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GENERALIZED BROWN REPRESENTABILITY IN HOMOTOPY CATEGORIES: ERRATUM

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ABSTRACT. Propositions 4.2 and 4.3 of the author's article (Theory Appl. Categ. 14 (2005), 451-479) are not correct. We show that their use can be avoided and all remaining results remain correct.

Propositions 4.2 and 4.3 of the author's [3] are not correct and I am grateful to J. F. Jardine for pointing it out. In fact, consider the diagram D sending the one morphism category to the point Δ_0 in the homotopy category $\mathrm{Ho}(\mathbf{SSet})$ of simplicial sets. The standard weak colimit of D is the standard weak coequalizer

$$\Delta_0 \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} \Delta_0$$

This weak coequalizer is the homotopy pushout

$$\begin{array}{ccc} \Delta_0 & \longrightarrow & P \\ \uparrow & & \uparrow \\ \Delta_0 \amalg \Delta_0 & \longrightarrow & \Delta_0 \end{array}$$

But P is weakly equivalent to the circle and thus it is not contractible. Hence the standard weak colimit of D is not weakly equivalent to the colimit of D.

A fatal error is in the last line of part I of the proof of Proposition 4.2. We could neglect the identity morphisms in the construction of weak colimits via coproducts and weak coequalizers. It means that we take the coproduct

$$\coprod_{e:d\to d'} Dd$$

indexed by all non-identity morphisms e of $\mathcal{D}$. Then, taking for $\mathcal{D}$ the three-element chain and for $D : \mathcal{D} \to \mathbf{SSet}$ the constant diagram at Δ_0, the modified standard weak colimit of D is not contractible again. In fact, R. Jardine showed that for each small category $\mathcal{D}$ and each diagram $D : \mathcal{D} \to \mathbf{SSet}$ constant at Δ_0, the weak standard colimit of D is

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weakly equivalent to the graph of $\mathcal{D}$. The graph of the three-element chain $0 < 1 < 2$ contains the boundary of the simplex Δ_1 but not Δ_1 itself. It does not seem possible to find a modification of standard weak colimits satisfying Proposition 4.2.

These incorrect results were used only for proving Theorems 5.4 and 5.7 (numbered references here and below are for results in [3]). Fortunately, we can avoid their use. Theorem 5.4 says that the functor

$$E_\lambda : \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$$

is essentially surjective on objects, i.e., that for each X in $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ there is K in $\mathrm{Ho}(\mathcal{K})$ with $E_\lambda K \cong X$. Theorem 5.7 then adds that E_λ is full. This formulation is not correct and one has to replace it by E_λ being *essentially surjective* in the sense that every morphism $f : X \to Y$ in $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ is isomorphic to $E_\lambda(g)$ in the category of morphisms of $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ for some $g : K \to L$. The latter means the commutativity of a square

$$\begin{array}{ccc} E_\lambda K & \xrightarrow{E_\lambda(g)} & E_\lambda L \\ \cong\uparrow & & \uparrow\cong \\ X & \xrightarrow[f]{} & Y \end{array}$$

An essentially surjective functor is essentially surjective on objects (by using $f = \mathrm{id}_X$).

Thus the correct formulation of Theorems 5.4 and 5.7 is the following statement.

1.1. Theorem. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^1_λ) and (G^2_λ). Then the functor*

$$E_\lambda : \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$$

is essentially surjective.

Proof. Since (G^1_λ) implies (G^4_λ) (by Remark 3.4), it follows from Corollary 5.2 that it suffices to prove that $\mathrm{Ind}_\lambda P_\lambda$ is essentially surjective. At first, we prove that it is essentially surjective on objects. We start in the same way as in the proof of Theorem 5.4. We consider an object X in $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ and express it as a λ-filtered colimit $(\delta_d : Dd \to X)$ of the canonical diagram $D : \mathcal{D} \to \mathrm{Ho}(\mathcal{K}_\lambda)$. Without any loss of generality, we can assume that Dd belong to $\mathcal{K}_{cf}$ for all d in $\mathcal{D}$. Then we take a standard weak colimit $(\overline{\delta}_d : \overline{D}d \to K)$ of the lifting $\overline{D}$ of D along E_λ. Standard weak colimits are given by a construction in $\mathcal{K}$ and let $\overline{K}$ be the resulting object. Thus $K = P\overline{K}$ and we can assume that $\overline{K}$ is in $\mathcal{K}_{cf}$. In $\mathcal{K}$, λ-filtered colimits commute both with coproducts and with homotopy pushouts. The second claim follows from the fact that homotopy pushouts are constructed via pushouts and (cofibration, trivial fibration) factorizations and the latter preserve λ-filtered colimits (by (G^1_λ)). Consequently, λ-filtered colimits commute in $\mathcal{K}$ with the construction of standard weak colimits and thus $\overline{K}$ is a λ-filtered colimit $\alpha_\mathcal{E} : \overline{K}_\mathcal{E} \to \overline{K}$ of objects $\overline{K}_\mathcal{E}$ giving standard weak colimits of λ-small subdiagrams $D_\mathcal{E} : \mathcal{E} \to \mathrm{Ho}(\mathcal{K}_\lambda)$

of D. All objects $\overline{K}_{\mathcal{E}}$ belong to $\mathcal{K}_\lambda$. We can even assume that each $\mathcal{E}$ has a terminal object $d_{\mathcal{E}}$. Let $u_{\mathcal{E}} : Dd_{\mathcal{E}} \to P_\lambda \overline{K}_{\mathcal{E}}$ be the corresponding component of a standard weak colimit cocone. Clearly, these morphisms form a natural transformation from D to the the diagram consisting of $P_\lambda \overline{K}_{\mathcal{E}}$. Since $d_{\mathcal{E}}$ is a terminal object of $\mathcal{E}$, there is a morphism $s_{\mathcal{E}} : P_\lambda \overline{K}_{\mathcal{E}} \to Dd_{\mathcal{E}}$ with $s_{\mathcal{E}} u_{\mathcal{E}} = \mathrm{id}_{Dd_{\mathcal{E}}}$. Thus each morphism $u_{\mathcal{E}}$ is a split monomorphism. Thus the colimit

$$u : X \to E_\lambda \overline{K} = (\mathrm{Ind}_\lambda P_\lambda)\overline{K}$$

of $E_\lambda(u_{\mathcal{E}})$ is a λ-pure monomorphism (see [1]). This argument is a correct part of the proof of Proposition 4.3.

Now, consider a pullback

$$\begin{array}{ccc}
\mathcal{K}_\lambda & \xrightarrow{P_\lambda} & \mathrm{Ho}(\mathcal{K}_\lambda) \\
{\scriptstyle \overline{\overline{D}}}\uparrow & & \uparrow{\scriptstyle D} \\
\overline{\overline{\mathcal{D}}} & \xrightarrow[\overline{P}]{} & \mathcal{D}
\end{array}$$

We will show that the functor $\overline{P} : \overline{\overline{\mathcal{D}}} \to \mathcal{D}$ is final. Observe that, for each object $\overline{d}$ in $\overline{\overline{\mathcal{D}}}$, we have $\overline{P}(\overline{d}) = QR\overline{d}$ and the same for morphisms. For each object d in $\mathcal{D}$, there is $\overline{d}$ in $\overline{\overline{\mathcal{D}}}$ with $d = \overline{P}(\overline{d})$. Consider two morphisms $f_1 : d \to \overline{P}(\overline{d}_1)$ and $g_1 : d \to \overline{P}(\overline{d}_2)$ in $\mathcal{D}$. Since $\mathcal{D}$ is λ-filtered, there is a commutative square (where we replace $\overline{P}$ by QR)

$$\begin{array}{ccc}
QR\overline{d}_1 & \xrightarrow{h_2} & e \\
{\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle h_1} \\
d & \xrightarrow[g_1]{} & QR\overline{d}_2
\end{array}$$

in $\mathcal{D}$. Let $\alpha : \mathrm{Id}_{\mathcal{K}} \to R_f$ and $\beta : R_c \to \mathrm{Id}_{\mathcal{K}}$ be natural transformation given by fibrant and cofibrant replacements. Since α is a pointwise trivial cofibration and β is a pointwise trivial fibration, both $QR\alpha$ and $QR\beta$ are natural isomorphisms. Thus we get the following zig-zags in the comma category $d \downarrow \overline{P}$

$$f_1 \xleftarrow{QR\beta_{\overline{d}_1}} f_2 \xrightarrow{QR\alpha_{R_c\overline{d}_1}} f_3 \xrightarrow{QRh_2} f_4$$

and

$$g_1 \xleftarrow{QR\beta_{\overline{d}_2}} g_2 \xrightarrow{QR\alpha_{R_c\overline{d}_2}} g_3 \xrightarrow{QRh_1} g_4$$

Here, $f_2 = QR(\beta_{\overline{d}_1})^{-1} \cdot f_1$ and f_3, f_4 are the corresponding compositions; analogously for g_1. Since $f_4 = g_4$, f_1 and g_1 are connected by a zig-zag in the category $d \downarrow \overline{P}$. Thus the functor $\overline{P}$ is final. Consequently,

$$X \cong \mathrm{colim}\, D \cong \mathrm{colim}(\overline{P} \cdot \overline{\overline{D}}).$$

The latter object is isomorphic to

$$(\mathrm{Ind}_\lambda P_\lambda)\,\mathrm{colim}\,\overline{\overline{D}}$$

provided that the diagram $\overline{\overline{D}}$ is λ-filtered. Evidently, this diagram is λ-filtered when $X \cong (\mathrm{Ind}_\lambda P_\lambda)L$ for some L in $\mathcal{K}$. Thus X belongs to the essential image of $\mathrm{Ind}_\lambda P_\lambda$ if and only if the corresponding diagram $\overline{\overline{D}}_X$ is λ-filtered (the index X denotes that $\overline{\overline{D}}$ belongs to X).

Consider X and a λ-small subcategory $\mathcal{A}$ of $\overline{\overline{\mathcal{D}}}_X$. Since the functor P_λ preserves λ-small coproducts, we can assume that $\mathcal{A}$ consists of morphisms

$$h_i : \overline{\overline{D}}_X d_1 \to \overline{\overline{D}}_X d_2,$$

$i \in I$ where $\mathrm{card}\, I < \lambda$. Then h_i, $i \in I$ are morphisms $u\delta_{d_1} \to u\delta_{d_2}$ in the diagram $\overline{\overline{D}}_Y$ for $Y = (\mathrm{Ind}_\lambda P_\lambda)\overline{K}$. Since the latter diagram is λ-filtered, there are $f : P_\lambda A \to Y$ and $h : \overline{\overline{D}}_X d_2 \to A$ such that h coequalizes all h_i, $i \in I$ and $fP_\lambda(h) = u\delta_{d_2}$. Since u is λ-pure, there is $g : P_\lambda A \to X$ such that $gP_\lambda(h) = \delta_{d_2}$. Thus $h : \delta_{d_2} \to g$ coequalizes h_i, $i \in I$ in $\overline{\overline{\mathcal{D}}}_X$. We have proved that $\overline{\overline{\mathcal{D}}}_X$ is λ-filtered and thus

$$X \cong (\mathrm{Ind}_\lambda P_\lambda)\,\mathrm{colim}\,\overline{\overline{D}}_X.$$

We have thus proved that $\mathrm{Ind}_\lambda P_\lambda$ is essentially surjective on objects.

Now, consider a morphism $h : X \to Y$ in $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$ and express X and Y as canonical λ-filtered colimits $(\delta_{Xd} : D_X d \to X)$ and $(\delta_{Yd} : D_Y d \to Y)$ of objects from $\mathrm{Ho}(\mathcal{K}_\lambda)$. Let $\overline{\overline{D}}_X : \overline{\overline{\mathcal{D}}}_X \to \mathcal{K}_\lambda$ and $\overline{\overline{D}}_Y : \overline{\overline{\mathcal{D}}}_Y \to \mathcal{K}_\lambda$ be the diagrams constructed above. The prescription $Hf = hf$ yields a functor

$$H : \mathcal{D}_X \to \mathcal{D}_Y$$

such that $D_Y H = D_X$. Since the diagrams $\overline{\overline{D}}_X$ and $\overline{\overline{D}}_Y$ are given by pullbacks, there is a functor

$$\overline{H} : \overline{\overline{\mathcal{D}}}_X \to \overline{\overline{\mathcal{D}}}_Y$$

such that $\overline{\overline{D}}_Y \overline{H} = \overline{\overline{D}}_X$. This gives a morphism

$$\overline{h} : \mathrm{colim}\,\overline{\overline{D}}_X \to \mathrm{colim}\,\overline{\overline{D}}_Y$$

such that $(\mathrm{Ind}_\lambda P_\lambda)\overline{h} \cong h$. Thus the functor $\mathrm{Ind}_\lambda P_\lambda$ is essentially surjective. ■

We say that a locally λ-presentable model category $\mathcal{K}$ is *essentially λ-Brown* provided that the functor E_λ is essentially surjective. Thus the correct formulation of Corollary 5.8 is as follows.

1.2. Corollary. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^1_λ) and (G^2_λ). Then $\mathcal{K}$ is essentially λ-Brown.*

Thus, for a combinatorial model category $\mathcal{K}$, there are arbitrarily large regular cardinals λ such that $\mathcal{K}$ is essentially λ-Brown. Recall that $\mathcal{K}$ is λ-Brown if E_λ is essentially surjective on objects and full. The original formulation of Corollary 5.9 remains true for stable model categories.

1.3. Proposition. *Let $\mathcal{K}$ be a locally λ-presentable model category satisfying the conditions (G^1_λ), (G^2_λ) and such that E_λ reflects isomorphisms. Then $\mathcal{K}$ is λ-Brown.*

Proof. Consider a morphism $h : E_\lambda PK_1 \to E_\lambda PK_2$. In the proof of the Theorem above, we have found a morphism $\overline{h} : L_1 \to L_2$ such that $E_\lambda P\overline{h} \cong h$. Following the construction of objects L_1 and L_2, there are morphisms $t_i : K_i \to L_i$, $i = 1, 2$, such that

$$(E_\lambda P(t_1), E_\lambda P(t_2)) : h \to E_\lambda P\overline{h}$$

is an isomorphism. The reason is that the canonical diagram of K_i with respect to $\mathcal{K}_\lambda$ is a subdiagram of the diagram $\overline{\overline{D}}_i$ constructed in the proof above and $L_i = \operatorname{colim} \overline{\overline{D}}_i$, $i = 1, 2$. Since E_λ reflects isomorphisms, $P(t_1)$ and $P(t_2)$ are isomorphisms. We get the morphism

$$h' = P(t_2)^{-1} P(\overline{h}t_1) : PK_1 \to PK_2$$

with $E_\lambda(h') = h$. Thus E_λ is full. ∎

1.4. Corollary. *Let $\mathcal{K}$ be a combinatorial stable model category. Then there are arbitrarily large regular cardinals λ such that $\mathcal{K}$ is λ-Brown.*

Proof. It follows from Proposition 6.1, the proof of Proposition 6.4 and the Proposition above. ∎

1.5. Remark. 5.11 (3) implies a correct substitute of Proposition 4.3 saying that, for a stable locally λ-presentable model category $\mathcal{K}$ satisfying the conditions (G^1_λ) and (G^2_λ), the functor $P : \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ sends λ-filtered colimits of λ-presentable objects to minimal weak λ-filtered colimits.

The distinction between being Brown and essentially Brown is important, which is manifested by the following strengthening of Proposition 6.11.

1.6. Proposition. *Let $\mathcal{K}$ be a model category which is λ-Brown for arbitrarily large regular cardinals λ. Then idempotents split in* $\mathrm{Ho}(\mathcal{K})$.

Proof. Let $f : PK \to PK$ be an idempotent in $\mathrm{Ho}(\mathcal{K})$. There is a regular cardinal λ such that $\mathcal{K}$ is λ-Brown and K is λ-presentable in $\mathcal{K}$. Since idempotents split in $\mathrm{Ind}_\lambda(\mathrm{Ho}(\mathcal{K}_\lambda))$, there are morphisms $p : E_\lambda K \to E_\lambda L$ and $u : E_\lambda L \to E_\lambda K$ such that $pu = \mathrm{id}_{E_\lambda}$ and $E_\lambda f = up$. Since E_λ is full, there are morphisms $\overline{p} : K \to L$ and $\overline{u} : L \to K$ with $E_\lambda \overline{p} = p$ and $E_\lambda \overline{u} = u$. Since E_λ is faithful on $\mathrm{Ho}(\mathcal{K}_\lambda)$, we have $pu = \mathrm{id}_K$ and $f = up$. Hence idempotents split in $\mathrm{Ho}(\mathcal{K})$. ∎

Now, Remark 6.12 implies that **SSet** cannot be λ-Brown for arbitrarily large regular cardinals λ.

The proof of Proposition 6.10 is lacking a verification that each object A from $\mathrm{Ho}(\mathcal{K}_\mu)$ is μ-small. Also, the proof that $\mathrm{Ho}(\mathcal{K}_\mu)$ is μ-perfect has to be corrected because the coproduct of fibrant objects does not need to be fibrant and its fibrant replacement destroys the coproduct structure in $\mathcal{K}$. I am grateful to B. Chorny for bringing these gaps to my attention. In what follows, a corrected proof of Proposition 6.12 is presented. Its first two paragraphs and the last one remain unchanged. This means that we are replacing the third paragraph of the proof.

We will show that each object A from $\mathrm{Ho}(\mathcal{K}_\mu)$ is μ-small. Recall that, following Remark 2.4 (2), $\mathcal{K}$ satisfies the conditions (G^i_μ) for $i = 1, 2$, as well, and thus it is μ-Brown (see the Corollary above). Consider a morphism

$$f : A \to \coprod_{i \in I} K_i$$

in $\mathrm{Ho}(\mathcal{K})$. We have

$$\coprod_{i \in I} K_i \cong \operatorname{colim} \coprod_{j \in J} K_j$$

where the colimit is taken over all subsets $J \subseteq I$ having cardinality smaller than μ. Since

$$E_\mu \operatorname{colim} \coprod_{j \in J} K_j \cong \operatorname{colim} E_\mu \coprod_{j \in J} K_j$$

(see 6.5), $E_\mu f$ factorizes through some $E_\mu \coprod_{j \in J} K_j$. Thus f factorizes through some $\coprod_{j \in J} K_j$.

It remains to show that $\mathrm{Ho}(\mathcal{K}_\mu)$ is μ-perfect. Consider a morphism $f : A \to \coprod_{i \in I} K_i$ in $\mathrm{Ho}(\mathcal{K})$ where $\mathit{card}\, I < \mu$. Each K_i, $i \in I$, is a minimal μ-filtered colimit

$$(k^i_j : A_{ij} \to K_i)_{j \in J_i}$$

of objects $A_{ij} \in \mathrm{Ho}(\mathcal{K}_\mu)$. Hence all subcoproducts $X = \coprod_{i \in I} A_{ij_i}$ belong to $\mathrm{Ho}(\mathcal{K}_\mu)$. Let Z be their minimal weak μ-filtered colimit. Since minimal weak colimits commute with coproducts, $Z \cong \coprod_{i \in I} K_i$.

Thus $E_\mu f$ factorizes through some $E_\mu X$ of some subcoproduct X and therefore f factorizes through X, which yields [2], 3.3.1.2 in the definition of perfectness.

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