(x,1) \sim *$ and the unit is $* \sim (x,1)$.

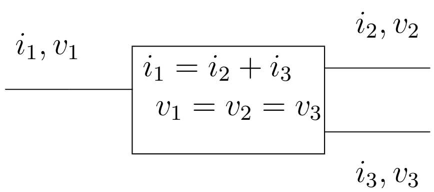
6.3. Analogue resistive circuits in $\mathbf{TRel}_{\mathbb{R},\mathbb{R}}$. We begin by describing circuits of resistors which may be described in $\mathbf{TRel}_{X,G}$ where $X = \mathbb{R}$ is the real numbers, and $G = \mathbb{R}$ as a group under addition. It is useful to use a graphical notation similar to that of Section 5 to do calculations in $\mathbf{TRel}_{\mathbb{R},\mathbb{R}}$. For example, we draw a relation $S : \mathbb{R} \times \mathbb{R} \longrightarrow X \times G$ as:

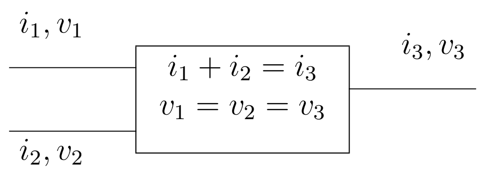


With this notation, where i denotes current and v denotes voltage, a resistor of resistance r is:

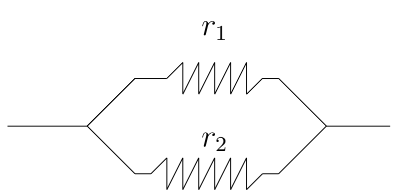


The multiplication and comultiplication, which we sometimes draw as forks, and which embody Kirchhoff's law of currents are:

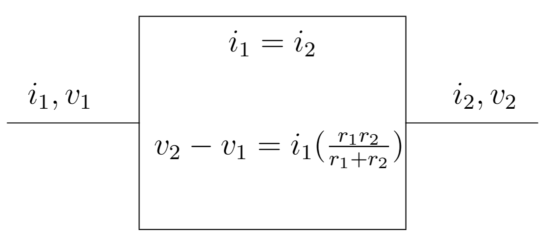




Using the operations of $\mathbf{TRel}_{\mathbb{R},\mathbb{R}}$ one can now evaluate a network of resistors. For example the circuit with two parallel resistors with resistances r_1, r_2 respectively



evaluates as:



6.4. Definition. *The category* $\mathbf{TSpan}_{X,G}$ *has objects* $(X\times G)^n$. *An arrow of* $\mathbf{TSpan}_{X,G}$ *is an isomorphism class of spans* S *in* **Set**: $(X \times G)^m \xleftarrow{\delta_0} S \xrightarrow{\delta_1} (Y \times G)^n$ *such that such that there exist a function* $G \times S \longrightarrow S$ *of* G *on* S *written* $(g, s) \mapsto gs$ *yielding a bijection for each* $g \in G$, *and satisfying:*

1) *if* $\delta_0(s) = (x_1, h_1, ..., x_m, h_m)$ *and* $\delta_1(s) = (y_1, k_1, ..., y_n, k_n)$ *then* $\delta_0(gs) = (x_1, h_1^g, ..., x_m, h_m^g)$ *and* $\delta_1(gs) = (y_1, k_1^g, ..., y_n, k_n^g)$ *for all* g *in* G,

2) *if* $\delta_0(s) = (x_1, h_1, ..., x_m, h_m)$ *and* $\delta_1(s) = (y_1, k_1, ...y_n, k_n)$ *then* $h_1...h_m(k_1..., k_n)^{-1} \in Z(G)$.

Composition and identities and tensor are defined as in **Span**(**Set**). *The braiding and Frobenius structure are as in* $\mathbf{TRel}_{X,G}$.

It is clear that this definition may be made in any category **C** with finite limits to give a category $\mathbf{TSpan}_{X,G}(\mathbf{C})$.

6.5. RLC circuits in $\mathbf{TSpan}_{X,G}(\mathbf{Graph})$. This example comes from the paper [9] where it is discussed in detail. However the Frobenius algebra structure was not noticed in that paper. The category is analogous to $\mathbf{TSpan}_G(\mathbf{Graph})$ where the group G is the real numbers under addition. The Frobenius algebra structure arises from the Kirchhoff law for currents. Since the group is abelian there is no information about the tangling of wires. We describe, as an example, circuits composed of resistors, capacitors and inductors.

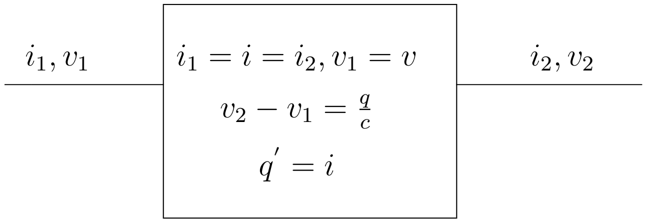
The algebra of RLC circuits we will describe was introduced in [9]. We will give a brief recapitulation without full details.

We need to say something first about the somewhat unusual interpretation of a graph in this setting. If the graph consists of the two (domain and codomain) functions $\phi : X \longrightarrow Y$ and $\psi : X \longrightarrow Y$ we will interpret this as the formal differential equation $\phi' = \psi$. For further explanation of this interpretation see [9]. In the examples we describe the interpretation will have a clear meaning. There is a notion of behaviour for such a system, namely a function $x : \mathbb{R} \longrightarrow X$ such that $\phi'(x(t)) = \psi(x(t))$ (only meaningful with smoothness assumptions).

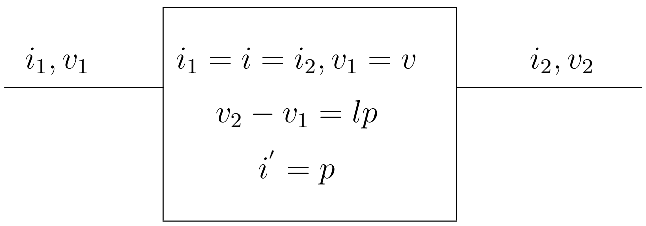
We will now consider $\mathbf{TSpan}_{X,G}(\mathbf{Graph})$ where both X and G are the graph with one object, and set of arrows $\mathbb{R}$; we will identify both X and G with the set $\mathbb{R}$, the group structure being addition.

Again it is useful to use a graphical notation similar to that of Section 5 to do calculations in $\mathbf{TSpan}_{\mathbb{R},\mathbb{R}}(\mathbf{Graph})$. For example, we draw the spans corresponding to the constants τ, Δ, ∇, η, ϵ, and the resistors of the algebra (in which all of the graphs have one object) exactly as in Section 6.3.

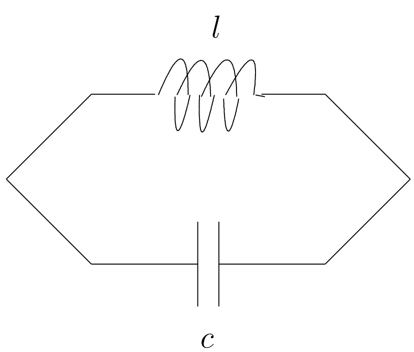
The graph of a capacitor with capacitance c is the pair of functions $\phi, \psi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $\phi(i, v, q) = q$ and $\psi(i, v, q) = i$; the interpretation is that a capacitor has state i, v, and also state q, the charge of the capacitor, and that $q' = i$. The boundary conditions (the morphism of the span) are on the left $v_1 = v$, and $i_1 = i$, and on the right $v_2 = v - \frac{q}{c}$ and $i_2 = i$. Hence we draw the capacitor as follows:



Similarly an inductor with inductance l has an extra variable of state p with graph $\mathbb{R}^3 \longrightarrow \mathbb{R}$, and pictures:



Using the operations of $\mathbf{TSpan}_{\mathbb{R},\mathbb{R}}(\mathbf{Graph})$ one can now evaluate a network of resistors, capacitors and inductors. For example the circuit of an inductance and a capacitance



evaluates as

i, v_1, v_2, p, q
$(-i)' = p$
$q' = i$
$\frac{q}{c} = v_1 - v_2 = lp$

A behaviour consists of five functions from $\mathbb{R}$ to $\mathbb{R}$, namely $i(t)$, $v_1(t)$, $v_2(t)$, $q(t)$, $p(t)$ such that $i^{'} = -p$, $q^{'} = i$ and $\frac{q}{c} = v_2 - v_1 = lp$.

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Department of Mathematics and Computer Science
Mount Allison University
Sackville, N. B. E4L 1E6 Canada

Dipartimento di Scienza e Alta Tecnologia
Università dell' Insubria
via Carloni, 78, Como, Italy

Dipartimento di Scienza e Alta Tecnologia
Università dell' Insubria
via Carloni, 78, Como, Italy

Email: rrosebrugh@mta.ca
nicoletta.sabadini@uninsubria.it
robert.walters@uninsubria.it

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