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MONADS AS EXTENSION SYSTEMS —NO ITERATION IS NECESSARY

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Abstract. We introduce a description of the algebras for a monad in terms of extension systems, similar to the one for monads given in [Manes, 1976]. We rewrite distributive laws for monads and wreaths in terms of this description, avoiding the iteration of the functors involved. We give a profunctorial explanation of why Manes' description of monads in terms of extension systems works.

1. Introduction

For adjoint functors $S \dashv H$, it has been well known since [Eilenberg & Moore, 1965] that monad structures on S are in bijective correspondence with comonad structures on H. Moreover, it is shown in [Eilenberg & Moore, 1965] that if $(\mathbb{S}, \mathbb{H})$ is a corresponding (monad, comonad) pair then the category of $\mathbb{S}$-algebras is *isomorphic* to the category of $\mathbb{H}$ coalgebras via a functor that identifies the forgetful functors. After [Street, 1972] it has been clear that these results of [Eilenberg & Moore, 1965] are actually part of the *formal theory of monads*, the definitions making sense in any 2-category and the theorems being provable in any suitably complete 2-category. It was acknowledged in [Lack & Street, 2002] that the formal theory of monads is easily adjusted to the greater generality of bicategories, although it suffices to prove most results in a general 2-category. Where possible we take the latter point of view in this paper.

The bijective correspondence of the nullary data

$$\frac{\eta : 1 \longrightarrow S}{\varepsilon : H \longrightarrow 1}$$

for monads and comonads is accomplished by a single application of taking mates with respect to the adjunction $S \dashv H$ (in any 2-category). For the binary components, it is useful to consider the correspondence of the data as a three-step mating process:

$$\begin{array}{c} \mu : SS \longrightarrow S \\ \hline \xi : S \longrightarrow HS \\ \hline \lambda : SH \longrightarrow H \\ \hline \delta : H \longrightarrow HH \end{array}$$

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This leads us to contemplate not only monads $\mathbb{S} = (S, \eta, \mu)$, and comonads $\mathbb{H} = (H, \varepsilon, \delta)$ but also 3-tuples $(S \dashv H, \eta, \xi)$ and $(S \dashv H, \varepsilon, \lambda)$. For each of the latter two, it is a simple matter to determine three equations so that the correspondences of the data above extend to the resulting equational structures. We give such equations for an $(S \dashv H, \eta, \xi)$, which we then call an *extension system*, in Section 9. The experienced reader will see immediately how to prescribe equations making an $(S \dashv H, \varepsilon, \lambda)$ what we would call a *lifting system*, although we will say little, explicitly, about these. We will speak of ξ as an *extension operator*, which terminology has already been used, for a special case, in [Manes & Mulry, 2007].

Suppose that $(S \dashv H, \eta, \xi)$ is an extension system on an object $\mathbf{C}$ in a bicategory $\mathcal{K}$. Then, for every $A, B : \mathbf{T} \to \mathbf{C}$ in $\mathcal{K}$, we have the composite functions

$$\mathcal{K}(\mathbf{T},\mathbf{C})(B, SA) \to \mathcal{K}(\mathbf{T},\mathbf{C})(B, HSA) \to \mathcal{K}(\mathbf{T},\mathbf{C})(SB, SA)$$

where the first factor is given by composition with ξA and the second by taking mates with respect to $S \dashv H$. This composite, which we will call $(-)^{\mathbb{S}}$, satisfies equations, which we will give in Section 2, but no longer requires that S have a right adjoint in $\mathcal{K}$. Accordingly, we generalize the definition of *extension system* to include the case where S does not necessarily have a right adjoint and show in Section 2 that, given $\eta : 1_{\mathbf{C}} \to S : \mathbf{C} \to \mathbf{C}$ in $\mathcal{K}$, there is a bijective correspondence between monads (S, η, μ) and extension systems $(S, \eta, (-)^{\mathbb{S}})$.

In [Manes, 1976], Exercise 1.3, p. 32, monads in **Cat** were presented as extension systems in which the data $\eta : 1_{\mathbf{C}} \to S : \mathbf{C} \to \mathbf{C}$ on a category $\mathbf{C}$ required only that S be initially given as an object function $|S| : |\mathbf{C}| \to |\mathbf{C}|$ and η as a function defined on $|\mathbf{C}|$ with no a priori naturality requirement. We are able to analyse this simplification in **Cat** by considering the canonical embedding of **Cat** in **Pro**, the bicategory of profunctors. Here we use the fact that any category $\mathbf{C}$ is, in **Pro**, the Kleisli object for a canonical monad $\mathbb{C}$ on $|\mathbf{C}|$. We discuss this in Section 9. We remark that when considering extension systems in **Cat**, we can always regard the situation as taking place in **Pro**, where every functor in **Cat** has a right adjoint, and exploit the simpler form that extension systems take in the presence of a right adjoint.

We define algebras for an extension system and, interpolating the aforementioned theorem of [Eilenberg & Moore, 1965], show that if (S, η, μ) and $(S, \eta, (-)^{\mathbb{S}})$ correspond then the categories of algebras for each are isomorphic via an arrow that identifies the forgetful arrows. Thus we are able to think of extension systems and their algebras as no more than an alternate presentation for monads. However, there is an important overarching reason to consider monads in this way. Extension systems allow us to completely dispense with the iterates SS and SSS of the underlying arrow. No iteration is necessary. A moment's reflection on the various terms of terms and terms of terms of terms that occur in practical applications suggest that this alone justifies the alternate approach. We give examples in Section 8.

We use the simplicity of the approach to further advance the general theory of monads with respect to composition of monads via both distributive laws, Sections 4 and 6, and wreaths, Sections 5 and 7. Here we are able to make use of alternate formulations of distributive laws first given in [Marmolejo, Rosebrugh, Wood, 2002]. In anticipation of further work, we note that extension systems in higher dimensional category theory provide an even more important simplification of monads. For even in dimension 2, some of the tamest examples are built on pseudofunctors that are difficult to iterate.

2. Extension systems in a 2-category

In the Introduction we motivated the idea of an *extension system* in terms of monad-like data with underlying arrow $S\colon \mathbf{C} \to \mathbf{C}$ in a 2-category, in the case that S is part of an adjunction $S \dashv H$. However, it is the non-elementary definition, that we can state without the assumption of a right adjoint for S, that is most useful for our work. After a preliminary Definition and Lemma we take this as our starting point.

We work in a 2-category $\mathcal{K}$. Let

$$\begin{tikzcd} & \mathbf{D} \arrow[dr, "T"] \arrow[d, phantom, "\varepsilon\Downarrow"] & \\ \mathbf{C} \arrow[ur, "S"] \arrow[rr, "U"'] & {} & \mathbf{E} \end{tikzcd} \tag{1}$$

be a 2-cell (in $\mathcal{K}$). For any span of arrows $(C, D)\colon \mathbf{T} \to \mathbf{C};\mathbf{D}$, pasting ε at S defines a family of functions

$$(-)^{\#}_{D,C}\colon \mathcal{K}(\mathbf{T},\mathbf{D})(D, SC) \to \mathcal{K}(\mathbf{T},\mathbf{E})(TD, UC)$$

whose effect on d in $\mathcal{K}(\mathbf{T},\mathbf{D})(D, SC)$ is the pasting composite

$$\begin{tikzcd} \mathbf{T} \arrow[rr, "D"] \arrow[dr, "C"'] & \arrow[d, phantom, "d\Downarrow"] & \mathbf{D} \arrow[dr, "T"] \arrow[d, phantom, "\varepsilon\Downarrow"] & \\ & \mathbf{C} \arrow[ur, "S"] \arrow[rr, "U"'] & {} & \mathbf{E} \end{tikzcd}$$

This 2-cell, whose full name is $d^{\#}_{D,C}$, will often be written simply as $d^{\#}$. The family of functions $(-)^{\#}_{D,C}$ respects *whiskering at* $\mathbf{T}$, meaning that for any $X\colon \mathbf{S} \to \mathbf{T}$,

$$d^{\#}X = (dX)^{\#} \tag{2}$$

and respects *blistering at* D, meaning that for any $b\colon B \to D\colon \mathbf{T} \to \mathbf{D}$,

$$(db)^{\#} = d^{\#} \cdot Tb \tag{3}$$

2.1. Definition. *A* pasting operator

$$(-)^{\#}\colon \mathcal{K}(\mathbf{T},\mathbf{D})(1, S) \to \mathcal{K}(\mathbf{T},\mathbf{E})(T, U)$$

is a family of functions

$$(-)^{\#}_{D,C}\colon \mathcal{K}(\mathbf{T},\mathbf{D})(D, SC) \to \mathcal{K}(\mathbf{T},\mathbf{E})(TD, UC)$$

which respects whiskering and blistering.

2.2. LEMMA. *For arrows S, T, U configured as is in (1), pasting operators*

$$\mathcal{K}(\mathbf{T},\mathbf{D})(1,S) \to \mathcal{K}(\mathbf{T},\mathbf{E})(T,U)$$

are in bijective correspondence with 2-cells $TS \to U$.

PROOF. Given a pasting operator $(-)^{\#} : \mathcal{K}(\mathbf{T},\mathbf{D})(1,S) \to \mathcal{K}(\mathbf{T},\mathbf{E})(T,U)$, we have

$$(1_S)^{\#}_{S,1_{\mathbf{C}}} : TS \to U.$$

Moreover, it is easy to see that any $d : D \to SC$ arises by whiskering 1_S at $\mathbf{C}$ by C and blistering the result at SC by d. Thus any $(-)^{\#} : \mathcal{K}(\mathbf{T},\mathbf{D})(1,S) \to \mathcal{K}(\mathbf{T},\mathbf{E})(T,U)$ is completely determined by $(1_S)^{\#}_{S,1_{\mathbf{C}}}$ and the latter can be any 2-cell $TS \to U$. It follows that the assignment $(-)^{\#} \mapsto (1_S)^{\#}_{S,1_{\mathbf{C}}}$ is a bijection. ■

2.3. DEFINITION. *Let $\mathbf{C}$ be an object in a 2-category $\mathcal{K}$. An* extension system *on $\mathbf{C}$ consists of an arrow $S : \mathbf{C} \to \mathbf{C}$, a 2-cell $\eta : 1_{\mathbf{C}} \to S$, and a pasting operator*

$$(-)^{\mathbb{S}} : \mathcal{K}(\mathbf{T},\mathbf{C})(1,S) \to \mathcal{K}(\mathbf{T},\mathbf{C})(S,S)$$

that we call the $\mathbb{S}$-extension operator. *This data is subject to the following equations, for every $C, B, A : \mathbf{T} \to \mathbf{C}$, $f : B \to SA$, and $g : C \to SB$,*

$$\eta^{\mathbb{S}} = 1_S, \tag{4}$$

$$\begin{tikzcd} B \arrow[r, "\eta B"] \arrow[dr, "f"'] & SB \arrow[d, "f^{\mathbb{S}}"] \\ & SA, \end{tikzcd} \tag{5}$$

and

$$\begin{tikzcd} SC \arrow[r, "g^{\mathbb{S}}"] \arrow[dr, "(f^{\mathbb{S}}g)^{\mathbb{S}}"'] & SB \arrow[d, "f^{\mathbb{S}}"] \\ & SA. \end{tikzcd} \tag{6}$$

2.4. THEOREM. *For $\eta : 1_{\mathbf{C}} \to S : \mathbf{C} \to \mathbf{C}$ in a 2-category $\mathcal{K}$, there is a bijective correspondence between extension systems $(S, \eta, (-)^{\mathbb{S}})$ and monads (S, η, μ).*

PROOF. By Lemma 2.2 we have a bijection between pasting operators $(-)^{\mathbb{S}}$ and 2-cells $\mu : SS \to S$. Let (S, η, μ) be a monad. The correspondence of Lemma 2.2 provides $f^{\mathbb{S}} = \mu A \cdot Sf : SB \to SA$. Now (4) is one of the unit monad axioms, while (5) is

$$f^{\mathbb{S}} \cdot \eta B = \mu A \cdot Sf \cdot \eta B = \mu A \cdot \eta SA \cdot f = f$$

using the other unit monad axiom, and (6) is

$$\begin{aligned} f^{\mathbb{S}} \cdot g^{\mathbb{S}} &= \mu A \cdot Sf \cdot \mu B \cdot Sg = \mu A \cdot \mu SA \cdot SSf \cdot Sg = \mu A \cdot S\mu A \cdot SSf \cdot Sg \\ &= \mu A \cdot S(\mu A \cdot Sf \cdot g) = (f^{\mathbb{S}} \cdot g)^{\mathbb{S}} \end{aligned}$$

using monad associativity; so that $(S, \eta, (-)^{\mathbb{S}})$ is an extension system.

On the other hand, if $(S, \eta, (-)^{\mathbb{S}})$ is an extension system, the correspondence of Lemma 2.2 provides $\mu = {1_S}^{\mathbb{S}}$. The first monad equation is $\mu \cdot \eta S = {1_S}^{\mathbb{S}} \cdot \eta S = 1_S$ by (5). The second is $\mu \cdot S\eta = {1_S}^{\mathbb{S}} \cdot S\eta = \eta^{\mathbb{S}} = 1_S$, by (3) and (4). Monad associativity is given by

$$\mu \cdot S\mu = {1_S}^{\mathbb{S}} \cdot S\mu = \mu^{\mathbb{S}} = ({1_S}^{\mathbb{S}})^{\mathbb{S}} = {1_S}^{\mathbb{S}} \cdot {1_{S^2}}^{\mathbb{S}} = {1_S}^{\mathbb{S}} \cdot {1_S}^{\mathbb{S}}S = \mu \cdot \mu S,$$

using (2), (3), and (6); so that (S, η, μ) is a monad. ∎

From now on we do not need to distinguish between monads and extension systems. If $(S, \eta, (-)^{\mathbb{S}})$ and (S, η, μ) correspond, we write $(S, \eta, (-)^{\mathbb{S}}) = \mathbb{S} = (S, \eta, \mu)$ and use freely the equations relating both. Note too that, for $b\colon B \to D\colon \mathbf{T} \to \mathbf{C}$, we have

$$Sb = (\eta D \cdot b)^{\mathbb{S}}, \tag{7}$$

which we leave as a simple exercise.

3. Algebras for extension systems

Notwithstanding the last paragraph, in the spirit of [Eilenberg & Moore, 1965], we give a definition of algebras for an extension system.

3.1. Definition. *For $(S, \eta, (-)^{\mathbb{S}})$ an extension system on $\mathbf{C}$ and $\mathbf{X}$ an object, both in $\mathcal{K}$, an $(S, \eta, (-)^{\mathbb{S}})$*-algebra with domain $\mathbf{X}$ *is a pair $\mathbb{B} = (B, (-)^{\mathbb{B}})$, where $B\colon \mathbf{X} \to \mathbf{C}$ and*

$$(-)^{\mathbb{B}}\colon \mathcal{K}(\mathbf{T}, \mathbf{C})(1, B) \to \mathcal{K}(\mathbf{T}, \mathbf{C})(S, B)$$

is a pasting operator that we call the $\mathbb{B}$-extension operator*, subject to the following equations, for every $h\colon Y \to BD$ and $k\colon Z \to SY\colon \mathbf{T} \to \mathbf{C}$,*

$$\begin{tikzcd} Y \arrow[r, "\eta Y"] \arrow[dr, "h"'] & SY \arrow[d, "h^{\mathbb{B}}"] \\ & BD, \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} SZ \arrow[r, "k^{\mathbb{S}}"] \arrow[dr, "(h^{\mathbb{B}} \cdot k)^{\mathbb{B}}"'] & SY \arrow[d, "h^{\mathbb{B}}"] \\ & BD. \end{tikzcd} \tag{9}$$

A homomorphism $p\colon (B, (-)^{\mathbb{B}}) \to (A, (-)^{\mathbb{A}})$ of $(S, \eta, (-)^{\mathbb{S}})$-algebras with domain $\mathbf{X}$ *is a 2-cell $p\colon B \to A$ subject to the following equation, for every $h\colon Y \to BD$,*

$$\begin{tikzcd} SY \arrow[r, "h^{\mathbb{B}}"] \arrow[dr, "(pD \cdot h)^{\mathbb{A}}"'] & BD \arrow[d, "pD"] \\ & AD. \end{tikzcd} \tag{10}$$

It is easy to see that $(S, \eta, (-)^{\mathbb{S}})$-algebras with domain $\mathbf{X}$ and their homomorphisms form a category $\mathcal{K}(\mathbf{X}, (\mathbf{C}, (S, \eta, (-)^{\mathbb{S}}))$ equipped with a forgetful functor to $\mathcal{K}(\mathbf{X}, \mathbf{C})$. We recall the (S, η, μ)-algebras with domain $\mathbf{X}$ as described in [Street, 1972] or [Marmolejo, 1997] and write $\mathcal{K}(\mathbf{X}, (\mathbf{C}, (S, \eta, \mu)))$ for these.

3.2. THEOREM. *The categories* $\mathcal{K}(\mathbf{X}, (\mathbf{C}, (S, \eta, (-)^{\mathbb{S}})))$ *and* $\mathcal{K}(\mathbf{X}, (\mathbf{C}, (S, \eta, \mu)))$ *are isomorphic via a functor that identifies the forgetful functors.*

PROOF. By Lemma 2.2, pasting operators $(-)^{\mathbb{B}} : \mathcal{K}(\mathbf{T}, \mathbf{C})(1, B) \to \mathcal{K}(\mathbf{T}, \mathbf{C})(S, B)$ are in bijective correspondence with 2-cells $\beta : SB \to B$. It suffices to show that the equations for algebras and their homomorphisms in either sense correspond to those in the other sense. Let $(B, (-)^{\mathbb{B}})$ be an $(S, \eta, (-)^{\mathbb{S}})$-algebra and consider $(B, {1_B}^{\mathbb{B}})$, where ${1_B}^{\mathbb{B}}$ arises from $(-)^{\mathbb{B}}$ as in Lemma 2.2. We have ${1_B}^{\mathbb{B}} \cdot \eta B = 1_B$ by (8), and

$$\begin{aligned} {1_B}^{\mathbb{B}} \cdot S\, {1_B}^{\mathbb{B}} &= {1_B}^{\mathbb{B}} \cdot (\eta B \cdot {1_B}^{\mathbb{B}})^{\mathbb{S}} = ({1_B}^{\mathbb{B}} \cdot \eta B \cdot {1_B}^{\mathbb{B}})^{\mathbb{B}} = ({1_B}^{\mathbb{B}})^{\mathbb{B}} \\ &= {1_B}^{\mathbb{B}} \cdot (1_{SB})^{\mathbb{S}} = {1_B}^{\mathbb{B}} \cdot {1_S}^{\mathbb{S}} B = {1_B}^{\mathbb{B}} \cdot \mu B, \end{aligned}$$

by (7), (9), (8), (9) and (2). If $p : (B, (-)^{\mathbb{B}}) \to (A, (-)^{\mathbb{A}})$ is a homomorphism of $(S, \eta, (-)^{\mathbb{S}})$-algebras then we have

$${1_A}^{\mathbb{A}} \cdot Sp = {1_A}^{\mathbb{A}} \cdot (\eta A \cdot p)^{\mathbb{S}} = ({1_A}^{\mathbb{A}} \cdot \eta A \cdot p)^{\mathbb{A}} = p^{\mathbb{A}} = p \cdot {1_B}^{\mathbb{B}},$$

by (7), (9), (8), and (10), showing that we also have $p : (B, (1_B)^{\mathbb{B}}) \to (A, (1_A)^{\mathbb{A}})$, a homomorphism of (S, η, μ)-algebras.

On the other hand, if (B, β) is an (S, η, μ)-algebra then, for $h : Y \to BD : \mathbf{Y} \to \mathbf{C}$, define $h^{\mathbb{B}} = SY \xrightarrow{Sh} SBD \xrightarrow{\beta D} BD$. Now (8) is $h^{\mathbb{B}} \cdot \eta Y = \beta D \cdot Sh \cdot \eta Y = \beta D \cdot \eta BD \cdot h = h$, and (9) is

$$\begin{aligned} (h^{\mathbb{B}} \cdot k)^{\mathbb{B}} &= \beta D \cdot S\beta D \cdot S^2 h \cdot Sk = \beta D \cdot \mu BD \cdot S^2 h \cdot Sk \\ &= \beta D \cdot Sh \cdot \mu Y \cdot Sk = h^{\mathbb{B}} \cdot k^{\mathbb{S}}. \end{aligned}$$

If $p : (B, \beta) \to (A, \alpha)$ is an (S, η, μ)-homomorphism, then $(p \cdot h)^{\mathbb{A}} = \alpha \cdot Sp \cdot Sh = p \cdot \beta \cdot Sh = p \cdot h^{\mathbb{B}}$ establishes (10) showing that we also have an $(S, \eta, (-)^{\mathbb{S}})$-homomorphism. ∎

Thus we do not need to distinguish between $(S, \eta, (-)^{\mathbb{S}})$-algebras and (S, η, μ)-algebras and speak simply of $\mathbb{S}$-algebras, freely using all the equations now at hand.

4. The 2-category Mnd($\mathcal{K}$)

Let $\mathcal{K}$ be a 2-category. Recall from [Street, 1972] that an object of the 2-category Mnd($\mathcal{K}$) consists of a pair $(\mathbf{C}, \mathbb{S})$ where $\mathbf{C}$ is an object of $\mathcal{K}$ and $\mathbb{S}$ is a monad on $\mathbf{C}$, that a 1-cell from $(\mathbf{D}, \mathbb{T})$ to $(\mathbf{C}, \mathbb{S})$ consists of a 1-cell $F : \mathbf{D} \to \mathbf{C}$ and a 2-cell $\lambda : SF \to FT$ in $\mathcal{K}$ such that the diagrams

$$\begin{tikzcd} & F \arrow[dl, "\eta_S F"'] \arrow[dr, "F\eta_T"] & \\ SF \arrow[rr, "\lambda"'] & & FT \end{tikzcd} \qquad \begin{tikzcd} S^2F \arrow[r, "S\lambda"] \arrow[d, "\mu_S F"'] & SFT \arrow[r, "\lambda T"] & FT^2 \arrow[d, "F\mu_T"] \\ SF \arrow[rr, "\lambda"'] & & FT \end{tikzcd}$$

commute, and that a 2-cell $(F, \lambda) \to (F', \lambda') : (\mathbf{D}, \mathbb{T}) \to (\mathbf{C}, \mathbb{S})$ consists of a 2-cell $\varphi : F \to F'$ in $\mathcal{K}$ such that the diagram

$$\begin{tikzcd} SF \arrow[r, "\lambda"] \arrow[d, "S\varphi"'] & FT \arrow[d, "\varphi T"] \\ SF' \arrow[r, "\lambda'"'] & F'T \end{tikzcd}$$

commutes.

In the spirit of Proposition 3.4 in [Marmolejo, Rosebrugh, Wood, 2002] (where it is done for distributive laws), we have the following lemma.

4.1. LEMMA. *Given* $F : \mathbf{D} \to \mathbf{C}$*, there is a bijection between 2-cells* $\lambda : SF \to FT$ *making* $(F, \lambda) : (\mathbf{D}, \mathbb{T}) \to (\mathbf{C}, \mathbb{S})$ *a 1-cell of* $\mathrm{Mnd}(\mathcal{K})$ *and 2-cells* $\alpha : SFT \to FT$ *making* (FT, α) *an* $\mathbb{S}$*-algebra satisfying the equation*

$$\begin{tikzcd} SFT^2 \arrow[r, "\alpha T"] \arrow[d, "SF\mu_T"'] & FT^2 \arrow[d, "F\mu_T"] \\ SFT \arrow[r, "\alpha"'] & FT. \end{tikzcd}$$

Moreover, if under the given bijection λ *and* α *correspond, given* $F : \mathbf{D} \to \mathbf{C}$*, and* λ' *and* α' *correspond, given* $F' : \mathbf{D} \to \mathbf{C}$*, then a 2-cell* $\varphi : F \to F'$ *gives a 2-cell* $\varphi : (F, \lambda) \to (F', \lambda')$ *of* $\mathrm{Mnd}(\mathcal{K})$ *if and only if* $\varphi T : FT \to F'T$ *is an* $\mathbb{S}$*-homomorphism.*

PROOF. If we start with α, then $\lambda = \alpha \cdot SF\eta_T$. In the opposite direction, given λ, define $\alpha = F\mu_T \cdot \lambda T$. ■

Denote the 2-category implicitly defined by the above lemma with 1-cells $(F, \alpha) : (\mathbf{D}, \mathbb{T}) \to (\mathbf{C}, \mathbb{S})$ by $\mathrm{Mnd}'(\mathcal{K})$. Observe that the composition of 1-cells $(G, \beta) : (\mathbf{E}, \mathbb{U}) \to (\mathbf{D}, \mathbb{T})$ and $(F, \alpha) : (\mathbf{D}, \mathbb{T}) \to (\mathbf{C}, \mathbb{S})$ is given by FG together with

$$SFGU \xrightarrow{SF\eta_T GU} SFTGU \xrightarrow{\alpha GU} FTGU \xrightarrow{F\beta} FGU.$$

4.2. COROLLARY. *The correspondences above define an identity-on-objects isomorphism of 2-categories* $\mathrm{Mnd}(\mathcal{K}) \to \mathrm{Mnd}'(\mathcal{K})$*, so that* $\mathrm{Mnd}'(\mathcal{K})$ *can be regarded as* $\mathrm{Mnd}(\mathcal{K})$*.* ■

4.3. Theorem. *Let* $\mathbb{T} = (T, \eta_T, (-)^{\mathbb{T}})$ *be a monad on* $\mathbf{D}$*, and let* $\mathbb{S} = (S, \eta_S, (-)^{\mathbb{S}})$ *be a monad on* $\mathbf{C}$*. A 1-cell in* $\mathrm{Mnd}(\mathcal{K})$ *from* $(\mathbf{D}, \mathbb{T})$ *to* $(\mathbf{C}, \mathbb{S})$ *can equivalently be defined as follows:* $(F, (-)^\lambda) : (\mathbf{D}, \mathbb{T}) \to (\mathbf{C}, \mathbb{S})$ *where* $F : \mathbf{D} \to \mathbf{C}$*, and* $(FT, (-)^\lambda)$ *is an* $\mathbb{S}$*-algebra, such that for every* $u : U \to TV : \mathbf{X} \to \mathbf{D}$ *and* $h : X \to FTU : \mathbf{X} \to \mathbf{C}$*, the diagram*

$$\begin{tikzcd}
SX \arrow[r, "h^\lambda"] \arrow[dr, swap, "(Fu^{\mathbb{T}} \cdot h)^\lambda"] & FTU \arrow[d, "Fu^{\mathbb{T}}"] \\
& FTV
\end{tikzcd} \tag{11}$$

commutes.

Furthermore, given $(F, (-)^\lambda), (F', (-)^{\lambda'}) : (\mathbf{D}, \mathbb{T}) \to (\mathbf{C}, \mathbb{S})$*, then* $\varphi : F \to F'$ *is a 2-cell in* $\mathrm{Mnd}(\mathcal{K})$ *if and only if* $\varphi T : (FT, (-)^\lambda) \to (F'T, (-)^{\lambda'})$ *is a morphism of* $\mathbb{S}$*-algebras.*

Proof. According to Theorem 3.2 we have that pasting operators $(-)^\lambda : \mathcal{K}(\mathbf{X}, \mathbf{C})(1, FT) \to \mathcal{K}(\mathbf{X}, \mathbf{C})(S, FT)$ that make $(FT, (-)^\lambda)$ an $\mathbb{S}$-algebra are in bijective correspondence with 2-cells $\alpha : SFT \to FT$ that make (FT, α) an $\mathbb{S}$-algebra. So we must show that the extra equation given in the statement of the theorem is satisfied if and only if the extra equation given in Lemma 4.1 is satisfied.

Given $(F, (-)^\lambda)$ as in the statement of the theorem, we have that

$$\alpha := (1_{FT})^\lambda : SFT \to FT.$$

The extra equation is $F\mu_T \cdot \alpha T = F(1_T{}^{\mathbb{T}}) \cdot (1_{FT^2})^\lambda = F(1_T{}^{\mathbb{T}})^\lambda = (1_{FT}{}^\lambda \cdot \eta_S FT \cdot F(1_T{}^{\mathbb{T}}))^\lambda = 1_{FT}{}^\lambda \cdot (\eta_S FT \cdot F(1_T{}^{\mathbb{T}}))^{\mathbb{S}} = \alpha \cdot SF\mu_T$. Thus $(F, \alpha) : (\mathbf{D}, \mathbb{T}) \to (\mathbf{C}, \mathbb{S})$ is a 1-cell in $\mathrm{Mnd}'(\mathcal{K})$.

In the opposite direction, assume that $(F, \alpha) : (\mathbf{D}, \mathbb{T}) \to (\mathbf{C}, \mathbb{S})$ is a 1-cell in $\mathrm{Mnd}'(\mathcal{K})$. Then for $h : X \to FTU : \mathbf{X} \to \mathbf{C}$ we have that

$$h^\lambda := SX \xrightarrow{Sh} SFTU \xrightarrow{\alpha U} FTU. \tag{12}$$

For $u : U \to TV$, the commutative diagram

$$\begin{tikzcd}
SX \arrow[r, "Sh"] \arrow[ddr, bend right, swap, "S(F(u^{\mathbb{T}}) \cdot h)"] & SFTU \arrow[r, "\alpha U"] \arrow[d, swap, "SFTu"] & FTU \arrow[d, "FTu"] \\
& SFT^2V \arrow[r, swap, "\alpha TV"] \arrow[d, "SF\mu_T V"] & FT^2V \arrow[d, "F\mu_T V"] \\
& SFTV \arrow[r, swap, "\alpha V"] & FTV
\end{tikzcd}$$

gives us (11). ∎

4.4. Remark. Observe that in the previous theorem we can obtain $\lambda : ST \to TS$ directly from the pasting operator $(-)^\lambda$ as $(F\eta_T)^\lambda$ to obtain a 1-cell $(F, \lambda) : (\mathbf{D}, \mathbb{T}) \to (\mathbf{C}, \mathbb{S})$ as described at the beginning of this section.

5. The 2-category EM($\mathcal{K}$)

Recall from [Lack & Street, 2002] that the 2-category EM($\mathcal{K}$) has the same objects and 1-cells as Mnd($\mathcal{K}$), but the 2-cells $(F,\lambda) \to (F',\lambda') : (\mathbf{D},\mathbb{T}) \to (\mathbf{C},\mathbb{S})$ are 2-cells $\rho : F \to F'T$ in $\mathcal{K}$ such that the diagram

$$\begin{tikzcd}
SF \arrow[r, "\lambda"] \arrow[d, "S\rho"'] & FT \arrow[r, "\rho T"] & F'T^2 \arrow[d, "F'\mu_T"] \\
SF'T \arrow[r, "\lambda' T"'] & F'T^2 \arrow[r, "F'\mu_T"'] & F'T
\end{tikzcd}$$

commutes.

5.1. Lemma. *If under the bijection given in Lemma 4.1, λ and α correspond, given $F : \mathbf{D} \to \mathbf{C}$, and λ' and α' correspond, given $F' : \mathbf{D} \to \mathbf{C}$, then a 2-cell $\rho : F \to F'T$ is a 2-cell $\rho : (F,\lambda) \to (F',\lambda')$ of* EM($\mathcal{K}$) *if and only if the diagram*

$$\begin{tikzcd}
SFT \arrow[r, "\alpha"] \arrow[d, "S\rho T"'] & FT \arrow[r, "\rho T"] & F'T^2 \arrow[d, "F'\mu_T"] \\
SF'T^2 \arrow[r, "\alpha' T"'] & F'T^2 \arrow[r, "F'\mu_T"'] & F'T
\end{tikzcd}$$

commutes. ∎

We present the following description of the 2-cells in EM($\mathcal{K}$).

5.2. Theorem. *Given 1-cells $(F,(-)^{\lambda}), (F',(-)^{\lambda'}) : (\mathbf{D},\mathbb{T}) \to (\mathbf{C},\mathbb{S})$ in* EM($\mathcal{K}$)*, a 2-cell $(F,(-)^{\lambda}) \to (F,(-)^{\lambda'})$ in* EM($\mathcal{K}$) *can be defined as an $\mathbb{S}$-algebra morphism $\beta : (FT,(-)^{\lambda}) \to (F'T,(-)^{\lambda'})$ such that for every $u : U \to TV : \mathbf{X} \to \mathbf{D}$, the diagram*

$$\begin{tikzcd}
FTU \arrow[r, "\beta U"] \arrow[d, "Fu^{\mathbb{T}}"'] & F'TU \arrow[d, "F'u^{\mathbb{T}}"] \\
FTV \arrow[r, "\beta V"'] & F'TV
\end{tikzcd}$$

commutes.

Proof. Assume we have $\beta : FT \to F'T$ as in the statement of the theorem. Define

$$\rho = (\, F \xrightarrow{F\eta_T} FT \xrightarrow{\beta} F'T \,). \tag{13}$$

The commutative diagram

$$\begin{tikzcd}
SFT \arrow[r, "\alpha"] \arrow[dd, "SF\eta_T T"'] \arrow[dr, "1_{SFT}"'] & FT \arrow[r, "F\eta_T T"] \arrow[ddr, "1_{FT}"] & FT^2 \arrow[r, "\beta T"] \arrow[dd, "F\mu_T"] & FT^2 \arrow[ddd, "F'\mu_T"] \\
& SFT \arrow[dr, "\alpha"] & & \\
SFT^2 \arrow[ur, "SF\mu_T"'] \arrow[r, "\alpha T"'] \arrow[dr, "S\beta T"'] & FT^2 \arrow[r, "F\mu_T"'] \arrow[dr, "\beta T"'] & FT \arrow[dr, "\beta"] & \\
& SF'T^2 \arrow[r, "\alpha' T"'] & F'T^2 \arrow[r, "F'\mu_T"'] & F'T
\end{tikzcd}$$

gives us the equation in Lemma 5.1.

In the opposite direction, assume that $\rho\colon(F,\lambda)\to(F',\lambda)\colon(\mathbf{D},\mathbb{T})\to(\mathbf{C},\mathbb{S})$ satisfies the equation in Lemma 5.1. Define

$$\beta := \left(FT \xrightarrow{\rho T} F'T^2 \xrightarrow{F'\mu_T} F'T\right). \tag{14}$$

Then the commutative diagram

$$\begin{tikzcd}
SFT \arrow[r, "\alpha"] \arrow[d, "S\rho T"'] & FT \arrow[r, "\rho T"] & F'T^2 \arrow[d, "F'\mu_T"] \\
SF'T^2 \arrow[r, "\alpha' T"] \arrow[dr, "SF'\mu_T"'] & F'T^2 \arrow[r, "F'\mu_T"] & F'T \\
& SF'T \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

tells us that $\beta:(FT,\alpha)\to(F'T,\alpha')$ is a morphism of $\mathbb{S}$-algebras, and for $u\colon U\to TV$ the commutative diagram

$$\begin{tikzcd}
FTU \arrow[r, "\rho TU"] \arrow[d, "FTu"'] & F'T^2U \arrow[r, "F'\mu_T U"] \arrow[d, "F'T^2u"'] & F'TU \arrow[d, "F'Tu"] \\
FT^2V \arrow[r, "\rho T^2V"] \arrow[d, "F\mu_T V"'] & F'T^3V \arrow[r, "F'\mu_T TV"] \arrow[d, "F'T\mu_T V"'] & F'T^2V \arrow[d, "F'\mu_T V"] \\
FTV \arrow[r, "\rho TV"'] & F'T^2V \arrow[r, "F'\mu_T V"'] & F'TV
\end{tikzcd}$$

tells us that $\beta V\cdot Fu^{\mathbb{T}} = F'u^{\mathbb{T}}\cdot\beta U$.

6. Distributive laws

We recall the characterization of distributive laws given in Proposition 3.5 of [Marmolejo, Rosebrugh, Wood, 2002].

6.1. Proposition. *Given monads* $\mathbb{T}$ *and* $\mathbb{S}$ *on* $\mathbf{C}$ *in a 2-category* $\mathcal{K}$*, there is a bijective correspondence between distributive laws* $\lambda\colon ST \to TS$ *of* $\mathbb{S}$ *over* $\mathbb{T}$ *and* $\mathbb{S}$*-algebras* $\alpha\colon STS \to TS$ *that satisfy the commutativity of the diagrams*

$$\begin{tikzcd}
STS^2 \arrow[r, "ST\mu_S"] \arrow[d, "\alpha S"'] & STS \arrow[d, "\alpha"] \\
TS^2 \arrow[r, "T\mu_S"'] & TS,
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2 \arrow[r, "S\eta_T S"] \arrow[d, "\mu_S"'] & STS \arrow[d, "\alpha"] \\
S \arrow[r, "\eta_T S"'] & TS,
\end{tikzcd}
\qquad
\begin{tikzcd}
ST^2S \arrow[r, "ST\eta_S TS"] \arrow[d, "S\mu_T S"'] & STSTS \arrow[r, "\alpha TS"] & TSTS \arrow[r, "T\alpha"] & T^2S \arrow[d, "\mu_T S"] \\
STS \arrow[rrr, "\alpha"'] & & & TS,
\end{tikzcd}$$

given by $\lambda \mapsto (STS \xrightarrow{\lambda S} TS^2 \xrightarrow{T\mu_S} TS)$ *with inverse* $\alpha \mapsto (ST \xrightarrow{ST\eta_S} STS \xrightarrow{\alpha} TS)$*.*

6.2. Theorem. *Let* $\mathbb{T} = (T, \eta_T, (-)^{\mathbb{T}})$ *and* $\mathbb{S} = (S, \eta_S, (-)^{\mathbb{S}})$ *be monads on* $\mathbf{C}$*. A distributive law of* $\mathbb{S}$ *over* $\mathbb{T}$ *can equivalently be given as follows. An* $\mathbb{S}$*-algebra* $(TS, (-)^{\lambda})$*, such that*

$$(T\eta_S \cdot \eta_T)^{\lambda} = \eta_T S, \tag{15}$$

and the commutativity of the diagram:

$$\begin{tikzcd}
SX \arrow[r, "h^{\lambda}"] \arrow[dr, "((r^{\lambda})^{\mathbb{T}} \cdot h)^{\lambda}"'] & TSU \arrow[d, "(r^{\lambda})^{\mathbb{T}}"] \\
& TSV,
\end{tikzcd} \tag{16}$$

for every $h\colon X \to TSU$ *and* $r\colon U \to TSV$*.*

Proof. Assume $(TS, (-)^{\lambda})$ given with the stated properties. We show first that $(T, (-)^{\lambda})\colon (\mathbf{C}, \mathbb{S}) \to (\mathbf{C}, \mathbb{S})$ is a 1-cell in $\mathrm{Mnd}(\mathcal{K})$. Indeed, for $u\colon U \to SV$ we have

$$T(u^{\mathbb{S}}) = (\eta_T SV \cdot u^{\mathbb{S}})^{\mathbb{T}} = ((T\eta_S V \cdot \eta_T V)^{\lambda} \cdot u^{\mathbb{S}})^{\mathbb{T}} = (((T\eta_S V \cdot \eta_T V)^{\lambda} \cdot u)^{\lambda})^{\mathbb{T}},$$

using (7), (15) and (9). Now a direct use of (16) produces (11).

The corresponding 2-cell $\alpha = {1_{TS}}^{\lambda}$ is, according to Theorem 4.3, an $\mathbb{S}$-algebra and it satisfies the first of the equations in Proposition 6.1. The second one is given by

$$\begin{aligned}
\alpha \cdot S\eta_T S &= {1_{TS}}^{\lambda} \cdot (\eta_S TS \cdot \eta_T S)^{\mathbb{S}} = ({1_{TS}}^{\lambda} \cdot \eta_S TS \cdot \eta_T S)^{\lambda} = (\eta_T S)^{\lambda} \\
&= ((T\eta_S \cdot \eta_T)^{\lambda})^{\lambda} = (T\eta_S \cdot \eta_T)^{\lambda} \cdot {1_S}^{\mathbb{S}} = \eta_T S \cdot \mu_S,
\end{aligned}$$

whereas the third is

$$\begin{aligned}
\alpha \cdot S\mu_T S &= {1_{TS}}^{\lambda} \cdot (\eta_S TS \cdot \mu_T S)^{\mathbb{S}} = ({1_{TS}}^{\lambda} \cdot \eta_S TS \cdot \mu_T S)^{\lambda} = (\mu_T S)^{\lambda} = ({1_{TS}}^{\mathbb{T}})^{\lambda} \\
&= (({1_{TS}}^{\lambda} \cdot \eta_S TS)^{\mathbb{T}})^{\lambda} = ((({1_{TS}}^{\lambda})^{\mathbb{T}} \cdot \eta_T STS \cdot \eta_S TS)^{\mathbb{T}})^{\lambda} \\
&= (({1_{TS}}^{\lambda})^{\mathbb{T}} \cdot (\eta_T STS \cdot \eta_S TS)^{\mathbb{T}})^{\lambda} = (({1_{TS}}^{\lambda})^{\mathbb{T}} \cdot T\eta_S TS)^{\lambda} \\
&= ((({1_{TS}}^{\lambda})^{\mathbb{T}})^{\lambda} \cdot \eta_S TSTS \cdot T\eta_S TS)^{\lambda} = (({1_{TS}}^{\lambda})^{\mathbb{T}})^{\lambda} \cdot (\eta_S TSTS \cdot T\eta_S TS)^{\mathbb{S}} \\
&= (({1_{TS}}^{\lambda})^{\mathbb{T}})^{\lambda} \cdot ST\eta_S TS = ({1_{FT}}^{\lambda})^{\mathbb{T}} \cdot {1_{TSTS}}^{\lambda} \cdot ST\eta_S TS = \alpha^{\mathbb{T}} \cdot \alpha TS \cdot ST\eta_S TS \\
&= ({1_{TS}}^{\mathbb{T}} \cdot \eta_T TS \cdot \alpha)^{\mathbb{T}} \cdot \alpha TS \cdot ST\eta_S TS = {1_{TS}}^{\mathbb{T}} \cdot (\eta_T TS \cdot \alpha)^{\mathbb{T}} \cdot \alpha TS \cdot ST\eta_S TS \\
&= \mu_T S \cdot T\alpha \cdot \alpha TS \cdot ST\eta_S TS.
\end{aligned}$$

In the opposite direction, assume we have an $\mathbb{S}$-algebra $\alpha : STS \to TS$ that satisfies the conditions of Proposition 6.1. Its corresponding pasting operator in $h : X \to TSU$ is given by

$$SX \xrightarrow{Sh} STSU \xrightarrow{\alpha U} TSU,$$

and produces a 1-cell $(T, (-)^{\lambda}) : (\mathbf{C}, \mathbb{S}) \to (\mathbf{C}, \mathbb{S})$. Then

$$(T\eta_S \cdot \eta_T)^{\lambda} = \alpha \cdot ST\eta_S \cdot S\eta_T = \alpha \cdot S\eta_T S \cdot S\eta_S = \eta_T S \cdot \mu_S \cdot S\eta_S = \eta_T S,$$

and for $h : X \to TSU$ and $r : U \to TSV$, the commutative diagram

$$\begin{tikzcd}
SX \arrow[r, "Sh"] \arrow[ddddr, bend right, "S((r^{\lambda})^{\mathbb{T}} \cdot h)"'] & STSU \arrow[rrr, "\alpha U"] \arrow[dd, "STSr"] & & & TSU \arrow[dd, "TSr"] \\
& & & STSTSV \arrow[dr, "\alpha TSV"] & \\
& STSTSV \arrow[r, "ST\eta_S STSV"'] \arrow[urr, bend left, "1_{STSTSV}"] \arrow[d, "ST\alpha V"'] & STS^2TSV \arrow[r, "\alpha STSV"'] \arrow[ur, "ST\mu_S TSV"] \arrow[d, "STS\alpha V"'] & TS^2TSV \arrow[r, "T\mu_S TSV"'] \arrow[d, "TS\alpha V"'] & TSTSV \arrow[d, "T\alpha V"'] \\
& ST^2SV \arrow[r, "ST\eta_S TSV"'] \arrow[d, "S\mu_T SV"] & STSTSV \arrow[r, "\alpha TSV"'] & TSTSV \arrow[r, "T\alpha V"'] & T^2SV \arrow[d, "\mu_T SV"'] \\
& STSV \arrow[rrr, "\alpha V"'] & & & TSV,
\end{tikzcd}$$

tells us that $(r^{\lambda})^{\mathbb{T}} \cdot h^{\lambda} = ((r^{\lambda})^{\mathbb{T}} \cdot h)^{\lambda}$. ■

The proofs of the next two propositions are left to the reader.

6.3. PROPOSITION. *Given a distributive law* λ *of* $\mathbb{S}$ *over* $\mathbb{T}$, *the composite monad is given by* $\mathbb{T} \circ_{\lambda} \mathbb{S} = (TS, T\eta_S \cdot \eta_T, ((-)^{\lambda})^{\mathbb{T}})$. ■

There is also a result closer to “compatible structures”:

6.4. PROPOSITION. *Let* $\mathbb{S}$ *and* $\mathbb{T}$ *be monads on* $\mathbf{C}$. *There is a bijection between distributive laws of* $\mathbb{S}$ *over* $\mathbb{T}$ *and monad structures* $(TS, T\eta_S \cdot \eta_T, (-)^{(\mathbb{TS})})$ *on* TS *such that for every* $k : Y \to SX$, $T(k^{\mathbb{S}}) = (\eta_T SX \cdot k)^{(\mathbb{TS})}$, *and for every* $m : Y \to TSX$ *and* $h : X \to TSU$, $h^{(\mathbb{TS})} = (h^{(\mathbb{TS})} \cdot \eta_T SX)^{\mathbb{T}}$, *and the diagram*

$$\begin{tikzcd}
TY \arrow[r, "m^{\mathbb{T}}"] \arrow[dr, "(h^{(\mathbb{TS})} \cdot m)^{\mathbb{T}}"'] & TSX \arrow[d, "h^{(\mathbb{TS})}"] \\
& TSU
\end{tikzcd}$$

commutes. ■

7. Wreaths

Recall from [Lack & Street, 2002] that given a monad $\mathbb{S} = (S, \eta, \mu)$ on an object $\mathbf{C}$ of $\mathcal{K}$ and a 1-cell $T : \mathbf{C} \to \mathbf{C}$, a wreath consists of 2-cells

$$\sigma : 1_{\mathbf{C}} \to TS, \quad \lambda : ST \to TS, \quad \nu : T^2 \to TS,$$

that satisfy the commutativity of the following diagrams:

$$\begin{tikzcd}
& T \arrow[dl, "\eta T"'] \arrow[dr, "T\eta"] & \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2T \arrow[r, "S\lambda"] \arrow[d, "\mu T"'] & STS \arrow[r, "\lambda S"] & TS^2 \arrow[d, "T\mu"] \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}$$

$$\begin{tikzcd}
S \arrow[rr, "\sigma S"] \arrow[d, "S\sigma"'] & & TS^2 \arrow[d, "T\mu"] \\
STS \arrow[r, "\lambda S"'] & TS^2 \arrow[r, "T\mu"'] & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
ST^2 \arrow[r, "\lambda T"] \arrow[d, "S\nu"'] & TST \arrow[r, "T\lambda"] & T^2S \arrow[r, "\nu S"] & TS^2 \arrow[d, "T\mu"] \\
STS \arrow[rr, "\lambda S"'] & & TS^2 \arrow[r, "T\mu"'] & TS
\end{tikzcd}$$

$$\begin{tikzcd}
T \arrow[r, "T\sigma"] \arrow[ddr, "T\eta"'] & T^2S \arrow[d, "\nu S"] & TST \arrow[l, "T\lambda"'] & T \arrow[l, "\sigma T"'] \arrow[ddll, "T\eta"] \\
& TS^2 \arrow[d, "T\mu"] & & \\
& TS & &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3 \arrow[r, "T\nu"] \arrow[d, "\nu T"'] & T^2S \arrow[r, "\nu S"] & TS^2 \arrow[dd, "T\mu"] \\
TST \arrow[d, "T\lambda"'] & & \\
T^2S \arrow[r, "\nu S"'] & TS^2 \arrow[r, "T\mu"'] & TS
\end{tikzcd}$$

7.1. **Proposition.** *Let $\mathbb{S}$ be a monad on $\mathbf{C}$ and $T : \mathbf{C} \to \mathbf{C}$ be a 1-cell in $\mathcal{K}$. Fix a 2-cell $\sigma : 1_{\mathbf{C}} \to TS$. There is a bijective correspondence between pairs of 2-cells $(\lambda : ST \to TS, \nu : T^2 \to TS)$ making (σ, λ, ν) a wreath, and pairs $(\alpha : STS \to TS, \gamma : T^2S \to TS)$ such that (TS, α) is an $\mathbb{S}$-algebra and the following diagrams commute:*

$$\begin{tikzcd}
STS^2 \arrow[r, "\alpha S"] \arrow[d, "ST\mu"'] & TS^2 \arrow[d, "T\mu"] \\
STS \arrow[r, "\alpha"'] & TS,
\end{tikzcd}
\quad
\begin{tikzcd}
S \arrow[r, "\sigma S"] \arrow[d, "S\sigma"'] & TS^2 \arrow[d, "T\mu"] \\
STS \arrow[r, "\alpha"'] & TS,
\end{tikzcd}
\quad
\begin{tikzcd}
T^2S^2 \arrow[r, "\gamma S"] \arrow[d, "T^2\mu"'] & TS^2 \arrow[d, "T\mu"] \\
T^2S \arrow[r, "\gamma"'] & TS,
\end{tikzcd}$$

$$\begin{tikzcd}
T \arrow[r, "T\sigma"] \arrow[dr, "T\eta"'] & T^2S \arrow[d, "\gamma"] \\
& TS,
\end{tikzcd}
\quad
\begin{tikzcd}
TSTS \arrow[r, "T\alpha"] & T^2S \arrow[d, "\gamma"] \\
TS \arrow[u, "\sigma TS"] \arrow[r, "1_{TS}"'] & TS,
\end{tikzcd}
\quad
\begin{tikzcd}
STSTS \arrow[r, "\alpha TS"] \arrow[d, "ST\alpha"'] & TSTS \arrow[r, "T\alpha"] & T^2S \arrow[d, "\gamma"] \\
ST^2S \arrow[r, "S\gamma"'] & STS \arrow[r, "\alpha"'] & TS,
\end{tikzcd}$$

$$\begin{tikzcd}
T^2STS \arrow[r, "\gamma TS"] \arrow[d, "T^2\alpha"'] & TSTS \arrow[r, "T\alpha"] & T^2S \arrow[d, "\gamma"] \\
T^3S \arrow[r, "T\gamma"'] & T^2S \arrow[r, "\gamma"'] & TS
\end{tikzcd}$$

■

7.2. Theorem. *Given a monad* $\mathbb{S} = (S, \eta, (-)^{\mathbb{S}})$ *on* $\mathbf{C}$ *in* $\mathcal{K}$ *and a 1-cell* $T\colon \mathbf{C} \to \mathbf{C}$, *a wreath can be equivalently defined as follows:*

1. *A 2-cell* $\sigma\colon 1_{\mathbf{C}} \to TS$ *in* $\mathcal{K}$.
2. *A 1-cell* $(T, (-)^s)\colon (\mathbf{C}, \mathbb{S}) \to (\mathbf{C}, \mathbb{S})$ *in* $\mathrm{Mnd}(\mathcal{K})$.
3. *A pasting operator* $(-)^t\colon \mathcal{K}(\mathbf{X}, \mathbf{C})(1, TS) \to \mathcal{K}(\mathbf{X}, \mathbf{C})(T, TS)$.
4. *For every* A, $(\sigma A)^t = T\eta A$, *and for every* $f\colon B \to TSA$, $h\colon B \to SA$ *the diagrams*

$$\begin{tikzcd} B \arrow[r, "\sigma B"] \arrow[d, "h"'] & TSB \arrow[d, "Th^{\mathbb{S}}"] & B \arrow[r, "\sigma B"] \arrow[dr, "f"'] & TSB \arrow[d, "(f^s)^t"] \\ SA \arrow[r, "(\sigma A)^s"'] & TSA & & TSA \end{tikzcd} \tag{17}$$

   commute.

5. *For every* $g\colon C \to TSB$, $h\colon B \to SA$, $k\colon C \to B$ *and* $f\colon B \to TSA$ *the diagrams*

$$\begin{tikzcd} TC \arrow[r, "g^t"] \arrow[dr, "(Th^{\mathbb{S}}\cdot g)^t"'] & TSB \arrow[d, "Th^{\mathbb{S}}"] \\ & TSA \end{tikzcd} \tag{18}$$

   commute.

6. *For every* $f\colon B \to TSA$ *and* $g\colon C \to TSB$, *the diagrams*

$$\begin{tikzcd} SC \arrow[r, "g^s"] \arrow[dr, "((f^s)^t g)^s"'] & TSB \arrow[d, "(f^s)^t"] & TC \arrow[r, "g^t"] \arrow[dr, "((f^s)^t g)^t"'] & TSB \arrow[d, "(f^s)^t"] \\ & TSA & & TSA \end{tikzcd} \tag{19}$$

   commute.

Proof. We know that 2-cells $\alpha\colon STS \to TS$ correspond to the pasting operators $(-)^s$, and that 2-cells $\gamma\colon T^2S \to TS$ correspond to the pasting operators $(-)^t$ according to Lemma 2.2.

Assume that we have the conditions of the statement of the theorem. Then $\alpha = 1_{TS}{}^s$ and $\gamma = 1_{TS}{}^t$. We check that the conditions of Proposition 7.1 are satisfied. The fact that $(T, (-)^s)\colon (\mathbf{C}, \mathbb{S}) \to (\mathbf{C}, \mathbb{S})$ is a 1-cell in $\mathrm{Mnd}(\mathcal{K})$ means, according to Theorem 4.3, that (TS, α) is an $\mathbb{S}$-algebra and that the first of the diagrams in Proposition 7.1 commutes. The second is $T\mu \cdot \sigma S = T(1_S{}^{\mathbb{S}}) \cdot \sigma S = \sigma^s = (1_{TS}{}^s \cdot \eta TS \cdot \sigma)^s = 1_{TS}{}^s \cdot (\eta TS \cdot \sigma)^{\mathbb{S}} = \alpha \cdot S\sigma$, using (17). The third is $T\mu \cdot \gamma S = T(1_S{}^{\mathbb{S}}) \cdot 1_{TS^2}{}^t = (T\mu)^t = 1_{TS}{}^t \cdot T^2\mu = \gamma \cdot T^2\mu$, using (18) and the fact that $(-)^t$ respects blistering. The fourth is $\gamma \cdot T\sigma = 1_{TS}{}^t \cdot T\sigma = \sigma^t = T\eta$ using the fact that $(-)^t$ respects blistering. The fifth is $\gamma\cdot T\alpha\cdot\sigma TS = 1_{TS}{}^t\cdot T(1_{TS}{}^s)\cdot\sigma TS = (1_{TS}{}^s)^t\cdot\sigma TS = 1_{TS}$, using (17). The sixth is $\gamma\cdot T\alpha\cdot\alpha TS = (1_{TS}{}^s)^t\cdot 1_{TSTS}{}^s = ((1_{TS}{}^s)^t)^s = (\alpha^t)^s = (\gamma\cdot T\alpha)^s = (\gamma^s\cdot\eta T^2S\cdot T\alpha)^s = \gamma^s\cdot(\eta T^2S\cdot T\alpha)^{\mathbb{S}} = \alpha\cdot S\gamma\cdot ST\alpha$. And the last is $\gamma\cdot T\alpha\cdot\gamma TS = \alpha^t\cdot 1_{TSTS}{}^t = (1_{TS}{}^s)^t\cdot 1_{TSTS}{}^t = ((1_{TS}{}^s)^t)^t = (\alpha^t)^t = (\gamma\cdot T\alpha)^t = \gamma^t\cdot T^2\alpha = \gamma\cdot T\gamma\cdot T^2\alpha$.

In the opposite direction assume we have $\sigma: 1_{\mathbf{C}}\to TS$, $\alpha: STS\to TS$ and $\gamma: T^2S\to TS$ that satisfy the conditions of Proposition 7.1. Then for $f: B\to TSA$ we have that $f^s = \alpha A\cdot Sf$, and $f^t = \gamma A\cdot Tf$, that is, $(-)^s$ is the pasting operator corresponding to α and $(-)^t$ is the pasting operator corresponding to γ. The fact that (TS,α) is an $\mathbb{S}$-algebra together with the first commutative diagram of Proposition 7.1 means that $(T,(-)^s): (\mathbf{C},\mathbb{S})\to(\mathbf{C},\mathbb{S})$ is a 1-cell in $\mathrm{Mnd}(\mathcal{K})$. Now, $(\sigma A)^t = \gamma A\cdot T\sigma A = T\eta A$, using the triangle from Proposition 7.1. Furthermore

$$T(h^{\mathbb{S}})\cdot\sigma B = T\mu A\cdot TSh\cdot\sigma B = T\mu A\cdot\sigma SA\cdot h = \alpha A\cdot S\sigma A\cdot h = (\sigma A)^s\cdot h,$$

using the second commutative diagram from Proposition 7.1, and

$$(f^s)^t\cdot\sigma B = \gamma A\cdot T\alpha A\cdot TSf\cdot\sigma B = \gamma A\cdot T\alpha A\cdot\sigma TSA\cdot f = f,$$

using the fifth commutative diagram of Proposition 7.1 give us (17). (18) is given by

$$\begin{aligned} T(h^{\mathbb{S}})\cdot g^t &= T\mu A\cdot TSh\cdot\gamma B\cdot Tg = T\mu A\cdot\gamma SA\cdot T^2Sh\cdot Tg \\ &= \gamma A\cdot T^2\mu A\cdot T^2Sh\cdot Tg = (T(h^{\mathbb{S}})\cdot g)^t \end{aligned}$$

using the third commutative diagram from Proposition 7.1. Finally, the commutative diagrams

$$\begin{tikzcd}
SC \arrow[r, "Sg"] \arrow[dddr, bend right, "S((f^s)^t\cdot g)"'] & STSB \arrow[r, "\alpha B"] \arrow[d, "STSf"] & TSB \arrow[d, "TSf"] \\
 & STSTSA \arrow[r, "\alpha TSA"] \arrow[d, "ST\alpha A"'] & TSTSA \arrow[d, "T\alpha A"] \\
 & ST^2SA \arrow[d, "S\gamma A"] & T^2SA \arrow[d, "\gamma A"] \\
 & STSA \arrow[r, "\alpha A"'] & TSA,
\end{tikzcd}
\qquad
\begin{tikzcd}
TC \arrow[r, "Tg"] \arrow[dddr, bend right, "T((f^s)^t\cdot g)"'] & T^2SB \arrow[r, "\gamma B"] \arrow[d, "T^2Sf"] & TSB \arrow[d, "TSf"] \\
 & T^2STSA \arrow[r, "\gamma TSA"] \arrow[d, "T^2\alpha A"'] & TSTSA \arrow[d, "T\alpha A"] \\
 & T^3SA \arrow[d, "T\gamma A"] & T^2SA \arrow[d, "\gamma A"] \\
 & T^2SA \arrow[r, "\gamma A"'] & TSA,
\end{tikzcd}$$

give us (19). ■

8. Monads, algebras, distributive laws and wreaths in **Cat**

What we have been calling extension systems were first described by E. Manes as an alternative definition for monads on categories in [Manes, 1976]. Manes recognized, in giving a monad $\mathbb{S}$ on a category $\mathbf{C}$ as an extension system, that a mere function $|S|:|\mathbf{C}| \to |\mathbf{C}|$ and a mere $\mathbf{C}$-arrow valued function η, defined on $|\mathbf{C}|$ with no a priori naturality requirement, sufficed in the presence of an extension operator. Thus fewer axioms are required for extension systems on categories but we do have to show the naturality of the transformations that we introduce. However, formally it is very similar to what we have done so far in a general 2-category, and most of the proofs are similar to the proofs already given. Thus, in this section we present the precise statements for this important particular case and give the extra arguments necessary for the naturality of the various transformations. In the next section, we analyze the extra structure on **Cat** that enables the description of extension systems as functions, in terms of profunctors.

First we recall Exercise 1.3.12, page 32, of [Manes, 1976]:

8.1. THEOREM. *A monad $\mathbb{S}$ on a category $\mathbf{C}$ can be defined as follows: A* function *$|S|:|\mathbf{C}| \to |\mathbf{C}|$, for every $A \in \mathbf{C}$, an arrow $\eta A : A \to SA$, and for every morphism $f : B \to SA$ in $\mathbf{C}$, an $\mathbb{S}$-extension $f^{\mathbb{S}} : SB \to SA$ subject to the axioms: for every A in $\mathbf{C}$,*

$$(\eta A)^{\mathbb{S}} = 1_{SA},$$

for every $f : B \to SA$ in $\mathbf{C}$ and $g : C \to SB$, the diagrams

$$\begin{tikzcd} B \arrow[r, "\eta B"] \arrow[dr, "f"'] & SB \arrow[d, "f^{\mathbb{S}}"] \\ & SA, \end{tikzcd} \qquad \begin{tikzcd} SC \arrow[r, "g^{\mathbb{S}}"] \arrow[dr, "(f^{\mathbb{S}}\cdot g)^{\mathbb{S}}"'] & SB \arrow[d, "f^{\mathbb{S}}"] \\ & SA \end{tikzcd}$$

commute. ■

Recall then that for $\ell : B \to A$, we can define S on ℓ by the formula $S\ell = (\eta A \cdot \ell)^{\mathbb{S}}$, and that this makes $S : \mathbf{C} \to \mathbf{C}$ a functor and $\eta : 1_{\mathbf{C}} \to S$ a natural transformation. And the definition of μA is given by $\mu A = {1_{SA}}^{\mathbb{S}}$.

8.2. THEOREM. *Given a monad $\mathbb{S} = (S, \eta, (-)^{\mathbb{S}})$ on the category $\mathbf{C}$, an $\mathbb{S}$-algebra can be defined as follows: $\mathbb{B} = (B, (-)^{\mathbb{B}})$, where B is an object of $\mathbf{C}$, and $(-)^{\mathbb{B}}$ assigns to every arrow of the form $h : X \to B$ in $\mathbf{C}$, an extension $h^{\mathbb{B}} : SX \to B$ subject to the commutativity of the following diagrams (with $h : X \to B$ and $y : Y \to SX$):*

$$\begin{tikzcd} X \arrow[r, "\eta X"] \arrow[dr, "h"'] & SX \arrow[d, "h^{\mathbb{B}}"] \\ & B, \end{tikzcd} \qquad \begin{tikzcd} SY \arrow[r, "y^{\mathbb{S}}"] \arrow[dr, "(h^{\mathbb{B}}y)^{\mathbb{B}}"'] & SX \arrow[d, "h^{\mathbb{B}}"] \\ & B. \end{tikzcd}$$

A morphism of $\mathbb{S}$-algebras $(B, (-)^{\mathbb{B}})$ to $(A, (-)^{\mathbb{A}})$ can be defined as an arrow $\ell : B \to A$ in $\mathbf{C}$ subject to the commutativity of the diagram (where $h : X \to B$):

$$\begin{tikzcd} SX \arrow[r, "h^{\mathbb{B}}"] \arrow[dr, "(\ell\cdot h)^{\mathbb{A}}"'] & B \arrow[d, "\ell"] \\ & A. \end{tikzcd}$$

PROOF. Given $(B,(-)^{\mathbb{B}})$ define the action by ${1_B}^{\mathbb{B}}\colon SB \to B$. On the other hand, given an $\mathbb{S}$-algebra (B,β) and $h\colon X \to B$ in $\mathbf{C}$, define $h^{\mathbb{B}} = \beta \cdot Sh$. ∎

8.3. THEOREM. *Let* $\mathbb{S} = (S,\eta_S,(-)^{\mathbb{S}})$ *and* $\mathbb{T} = (T,\eta_T,(-)^{\mathbb{T}})$ *be monads on the category* $\mathbf{C}$. *A distributive law of* $\mathbb{S}$ *over* $\mathbb{T}$ *can be defined as follows. For every* A *in* $\mathbf{C}$ *an* $\mathbb{S}$-*algebra* $(TSA,(-)^{\lambda})$ *such that for every* A *in* $\mathbf{C}$, $(T\eta_S A \cdot \eta_T A)^{\lambda} = \eta_T SA$, *and for every* $f\colon B \to TSA$, $(f^{\lambda})^{\mathbb{T}}\colon (TSB,(-)^{\lambda}) \to (TSA,(-)^{\lambda})$ *is a morphism of* $\mathbb{S}$-*algebras.*

PROOF. Given the conditions of the theorem define $\alpha A = {1_{TSA}}^{\lambda}$. We show that $\alpha\colon STS \to TS$ is natural. So take $\ell\colon B \to A$. Observe that

$$\begin{aligned} TS\ell &= (\eta_T SA \cdot S\ell)^{\mathbb{T}} = ((T\eta_S A \cdot \eta_T A)^{\lambda} \cdot (\eta_S A \cdot \ell)^{\mathbb{S}})^{\mathbb{T}} \\ &= ((T\eta_S A \cdot \eta_T A)^{\lambda} \cdot \eta_S A \cdot \ell)^{\lambda})^{\mathbb{T}} = ((T\eta_S A \cdot \eta_T A \cdot \ell)^{\lambda})^{\mathbb{T}}. \end{aligned}$$

Thus

$$\begin{aligned} TS\ell \cdot \alpha B &= ((T\eta_S A \cdot \eta_T A \cdot \ell)^{\lambda})^{\mathbb{T}} \cdot {1_{TSB}}^{\lambda} = (((T\eta_S A \cdot \eta_T A \cdot \ell)^{\lambda})^{\mathbb{T}})^{\lambda} = (TS\ell)^{\lambda} \\ &= ({1_{TSA}}^{\lambda} \cdot \eta_S TSA \cdot TS\ell)^{\lambda} = {1_{TSA}}^{\lambda} \cdot (\eta_S TSA \cdot TS\ell)^{\mathbb{S}} = \alpha A \cdot STS\ell, \end{aligned}$$

and α is natural. ∎

8.4. THEOREM. *Given a monad* $\mathbb{S} = (S,\eta,(-)^{\mathbb{S}})$ *on a category* $\mathbf{C}$ *and an endofunctor* $T\colon \mathbf{C} \to \mathbf{C}$, *a wreath can be equivalently given as follows:*

1. *For every* A *in* $\mathbf{C}$, *an arrow* $\sigma A\colon A \to TSA$.

2. *For every* A, B *in* $\mathbf{C}$, *functions*

$$\begin{tikzcd} \mathbf{C}(SB,TSA) & \mathbf{C}(B,TSA) \arrow[l, "(-)^s"'] \arrow[r, "(-)^t"] & \mathbf{C}(TB,TSA) \end{tikzcd}$$

3. *For every* A, $(TSA,(-)^s)$ *is an* $\mathbb{S}$-*algebra and* $T(h^{\mathbb{S}})\colon (TSB,(-)^s) \to (TSA,(-)^s)$ *is a morphism of* $\mathbb{S}$-*algebras for every* $h\colon B \to SA$. *That is, the diagrams*

$$\begin{tikzcd} B \arrow[r, "\eta B"] \arrow[dr, "f"'] & SB \arrow[d, "f^s"] \\ & TSA \end{tikzcd} \qquad \begin{tikzcd} SC \arrow[r, "k^{\mathbb{S}}"] \arrow[dr, "(f^s k)^s"'] & SB \arrow[d, "f^s"] \\ & TSA \end{tikzcd} \qquad \begin{tikzcd} SC \arrow[r, "g^s"] \arrow[dr, "(T(h^{\mathbb{S}})g)^s"'] & TSB \arrow[d, "T(h^{\mathbb{S}})"] \\ & TSA \end{tikzcd} \tag{20}$$

commute for every $f\colon B \to TSA$, $g\colon C \to TSB$, *and* $k\colon C \to SB$.

4. *For every* A *we have that* $(\sigma A)^t = T\eta A$, *and for every* $f\colon B \to TSA$, $h\colon B \to SA$ *the diagrams*

$$\begin{tikzcd} B \arrow[r, "\sigma B"] \arrow[d, "h"'] & TSB \arrow[d, "T(h^{\mathbb{S}})"] \\ SA \arrow[r, "(\sigma A)^s"'] & TSA \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\sigma B"] \arrow[dr, "f"'] & TSB \arrow[d, "(f^s)^t"] \\ & TSA \end{tikzcd} \tag{21}$$

commute.

5. *For every* $g : C \to TSB$, $h : B \to SA$, $k : C \to B$ *and* $f : B \to TSA$, *the diagrams*

$$\begin{tikzcd} TC \arrow[r, "g^t"] \arrow[dr, "(T(h^{\mathbb{S}})g)^t"'] & TSB \arrow[d, "T(h^{\mathbb{S}})"] \\ & TSA \end{tikzcd} \qquad \begin{tikzcd} TC \arrow[r, "Tk"] \arrow[dr, "(f \cdot k)^t"'] & TB \arrow[d, "f^t"] \\ & TSA \end{tikzcd} \tag{22}$$

commute.

6. *For every* $f : B \to TSA$ *and* $g : C \to TSB$, *the diagrams*

$$\begin{tikzcd} SC \arrow[r, "g^s"] \arrow[dr, "((f^s)^t g)^s"'] & TSB \arrow[d, "(f^s)^t"] \\ & TSA \end{tikzcd} \qquad \begin{tikzcd} TC \arrow[r, "g^t"] \arrow[dr, "((f^s)^t g)^t"'] & TSB \arrow[d, "(f^s)^t"] \\ & TSA \end{tikzcd} \tag{23}$$

commute.

Proof. For $A \in \mathbf{C}$, take σA as given and define $\alpha A := {1_{TSA}}^s$ and $\gamma A := {1_{TSA}}^t$. We show first that σ is natural. Take $\ell : B \to A$ in $\mathbf{A}$, then

$$TS\ell \cdot \sigma B = T((\eta A \cdot \ell)^{\mathbb{S}}) \cdot \sigma B = (\sigma A)^s \cdot \eta A \cdot \ell = \sigma A \cdot \ell,$$

using (21) and (20). For the naturality of α we have:

$$\begin{aligned} TS\ell \cdot \alpha B &= T((\eta_S A \cdot \ell)^{\mathbb{S}}) \cdot {1_{TSA}}^s = (TS\ell)^s = ({1_{TSA}}^s \cdot \eta_S TSA \cdot TS\ell)^s \\ &= {1_{TSA}}^s \cdot (\eta_S TSA \cdot TS\ell)^{\mathbb{S}} = \alpha A \cdot STS\ell, \end{aligned}$$

using the equations of (20). And for the naturality of γ we have

$$TS\ell \cdot \gamma B = T((\eta_S A \cdot \ell)^{\mathbb{S}}) \cdot {1_{TSB}}^t = (TS\ell)^t = \gamma A \cdot T^2 S\ell.$$

■

8.5. Example. Beck's original example [Beck, 1969] has $\mathbb{S}$ the free monoid monad and $\mathbb{T}$ the free abelian group monad. Thus SA is the underlying set of the free monoid in A,

$$SA = \{[a_1, \dots, a_n] | n \in \mathbb{N}, a_j \in A, j = 1, \dots, n\},$$

$\eta_S : A \to SA$ is such that

$$\eta_S(a) = [a],$$

and for $h : A \to SB$, we have that

$$h^{\mathbb{S}}([a_1, \dots, a_n]) = h(a_1) \sim h(a_2) \sim \cdots \sim h(a_n),$$

where $\sim$ denotes concatenation ($[b_1, \ldots, b_k] \sim [c_1, \ldots, c_\ell] = [b_1, \ldots, b_k, c_1, \ldots, c_\ell]$). On the other hand, TA is the underlying set of the free group with A generators. Thus the elements of TA are formal sums

$$\sum_{a \in A} n_a \cdot a$$

with $n_a \in \mathbb{Z}$ for every $a \in A$, and only a finite number of the n_a are non-zero. $\eta_{_T} A : A \to TA$ is such that

$$\eta_{_T} A(c) = 1 \cdot c,$$

and for $k : A \to TB$, $k^{\mathbb{T}} : TA \to TB$ is

$$k^{\mathbb{T}} \left(\sum_{a \in A} n_a \cdot a \right) = \sum_{a \in A} n_a k(a),$$

where this last sum is taken in the abelian group TB.

Now, TSB has a monoid structure given by

$$\left(\sum_{w \in SB} m_w \cdot w \right) * \left(\sum_{w \in SB} n_w \cdot w \right) = \sum_{w \in SB} \left(\sum_{u \sim v = w} m_u n_u \right) \cdot w.$$

Thus, given $f : A \to TSB$, we define f^λ the unique monoid morphism $f^\lambda : SA \to TSB$ such that

$$\begin{tikzcd}
A \arrow[r, "\eta_S A"] \arrow[dr, "f"'] & SA \arrow[d, "f^\lambda"] \\
& TSB
\end{tikzcd}$$

commutes. That is, $f^\lambda([a_1, \ldots, a_n]) = f(a_1) * f(a_2) * \cdots * f(a_n)$. Let $g : C \to TSB$ and take $[c_1, \ldots, c_n] \in SC$. Assume that $g(c_i) = \sum_{w \in SB} k_w^{(i)} \cdot w$ for $i = 1, \ldots, n$. Then

$$((f^\lambda)^{\mathbb{T}} g)^\lambda([c_1, \ldots, c_n]) = \left(\sum_{w \in SB} k_w^{(1)} f^\lambda(w) \right) * \cdots * \left(\sum_{w \in SB} k_w^{(n)} f^\lambda(w) \right).$$

On the other hand

$$(f^\lambda)^{\mathbb{T}}(g^\lambda([c_1, \ldots, c_n])) = \sum_{w \in SB} \left(\sum_{w_1 \sim \cdots \sim w_n = w} k_{w_1}^{(1)} \cdots k_{w_n}^{(n)} \right) f^\lambda(w).$$

It takes just a moment to realize that to see that these are the same, it suffices to show that the operation $*$ distributes over the addition in TSA, but this is easy.

Then the calculation

$$\begin{aligned}
(\eta_{_T} SA \cdot \eta_{_S} A)^\lambda([a_1, \ldots, a_n]) &= \eta_{_T} SA \cdot \eta_{_S} A(a_1) * \cdots * \eta_{_T} SA \cdot \eta_{_S} A(a_n) \\
&= (1 \cdot [a_1]) * \cdots * (1 \cdot [a_n]) \\
&= 1 \cdot [a_1, \ldots, a_n] \\
&= \eta_{_T} SA([a_1, \ldots, a_n])
\end{aligned}$$

shows that we have a distributive law.

We have that $\lambda A = (T\eta_S A)^\lambda : STA \to TSA$,

$$\begin{aligned}\lambda A\left(\left[\sum_{a\in A} n_a^{(1)}\cdot a, \ldots, \sum_{a\in A} n_a^{(r)}\cdot a\right]\right) &= T\eta_S A\left(\sum_{a\in A} n_a^{(1)}\cdot a\right) * \cdots * T\eta_S A\left(\sum_{a\in A} n_a^{(r)}\cdot a\right)\\ &= \left(\sum_{a\in A} n_a^{(1)}\cdot [a]\right) * \cdots * \left(\sum_{a\in A} n_a^{(r)}\cdot [a]\right)\\ &= \sum_{[a_1,\ldots,a_r]} n_{a_1}^{(1)}\cdots n_{a_r}^{(r)}\cdot [a_1,\ldots,a_r]\end{aligned}$$

as expected.

8.6. Example. Another example from [Beck, 1969] has $\mathbb{T}$ arbitrary on $\mathbf{C}$ with coproducts, and $\mathbb{S}$ the constants monad. (It is done for the category of sets in Beck's paper but it works in the given context.) That is, take a fixed object C in $\mathbf{C}$, define for every A, $SA = A + C$, $\eta_S A = i_A : A \to A + C$ the canonical injection of the first summand, and for $h : A \to SB$ define $h^{\mathbb{S}} : SA \to SB$ as the unique arrow such that the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_S A"] \arrow[dr, "h"'] & A + C \arrow[d, "h^{\mathbb{S}}"] & C \arrow[l, "i_C"'] \arrow[dl, "i_C"] \\ & B + C & \end{tikzcd}$$

commutes.

Given $f : A \to TSB = T(B + C)$, define $f^\lambda : A + C \to TSB$ as the unique arrow that makes the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_S A"] \arrow[dr, "f"'] & A + C \arrow[d, "f^\lambda"] & C \arrow[l, "i_C"'] \arrow[d, "i_C"] \\ & T(B + C) & B + C \arrow[l, "\eta_T(B+C)"] \end{tikzcd}$$

commute. The commutative diagram

$$\begin{tikzcd} A \arrow[r, "\eta_S A"] \arrow[d, "\eta_S A"'] & A + C \arrow[d, "(\eta_T(A+C)\cdot\eta_S A)^\lambda"] & C \arrow[l, "i_C"'] \arrow[d, "i_C"] \\ A + C \arrow[r, "\eta_T(A+C)"'] & T(A + C) & A + C \arrow[l, "\eta_T(A+C)"] \end{tikzcd}$$

tells us that $(\eta_T(A + C)\cdot\eta_S A)^\lambda = \eta_T(A + C)$. For $h : A \to SB$ and $g : B \to T(D + C)$, the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta_S A"] \arrow[dr, "h"'] \arrow[ddr, bend right, "g^\lambda h"'] & A+C \arrow[d, "h^{\mathbb{S}}"] & C \arrow[l, "i_C"'] \arrow[dl, "i_C"] \arrow[d, "i_C"] \\
 & B+C \arrow[d, "g^\lambda"] & D+C \arrow[dl, "\eta_T(D+C)"] \\
 & T(D+C) &
\end{tikzcd}$$

tells us that $(g^\lambda \cdot h)^\lambda = g^\lambda \cdot h^{\mathbb{S}}$. And the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta_S A"] \arrow[dr, "f"'] \arrow[ddr, bend right, "(g^\lambda)^{\mathbb{T}} f"'] & A+C \arrow[d, "f^\lambda"] & & C \arrow[lll, "i_C"'] \arrow[dl, "i_C"'] \arrow[dd, "i_C"] \\
 & T(B+C) \arrow[d, "(g^\lambda)^{\mathbb{T}}"'] & B+C \arrow[l, "\eta_T(B+C)"'] \arrow[dl, "g^\lambda"] & \\
 & T(D+C) & & D+C \arrow[ll, "\eta_T(D+C)"]
\end{tikzcd}$$

tells us that $((g^\lambda)^{\mathbb{T}} \cdot f)^\lambda = (g^\lambda)^{\mathbb{T}} \cdot f^\lambda$. Therefore we have a distributive law, where $\lambda A : STA \to TSA$ is the unique arrow that makes the diagram

$$\begin{tikzcd}
TA \arrow[r, "i_{TA}"] \arrow[dr, "Ti_A"'] & TA+C \arrow[d, "\lambda A"] & C \arrow[l, "i_C"'] \arrow[d, "i_C"] \\
 & T(A+C) & A+C \arrow[l, "\eta_T(A+C)"]
\end{tikzcd}$$

commute.

8.7. Example. Another example from [Beck, 1969] is to take a monoid M and the monad $\mathbb{M}$ it defines on **Set**. That is, to every set A we assign the set $M \times A$, $\eta_M A : A \to M \times A$ is $\eta_M(a) = (e, a)$, where e is the unit element of M, and for $f = \langle f_1, f_2 \rangle : B \to M \times A$ we define

$$f^{\mathbb{M}}(m, b) = (m * f_1(b), f_2(b))$$

where $* : M \times M \to M$ is the multiplication of the monoid. It is clear that $f^{\mathbb{M}} \cdot \eta_M B = f$, and if $g = \langle g_1, g_2 \rangle : C \to M \times B$ and $(m, c) \in M \times C$, we have that

$$f^{\mathbb{M}}(g^{\mathbb{M}}(m, c)) = f^{\mathbb{M}}(m * g_1(c), g_2(c)) = ((m * g_1(c)) * f_1 g_2(c), f_2 g_2(c)),$$

whereas $(f^{\mathbb{M}} g)^{\mathbb{M}}(m, c) = (m * (g_1(c) * f_1 g_2(c)), f_2 g_2(c))$, so $g^{\mathbb{M}} f^{\mathbb{M}} = (g^{\mathbb{M}} f)^{\mathbb{M}}$. Given any monad $\mathbb{T}$ on **Set** and $f : A \to T(M \times B)$ define $f^\lambda : M \times A \to T(M \times B)$ as

$$f^\lambda(m, a) = T(m * (-) \times B)(f(a)).$$

By the naturality of η_T, we have

$$\begin{aligned}(\eta_T(M \times A) \cdot \eta_M A)^\lambda(m,a) &= T(m * (-) \times A)((\eta_T(M \times A) \cdot \eta_M A)(a)) \\ &= (T(m * (-) \times A) \cdot \eta_T(M \times A))(e,a) \\ &= \eta_T(M \times A)(m * (-) \times A)(e,a) \\ &= \eta_T(M \times A)(m,a).\end{aligned}$$

Furthermore $(f^\lambda \cdot \eta_M A)(a) = f^\lambda(e,a) = T(e*(-)\times B)(f(a)) = f(a)$. For $g = \langle g_1, g_2\rangle : C \to M \times A$ and $(m,c) \in M \times C$ we have

$$\begin{aligned}(f^\lambda g^{\mathbb{M}})(m,c) = f^\lambda(g^{\mathbb{M}}(m,c)) = f^\lambda(m * g_1(c), g_2(c)) &= T(m * g_1(c) * (-) \times A)(f(g_2(c))) \\ &= T(m * (-) \times A)(T(g_1(c) * (-) \times A)(f(g_2(c)))) \\ &= T(m * (-) \times A)((f^\lambda g)(c)) = (f^\lambda g)^\lambda(m,c).\end{aligned}$$

It is not hard to show that for any $m \in M$, $f^\lambda \circ (m * (-) \times B) = T(m * (-) \times A) \circ f^\lambda$. Then for $h : C \to T(M \times A)$ and $(m,c) \in M \times C$ we have

$$\begin{aligned}((f^\lambda)^{\mathbb{T}} h)^\lambda(m,c) &= (T(m * (-) \times A) \cdot (f^\lambda)^{\mathbb{T}})(h(c)) \\ &= ((\eta_T(M \times A) \cdot (m * (-) \times A))^{\mathbb{T}} \cdot (f^\lambda)^{\mathbb{T}})(h(c)) \\ &= (((\eta_T(M \times A) \cdot (m * (-) \times A))^{\mathbb{T}} \cdot f^\lambda)^{\mathbb{T}})(h(c)) \\ &= ((T(m * (-) \times A) \cdot f^\lambda)^{\mathbb{T}})(h(c)) = (f^\lambda \cdot (m * (-) \times B))^{\mathbb{T}}(h(c)) \\ &= ((f^\lambda)^{\mathbb{T}} \cdot \eta_T(M \times B) \cdot (m * (-) \times B))^{\mathbb{T}}(h(c)) \\ &= ((f^\lambda)^{\mathbb{T}} \cdot T(m * (-) \times B))(h(c)) = ((f^\lambda)^{\mathbb{T}} \cdot h^\lambda)(m,c).\end{aligned}$$

We thus have a distributive law of $\mathbb{M}$ over $\mathbb{T}$.

8.8. Example. An example taken from [Varacca, 2003] has the monad $\mathbb{P}$ on **Set** of finite non-empty subsets, whose structure is given by, for any set X, $PX = \{X_0 \subseteq X | X_0$ finite non-empty$\}$, $\eta_P X : X \to PX$ is $\eta_P X(x) = \{x\}$, and for any function $f : Y \to PX$, $f^{\mathbb{P}}(Y_0) = \bigcup_{y \in Y_0} f(y)$.

On the other hand we have the monad $\mathbb{V}$ on **Set** of *indexed valuations*, whose structure is given as follows. For a set X, $V(X)$ has as elements equivalent classes of spans $(0,\infty) \xleftarrow{r} K \xrightarrow{\chi} X$ in **Set** with K finite, and the given span is equivalent to the span $(0,\infty) \xleftarrow{r'} K' \xrightarrow{\chi'} X$ iff there is a bijection $\kappa : K \to K'$ such that the diagram

$$\begin{tikzcd}
& K \arrow[ld, "r"'] \arrow[rd, "\chi"] \arrow[dd, "\kappa"] & \\
(0,\infty) & & X \\
& K' \arrow[lu, "r'"] \arrow[ru, "\chi'"'] &
\end{tikzcd}$$

commutes. The arrow $\eta_V X : X \to V(X)$ is such that the span associated to $x \in X$ is

$$(0,\infty) \xleftarrow{\ulcorner 1 \urcorner} 1 \xrightarrow{\ulcorner x \urcorner} X.$$

Take a function $f : Y \to V(X)$. For $y \in Y$ denote the span $f(y)$ by $(0,\infty) \xleftarrow{r^y} K_y \xrightarrow{\chi^y} X$. Define $f^{\mathbb{V}} : V(Y) \to V(X)$ on the span $(0,\infty) \xleftarrow{q} J \xrightarrow{\psi} Y$ as the span

$$(0,\infty) \xleftarrow{r} \textstyle\coprod_{j\in J} K_{\psi(j)} \xrightarrow{\chi} X,$$

where r and χ are the unique functions such that the diagram

$$\begin{tikzcd}
(0,\infty) \arrow[d, "q(j)\cdot -"'] & K_{\psi(j)} \arrow[l, "r^{\psi(j)}"'] \arrow[d, "inc_j"'] \arrow[dr, "\chi^{\psi(j)}"] & \\
(0,\infty) & \coprod_{j\in J} K_{\psi(j)} \arrow[l, "r"] \arrow[r, "\chi"'] & X
\end{tikzcd}$$

commutes for every $i \in I$. Thus for $k \in K_{\psi(j)}$ we have $\chi(k) = \chi^{\psi(j)}(k)$ and $r(k) = q(j) \cdot r^{\psi(j)}(k)$.

It is clear that $f^{\mathbb{V}} \circ \eta_V Y = f$, and for $g : Z \to V(Y)$ write $g(z)$ as

$$(0,\infty) \xleftarrow{q^z} J_z \xrightarrow{\psi^z} Y. \tag{24}$$

Then $f^{\mathbb{V}} g^{\mathbb{V}}$ on the span $(0,\infty) \xleftarrow{p} I \xrightarrow{\zeta} Z$ is the span

$$(0,\infty) \xleftarrow{\quad r \quad} \coprod_{j\in\coprod_{i\in I} J_{\zeta(i)}} K_{\psi^{\zeta(i)}(j)} \xrightarrow{\quad \chi \quad} X,$$

where for a $k \in K_{\psi(j)}$ with $j \in J_{\zeta(i)}$ we have

$$\xi(k) = \xi^{\psi^{\zeta(i)}(j)}(k) \quad \text{ and } \quad r(k) = p(i) \cdot q^{\zeta(i)}(j) \cdot r^{\psi^{\zeta(i)}(j)}(k).$$

On the other hand, $(f^{\mathbb{V}} g)^{\mathbb{V}}$ on the same span gives us

$$(0,\infty) \xleftarrow{\quad r' \quad} \coprod_{i\in I} \coprod_{j\in J_{\zeta(i)}} K_{\psi^{\zeta(i)}(j)} \xrightarrow{\quad \chi' \quad} X$$

where for a $k \in K_{\psi^{\zeta(i)}(j)}$ with $j \in J_{\zeta(i)}$ and $i \in I$ we have

$$\xi'(k) = \xi^{\psi^{\zeta(i)}(j)}(k) \quad \text{ and } \quad r'(k) = p(i) \cdot q^{\zeta(i)}(j) \cdot r^{\psi^{\zeta(i)}(j)}(k).$$

Then the canonical isomorphism

$$\coprod_{j\in\coprod_{i\in I} J_{\zeta(i)}} K_{\psi^{\zeta(i)}(j)} \xrightarrow{\quad \simeq \quad} \coprod_{i\in I} \coprod_{j\in J_{\zeta(i)}} K_{\psi^{\zeta(i)}(j)}$$

shows that these spans are the same element of $V(X)$. That is, the diagram

$$\begin{tikzcd}
V(Z) \arrow[r, "g^{\mathbb{V}}"] \arrow[dr, "(f^{\mathbb{V}}g)^{\mathbb{V}}"'] & V(Y) \arrow[d, "f^{\mathbb{V}}"] \\
 & V(X)
\end{tikzcd}$$

commutes. We have then shown that we have a monad $\mathbb{V}$.

We now give the distributive law. Take $h\colon Y \to P(V(X))$, and take a span

$$(0,\infty) \xleftarrow{q} J \xrightarrow{\psi} Y.$$

For every choice function $\ell\colon J \to \bigcup_{j\in J} h(\psi(j))$ (that is, $\ell(j) \in h(\psi(j))$), if $\ell(j)$ is the span

$$(0,\infty) \xleftarrow{r^j} K_j \xrightarrow{\chi^j} X,$$

for $j \in J$, we form the span $S(\ell, q, \psi)$ as

$$(0,\infty) \xleftarrow{r} \textstyle\coprod_{j\in J} K_j \xrightarrow{\chi} X,$$

where for $k \in K_j$ we define $\chi(k) = \chi^j(k)$ and $r(k) = q(j) \cdot r^j(k)$. We define

$$h^\lambda((q,\psi)) = \{S(\ell, q, \psi) | \ell\colon J \to \bigcup_{j\in J} h(\psi(j)) \text{ is a choice function}\}.$$

We show that this defines a distributive law of $\mathbb{V}$ over $\mathbb{P}$. We must show first that this definition of h^λ produces a structure of $\mathbb{V}$-algebra on $P(V(X))$. We know that $\eta_V Y(y) = (\ulcorner 1 \urcorner, \ulcorner y \urcorner)$. Then a choice function $\ell\colon 1 \to h(y)$ is simply to pick an element in $h(y)$. Then $S(\ell, \ulcorner 1 \urcorner, \ulcorner y \urcorner)$ is this chosen element, thus $h^\lambda \circ \eta_V Y = h$.

Then a typical element in $h^\lambda(g^{\mathbb{V}}(\,(0,\infty) \xleftarrow{p} I \xrightarrow{\zeta} Z))$ is formed as follows. Assume $g(z)$ is given by (24) for every $z \in Z$. Then for every $i \in I$ and every $j \in J_{\zeta(i)}$ we take an element

$$(0,\infty) \xleftarrow{r^{ij}} K_{ij} \xrightarrow{\chi^{ij}} X$$

in $h(\psi^{\zeta(i)}(j))$. Then the element of $h^\lambda(g^{\mathbb{V}}(p,\zeta))$ is

$$(0,\infty) \xleftarrow[r]{\quad} \textstyle\coprod_{(i,j)\in\coprod_{i\in I} J_{\zeta(i)}} K_{ij} \xrightarrow[\chi]{\quad} X,$$

where for $k \in K_{ij}$, $\chi(k) = \chi^{ij}(k)$ and $r(k) = q^{\zeta(i)}(j) \cdot r^{ij}(k)$.

On the other hand, a typical element of $(h^\lambda g^{\mathbb{V}})^\lambda(p,\zeta)$ is formed as follows. For every $i \in I$ and $j \in J_{\zeta(i)}$ take an element

$$(0,\infty) \xleftarrow{r^{ij}} K_{ij} \xrightarrow{\chi^{ij}} X$$

in $h(\psi^{\zeta(i)}(j))$. Then the element is formed as

$$(0,\infty) \xleftarrow[r']{} \coprod_{i\in I}\coprod_{j\in J_{\zeta(i)}} K_{ij} \xrightarrow[\chi']{} X,$$

where for $k \in K_{ij}$ $(j \in J_{\zeta(i)})$ we have that $\chi'(k) = \chi^{ij}(k)$ and $r'(k) = p(i)\cdot q^{\zeta(i)}(j)\cdot r^{ij}(k)$.

The canonical isomorphism $\coprod_{i\in I}\coprod_{j\in J_{\zeta(i)}} K_{ij} \to \coprod_{(i,j)\in\coprod_{i\in I} J_{\zeta(i)}} K_{ij}$ then tell us that $h^\lambda g^{\mathbb{V}} = (h^\lambda g)^\lambda$.

The condition $(\eta_P V(X)\cdot \eta_V X)^\lambda = \eta_P V(X)$ is easy, since there is only one possible choice function for a given element in $V(X)$.

Finally, take $k\colon Z \to P(V(Y))$ and $h\colon Y \to P(V(X))$. We must check that $(h^\lambda)^P k^\lambda = ((h^\lambda)^{\mathbb{P}} k)^\lambda$. Take $(0,\infty) \xleftarrow{p} I \xrightarrow{\zeta} Z$ in $V(Z)$. Then a typical element of $((h^\lambda)^P k^\lambda)(p,\zeta)$ is formed as follows: for every $i \in I$ chose a span $(0,\infty) \xleftarrow{q^i} J_i \xrightarrow{\psi^i} Y$ in $k(\zeta(i))$, then for every $j \in J_i$ choose a span $(0,\infty) \xleftarrow{r^{ij}} K_{ij} \xrightarrow{\chi^{ij}} X$ in $h(\psi^i(j))$; then the element in question is

$$(0,\infty) \xleftarrow{r} \coprod_{(i,j)\in\coprod_{i\in I} J_i} K_{ij} \xrightarrow{\chi} X$$

where for $k \in K_{ij}$, $(j \in J_i)$ we have $\chi(k) = \chi^{ij}(k)$ and $r(k) = p(i)\cdot q^i(j)\cdot r^{ij}(k)$.

On the other hand, a typical element in $((h^\lambda)^P k)^\lambda(p,\zeta)$ is formed as follows: for every $i \in I$ we take a span $(0,\infty) \xleftarrow{q^i} J_i \xrightarrow{\psi^i} Y$ in $k(\zeta(i))$, then for every $j \in J_i$ we take a span $(0,\infty) \xleftarrow{r^{ij}} K_{ij} \xrightarrow{\chi^{ij}} X$ in $h(\psi^i(j))$; then the element in question is

$$(0,\infty) \xleftarrow{r'} \coprod_{i\in I}\coprod_{j\in J_i} K_{ij} \xrightarrow{\chi'} X$$

where for $k \in K_{ij}$, $(j \in J_i)$ we have $\chi'(k) = \chi^{ij}(k)$ and $r'(k) = p(i)\cdot q^i(j)\cdot r^{ij}(k)$. Therefore, $(h^\lambda)^P k^\lambda = (h^\lambda k)^\lambda$. Thus we have a distributive law.

Compare with the proof given in [Varacca, 2003].

8.9. Example. This one is taken from [Manes & Mulry, 2007]. Let $\mathbb{S}$ be the free monoid monad (described in example 8.5) and let $\mathbb{T}$ the submonad of $\mathbb{S}$ of nonempty words:

$$TA = \{[a_1,\dots,a_n] | n \in \mathbb{N}\backslash\{0\}, a_j \in A, j = 1,\dots,n\},$$

for any set A. Give TSA the following binary operation

$$[U_1,\dots,U_k] * [V_1,\dots,V_\ell] = [U_1,\dots,U_{k-1},U_k \sim V_1,V_2,\dots,V_\ell]$$

where $U_1,\dots,U_k,V_1,\dots,V_\ell$ are elements of SA, and $\sim$ denotes concatenation. It is immediate to see that TSA with this operation is a monoid. Therefore, if for an $f\colon B \to TSA$ we define $f^\lambda\colon SA \to TSA$ such that

$$f^\lambda([a_1,\dots,a_n]) = f(a_1) * \cdots * f(a_n),$$

we obtain an $\mathbb{S}$-algebra $(TSA, (\ )^{\lambda})$. Now, for $[a_1, \ldots, a_k] \in SA$ we have

$$(T\eta_S A \cdot \eta_T A)^{\lambda}([a_1, \ldots, a_k]) = [[a_1]] * \cdots * [[a_k]] = [[a_1, \ldots, a_k]] = \eta_T SA([a_1, \ldots, a_k]).$$

Finally we must show that for $f : B \to TSA$, $(f^{\lambda})^{\mathbb{T}} : TSB \to TSA$ is a monoid morphism with respect to the operation $*$ on both, TSB and TSA. Let $[U_1, \ldots, U_k], [V_1, \ldots, V_\ell] \in TSB$. Then

$$\begin{aligned}(f^{\lambda})^{\mathbb{T}}([U_1, \ldots, U_k] * [V_1, \ldots, V_\ell]) &= (f^{\lambda})^{\mathbb{T}}([U_1, \ldots, U_{k-1}, U_k \sim V_1, V_2, \ldots, V_\ell]) \\ &= f^{\lambda}(U_1) \sim \cdots \sim f^{\lambda}(U_{k-1}) \sim f^{\lambda}(U_k \sim V_1) \sim f^{\lambda}(V_2) \sim f^{\lambda}(V_\ell)\end{aligned}$$

and

$$\begin{aligned}&(f^{\lambda})^{\mathbb{T}}([U_1, \ldots, U_k]) * (f^{\lambda})^{\mathbb{T}}([V_1, \ldots, V_\ell]) \\ &= (f^{\lambda}(U_1) \sim \cdots \sim f^{\lambda}(U_k)) * (f^{\lambda}(V_1) \sim \cdots \sim f^{\lambda}(V_\ell)) \\ &= f^{\lambda}(U_1) \sim \cdots \sim (f^{\lambda}(U_k) * f^{\lambda}(V_1)) \sim \cdots \sim f^{\lambda}(V_\ell).\end{aligned}$$

Since it direct to see that $f^{\lambda}(U_k) * f^{\lambda}(V_1) = f^{\lambda}(U_k \sim V_1)$, we have a distributive law of $\mathbb{S}$ over $\mathbb{T}$.

8.10. Example. We close our set of examples with a wreath from [Lack & Street, 2002]. Take a short exact sequence in the category of groups

$$1 \to A \to E \to G \to 1.$$

For $g \in G$ and $a \in A$ denote the action of g on a by $a^g = g^{-1}ag \in A$, and let $\rho : G \times G \to A$ be a normalized cocycle corresponding to the extension. Let $\mathbb{A}$ be the monad on **Set** determined by the group A (as described in example 8.7, there for a monoid M), and take the functor $T = G \times - : \mathbf{Set} \to \mathbf{Set}$. For the wreath define $\sigma X : X \to G \times A \times X$ such that $\sigma A(x) = (e, e, x)$, and for $f = \langle f_1, f_2, f_3 \rangle : Y \to G \times A \times X$, define $f^s : A \to G \times A \times X$ as

$$f^s(a, y) = (f_1(y), a^{f_1(y)} \cdot f_2(y), f_3(y)),$$

and $f^t : G \times Y \to G \times A \times X$ as

$$f^t(g, y) = (g \cdot f_1(y), \rho(g, f_1(y)) \cdot f_2(y), f_3(y)).$$

We only verify that the last condition from Theorem 8.4 is satisfied and leave the rest to the reader. We have that

$$(f^s)^t(g, a, y) = (g \cdot f_1(y), \rho(g, f_1(y)) \cdot a^{f_1(y)} \cdot f_2(y), f_3(y)).$$

Thus for $\ell = \langle \ell_1, \ell_2, \ell_3 \rangle : Z \to G \times A \times X$ we have that $(f^s)^t(\ell^t(g, z))$ is

$$(g \cdot \ell_1(z) \cdot f_1\ell_3(z), \rho(g \cdot \ell_1(y), f_1\ell_3(z)) \cdot (\rho(g, \ell_1(z)) \cdot \ell_2(z))^{f_1\ell_3(z)} \cdot f_2\ell_3(z), f_3\ell_3(z)),$$

whereas $((f^s)^t\ell)^t(g, z)$ is

$$(g \cdot \ell_1(z) \cdot f_1\ell_3(z), \rho(g, \ell_1(z) \cdot f_1\ell_3(z) \cdot \rho(\ell_1(z), f_1\ell_3(z)) \cdot \ell_2(z)^{f_1\ell_3(z)} \cdot f_2\ell_3(z), f_3\ell_3(z)).$$

The fact that they are equal follows from the cocycle condition

$$\rho(g \cdot h, k) \cdot \rho(g, h)^k = \rho(g, h \cdot k)\rho(h, k)$$

for $g, h, k \in G$.

9. Extension systems in the bicategory of profunctors

In the Introduction we also mentioned *extension systems* of the form $(\varphi, \psi : S \dashv H, \eta, \xi)$ on an object $\mathbf{C}$ in a 2-category $\mathcal{K}$. We recall that $\eta : 1_{\mathbf{C}} \to S : \mathbf{C} \to \mathbf{C}$ and that $\xi : S \to HS$ is to be the mate of a 2-cell $\mu : SS \to S$ so that explicitly we have

$$\xi = (S \xrightarrow{\varphi S} HS^2 \xrightarrow{H\mu} HS)$$

and

$$\mu = (S^2 \xrightarrow{S\xi} SHS \xrightarrow{\psi S} S).$$

We begin this section by reconciling our two usages of the term *extension system.*

9.1. Lemma. *The data (S, η, μ) constitute a monad if and only if the data $(\varphi, \psi : S \dashv H, \eta, \xi)$ satisfy the following three equations:*

$$\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & S \arrow[d, "\xi"] & S^2 \arrow[r, "S\xi"] & SHS \arrow[d, "\psi S"] & S^2 \arrow[r, "S\xi"] & SHS \arrow[r, "\psi S"] \arrow[d, "\xi HS"'] & S \arrow[d, "\xi"] \\
 & HS, & S \arrow[u, "\eta S"] \arrow[r, "1_S"'] & S, & & HSHS \arrow[r, "H\psi S"'] & HS.
\end{tikzcd} \tag{25}$$

Proof. Apply S to the first of these equations and post-compose the result with ψS. From the definition of μ this gives the monad equation $\mu \cdot S\eta = 1_S$. Conversely, given $\mu \cdot S\eta = 1_S$, the mate $1_{\mathbf{C}} \to HS$ with respect to the given adjunction of 1_S is φ while that of $\mu \cdot S\eta$ is $H\mu \cdot HS\eta \cdot \varphi = H\mu \cdot \varphi S \cdot \eta = \xi \cdot \eta$, from the definition of ξ. The second of the equations given above is immediately the monad equation $\mu \cdot \eta S = 1_S$ and conversely. Starting with the third equation, apply $S-$ and note the commutativity of each of the three squares pasted west or south of the result, as shown in the diagram below. From the definition of μ, the outer square gives the associativity equation $\mu \cdot S\mu = \mu \cdot \mu S$.

$$\begin{tikzcd}
S^3 \arrow[r, "S^2\xi"] \arrow[d, "S\xi S"'] & S^2HS \arrow[r, "S\psi S"] \arrow[d, "S\xi HS"'] & S^2 \arrow[d, "S\xi"] \\
SHS^2 \arrow[r, "SHS\xi"] \arrow[d, "\psi S^2"'] & SHSHS \arrow[r, "SH\psi S"'] \arrow[d, "\psi SHS"'] & SHS \arrow[d, "\psi S"] \\
S^2 \arrow[r, "S\xi"] & SHS \arrow[r, "\psi S"] & S.
\end{tikzcd}$$

Finally, starting with $\mu \cdot S\mu = \mu \cdot \mu S$, get the third equation of the statement from

$$\begin{tikzcd}
 & & SHS \arrow[ddr, "\psi S"] & & & \\
 & SHS^2 \arrow[ur, "SH\mu"] \arrow[dr, "\psi S^2"] & & & & \\
 & & S^2 \arrow[r, "\mu"'] & S \arrow[ddr, "\varphi S"] & & \\
S^2 \arrow[uur, "S\varphi S"] \arrow[urr, "1_{S^2}"'] \arrow[ddrr, "\varphi S^2"] \arrow[ddd, "S\varphi S"'] & & & & & \\
 & & & & HS^2 \arrow[dddr, "H\mu"] & \\
 & & HS^3 \arrow[urr, "HS\mu"] \arrow[r, "H\mu S"'] \arrow[d, "HS^2\varphi S"'] & HS^2 \arrow[d, "HS\varphi S"'] \arrow[dr, "1_{HS^2}"] & & \\
SHS^2 \arrow[rr, "\varphi SHS^2"] \arrow[d, "SH\mu"'] & & HS^2HS^2 \arrow[r, "H\mu HS^2"] \arrow[d, "HS^2H\mu"'] & HSHS^2 \arrow[r, "H\psi S^2"] \arrow[d, "HSH\mu"'] & HS^2 \arrow[dr, "H\mu"] & \\
SHS \arrow[rr, "\varphi SHS"'] & & HS^2HS \arrow[r, "H\mu HS"'] & HSHS \arrow[rr, "H\psi S"'] & & HS
\end{tikzcd}$$

■

Now recall, from [Borceux, 1994] say, the bicategory **Pro** of categories, profunctors, and equivariant 2-cells. There is a pseudofunctor $(-)_* : \mathbf{Cat} \to \mathbf{Pro}$ which is the identity on objects and takes a functor $F : \mathbf{X} \to \mathbf{A}$ to the profunctor with $F_*(A, X) = \mathbf{A}(A, FX)$. For any parallel pair of functors $F, G : \mathbf{X} \to \mathbf{A}$, there is a bijection between equivariant 2-cells $F_* \to G_*$ and natural transformations $F \to G$. That is, $(-)_*$ is locally fully faithful. Most importantly, for any functor F, there is an adjunction $F_* \dashv F^*$ in **Pro**, where $F^*(X, A) = \mathbf{A}(FX, A)$. It follows that to describe a monad (S, η, μ) in **Cat** via an extension system, we can either use the general theory of the bulk of this paper or avail ourselves of the adjunction $S_* \dashv S^*$ in **Pro** and proceed using the elementary definition provided by the data of Lemma 9.1, subject to the equations (25).

Now, note that in **Pro** the composite $S^*S_* : \mathbf{C} \to \mathbf{C}$ has $S^*S_*(B, A) = \mathbf{C}(SB, SA)$ so that to give a two cell $\xi : S_* \to S^*S_*$ in **Pro** is to give an equivariant family of functions $(\xi_{B,A})_{B,A}$ taking arrows $B \to SA$ to $SB \to SA$, whose effect is denoted, as usual, $f \mapsto f^{\mathbb{S}}$. Equivariance of ξ in the variable B means that, for all $g : C \to B$, we have $(fg)^{\mathbb{S}} = f^{\mathbb{S}} \cdot Sg$, which is just the blister equation (3) of Section 2. Equivariance of ξ in the variable A means that, for all $u : A \to X$, we have $(Su \cdot f)^{\mathbb{S}} = Su \cdot f^{\mathbb{S}}$ which follows easily from (6). The point here is that when the data for the elementary definition is expanded in **Pro**, it amounts to the same data required for the non-elementary definition in **Cat**. This reconciles our usage of the term *extension system* in both cases.

In Section 8, we remarked that Manes' original presentation of monads in **Cat** does not require that $S : \mathbf{C} \to \mathbf{C}$ be given as a functor nor that $\eta : 1_{\mathbf{C}} \to S$ be given as a natural transformation. It is interesting to note that the bicategory **Pro** also provides a venue to discuss this simplification, in terms of the formal theory of monads. For any category $\mathbf{C}$, write $|\mathbf{C}|$ for the set of objects of $\mathbf{C}$ regarded as a discrete category. In **Pro** we have, on the object $|\mathbf{C}|$, the monad $\mathbb{C}$ where $\mathbb{C}(B, A) = \mathbf{C}(B, A)$. The canonical, identity on objects, functor $K\colon|\mathbf{C}|\to\mathbf{C}$ admits an (op)action of $\mathbb{C}$ that we call $\kappa\colon K_*\mathbb{C}\to K_*$. The (C,A) component of this action, $\sum_{B\in|\mathbf{C}|}K_*(C,B)\times\mathbb{C}(B,A)\to K_*(C,A)$ is determined by the $\mathbf{C}(C,B)\times\mathbf{C}(B,A)\to\mathbf{C}(C,A)$ which are given by the composition of $\mathbf{C}$.

The profunctor K_*, together with κ, exhibits $\mathbf{C}$ as the Kleisli object in **Pro** for the monad $\mathbb{C}$. Moreover, the equivalence mediated by this data restricts to functors, in the sense that to give a functor $F\colon\mathbf{C}\to\mathbf{D}$ is to give a functor $\widehat{F}\colon|\mathbf{C}|\to\mathbf{D}$ together with an action $\widehat{F}_*\mathbb{C}\to\widehat{F}_*$. Similarly, to give a natural transformation $\tau\colon F\to G\colon\mathbf{C}\to\mathbf{D}$ is to give a natural transformation $\widehat{\tau}\colon\widehat{F}\to\widehat{G}\colon|\mathbf{C}|\to\mathbf{D}$ that respects the actions. An action $\widehat{F}_*\mathbb{C}\to\widehat{F}_*$ has a mate $\mathbb{C}\to\widehat{F}^*\widehat{F}_*$. The two action equations translate to make $\mathbb{C}\to\widehat{F}^*\widehat{F}_*$ a morphism of monads on $|\mathbf{C}|$ and conversely.

In the case of the data given originally by Manes, we have in these terms: $\eta\colon K\to\widehat{S}\colon|\mathbf{C}|\to\mathbf{C}$ and now the requisite morphism of monads $\mathbb{C}\to\widehat{S}^*\widehat{S}_*$ is given by the composite

$$\mathbb{C}=K^*K_*\xrightarrow{K^*\eta_*}K^*\widehat{S}_*\xrightarrow{(-)^{\mathbb{S}}}\widehat{S}^*\widehat{S}_*$$

— the monad morphism equations arising from two of those satisfied by the extension operator $(-)^{\mathbb{S}}$.

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