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ALGEBRAIC REAL ANALYSIS

PETER FREYD

ABSTRACT. An effort to initiate the subject of the title: the basic tool is the study of the abstract closed interval equipped with certain equational structures.

The title is wishful thinking; there ought to be a subject that deserves the name "algebraic real analysis."

Herein is a possible beginning.

For reasons that can easily be considered abstruse we were led to the belief that the closed interval—not the entire real line—is the basic structure of interest. Before describing those abstruse reasons, a theorem:

Let G be a compact group and $\mathtt{I}$ the closed interval. (We will not say which closed interval; to do so would define it as a part of the reals, belying the view of the closed interval as the fundamental structure.) Let $\boldsymbol{\mathcal{C}(G)}$ be the set of continuous maps from G to $\mathtt{I}$. We wish to view this as an algebraic structure, where the word "algebra" is in the very general sense, something described by operations and equations. In the case at hand, the only operators that will be considered right now are the constants, "top" and "bottom," denoted $\top$ and $\bot$, and the binary operation of "midpointing," denoted $\boldsymbol{x|y}$. (There are axioms that will define the notion of "closed midpoint algebra" but since the theorem is about specific examples they're not now needed.) $\mathcal{C}(G)$ inherits this algebraic structure in the usual way ($f|g$, for example, is the map that sends $\sigma \in G$ to $(f\sigma)\big|(g\sigma) \in \mathtt{I}$). We use the group structure on G to define an action of G on $\mathcal{C}(G)$, thus obtaining a representation of G on the group of automorphisms of the closed midpoint algebra. (Fortunately no knowledge of the axioms is necessary for the definition of automorphism, or even homomorphism.) Let $(\mathcal{C}(G), \mathtt{I})$ be the set of closed-midpoint-algebra homomorphisms from $\mathcal{C}(G)$ to $\mathtt{I}$. Again we obtain an action of G.

0.1. THEOREM. *There is a unique G-fixed point in* $(\mathcal{C}(G), \mathtt{I})$

There is an equivalent way of stating this:

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0.2. THEOREM. *There is a unique G-invariant homomorphism from the closed midpoint algebra $\mathcal{C}(G)$ to $\mathtt{I}$.*

This theorem is mostly von Neumann's: the unique G-invariant homomorphism is integration, that is, it is the map that sends $f : G \to \mathtt{I}$ to $\int f \,\mathrm{d}\sigma$. But it is not entirely von Neumann's: we have just characterized integration on compact groups without a single mention of inequalities or limits. The only non-algebraic notion that appeared was at the very beginning in the definition of $\mathcal{C}(G)$ as the set of *continuous* maps (in time we will obtain a totally algebraic definition).

The fact that we are stating this theorem for $\mathtt{I}$ and not the reals, $\mathbb{R}$ is critical. Consider the special case when G is the one-element group; the theorem says that the identity map on $\mathtt{I}$ is the only midpoint-preserving endomorphism that fixes $\top$ and $\bot$ (we said that the theorem is mostly von Neumann's; this part is not). It actually suffices to assume that the endomorphism fixes any two points but with the axiom of choice and a standard Hamel-basis argument we can find $2^{2^{\aleph_0}}$ counterexamples for this assertion if $\mathtt{I}$ is replaced with $\mathbb{R}$.

We do not need a group structure or even von Neumann to make the point. Consider this remarkably simple characterization of definite integration on $\mathcal{C}(\mathtt{I})$, continuous functions from $\mathtt{I}$ to $\mathtt{I}$:

$$\fint \top \,\mathrm{d}x = \top \qquad\qquad \fint \bot \,\mathrm{d}x = \bot$$

$$\fint f(x) \mid g(x) \,\mathrm{d}x = \fint f(x) \,\mathrm{d}x \;\;\mid\;\; \fint g(x) \,\mathrm{d}x$$

$$\fint f(x) \,\mathrm{d}x = \fint f(\bot|x) \,\mathrm{d}x \;\;\mid\;\; \fint f(x|\top) \,\mathrm{d}x$$ [1]

No inequalities. No limits. The first three equations say just that integration is a homomorphism of closed midpoint algebras. The fourth equation says that the mean value of a function on $\mathtt{I}$ is the midpoint of the two mean-values of the function on the two halves of $\mathtt{I}$.

The fourth equation—as any numerical or theoretical computer scientist will tell you—is a "fixed-point characterization." When Church proved that his and Gödel's notion of computability were coextensive he used the fact that all computation can be reduced to finding fixed points. (The word "point" here is traditional but misleading. The fixed point under consideration here is, as it was for Church, an operator on functions—rather far removed from the public notion of point.)

If we seek a fixed point of an operator the first thing to try, of course, is to iterate the operator on some starting point and to hope that the iterations converge. So let

$$\fint_0 f(x) \,\mathrm{d}x$$

[1]There was an appendix for Latex macros, but the powers that be deemed such to be beneath the dignity of this journal.

denote a "first approximation" operator, to wit, an arbitrary operator from $\mathcal{C}(\mathtt{I})$ to $\mathtt{I}$ that satisfies the first three equations. Define a sequence of operators, iteratively, as follows:

$$\fint_{n+1} f(x)\ \mathrm{d}x = \fint_{n} f(\bot|x)\ \mathrm{d}x \ \ \Big| \ \ \fint_{n} f(x|\top)\ \mathrm{d}x$$

where each new operator is to be considered an improvement of the previous. (One should verify that we automatically maintain the first three equations in each iteration.) Thus the phrase "fixed point" here turns out to mean an operator so good that it can not be improved. (What is being asserted is that there is a unique such operator.) Wonderfully enough: *no matter what closed-midpoint homomorphism is chosen as a first approximation, the values of these operators are guaranteed to converge.*

If we take the initial approximation to be evaluation on $\bot$, that is, if we take

$$\fint_{0} f(x)\ \mathrm{d}x = f(\bot)$$

then what we are saying turns out to be only that "left Riemann sums" work for integration. If we take the initial approximation to be $f(\top)$ we obtain "right Riemann sums." If we start with the midpoint of these two initial operators, that is if we take the initial approximation to be $f(\bot)|f(\top)$ then we are saying that "trapezoid sums" work. For "Simpson's rule" take it to be $\frac{1}{6}(f(\bot) + 4f(\bot|\top) + f(\top))$. [2]

1. Diversion: The Proximate Origins, or: Coalgebraic Real Analysis

The point of departure for this approach to analysis is the use of the closed interval as the fundamental structure; the reals are constructed therefrom. A pause to describe how I was prompted to explore such a view.

[2] If this paper's title is to be taken seriously we will be obliged to give an algebraic description of the limits used in the previous paragraph. Here's one way: let $\mathtt{I}^{\mathbb{N}} = \prod_{\mathbb{N}} \mathtt{I}$ denote the closed midpoint algebra of all sequences in $\mathtt{I}$. The first step is to identify sequences that agree almost everywhere to obtain the quotient algebra $\mathtt{I}^{\mathbb{N}} \to A$. The latter will be shown—for entirely algebraic reasons—to have a closed-midpoint homomorphism to $\mathtt{I}$ and we could use any such homomorphism to define the sequential limits appearing in the previous paragraph. There is, of course, an obvious objection: we would be assigning limits to all sequences not just convergent ones; worse, the homomorphism would be not at all unique. Remarkably enough we can turn this inside out: an element in $\mathtt{I}^{\mathbb{N}}$ is convergent iff it is in the joint equalizer of all homomorphisms of the form $\mathtt{I}^{\mathbb{N}} \to A \to \mathtt{I}$. Put another way, $\lim a_n = b$ iff $h\{a_n\} = b$ whenever $h : \mathtt{I}^{\mathbb{N}} \to \mathtt{I}$ is a closed-midpoint homomorphism that respects almost-everywhere equivalence.

This approach can be easily modified to supply limits of functions at points in arbitrary topological spaces. More interesting: it can be used to define derivatives. Let F be the set of all functions from the standard interval $[-1. +1]$ to itself such that $|f(x)| \le |x|$. We will regard F as a closed midpoint algebra where the identity function is taken as the top and its negation as bottom. Now identify functions that name the same germ at 0 (that is, that agree on some neighborhood of 0) to obtain a quotient algebra $F \to A$. The joint equalizer of all homomorphisms of the form $F \to A \to [-1, +1]$ is precisely the set of functions differentiable at 0; the common values delivered by all such homomorphisms are the derivatives of those functions. That is, $f'(0) = b$ iff $H(f) = b$ whenever $H : F \to [-1, +1]$ is a closed-midpoint homomorphism that depends only on the germs at 0 of its arguments.

The community of theoretical computer scientists (in the European sense of the phrase) had found something called "initial-algebra" definitions of data types to be of great use. Such definitions typically tell one how inductive programs—and then recursive programs—are to be defined and executed. It then became apparent that some types were better handled in a dual fashion: something called "final-coalgebra" definitions. Such can tell one how "co-inductive" and "co-recursive" programs are to be defined and executed. (One must really resist here the temptation to say "co-defined" and "co-executed.")

Thus began a search for a final-coalgebra definition of that ancient data type, the reals. There is, actually, always a trivial answer to the question: every object is automatically the final coalgebra of the functor constantly equal to that object. What was being sought was not just a functor with a final coalgebra isomorphic to the object in question but a functor that supplies its final coalgebra with the structure of interest. In 1999 an answer was found not for the reals but for the closed interval.[3] (To this date, no one has found a functor whose final coalgebra is *usefully* the reals.)

Consider, then, the category whose objects are sets with two distinguished points, **top** and **bottom**, denoted $\top$ and $\bot$ and whose maps are the functions that preserve $\top$ and $\bot$. Given a pair of objects, X and Y, we define their **ordered wedge**, denoted $\boldsymbol{X} \vee \boldsymbol{Y}$ to be the result of identifying the top of X with the bottom of Y. [4] This construction can clearly be extended to the maps to obtain the "ordered-wedge functor."

The closed interval can be defined as the final coalgebra of the functor that sends X to $X \vee X$. Let me explain.

First (borrowing from the topologists' construction of the ordinary wedge), $X \vee Y$ is taken as the subset of pairs, $\langle x, y\rangle$, in the product $X \times Y$ that satisfy the disjunctive condition: $x = \top$ *or* $y = \bot$. A map, then, from X to $X \vee X$ may be construed as a pair of self-maps, denoted $\overset{\wedge}{\boldsymbol{x}}$ and $\overset{\vee}{\boldsymbol{x}}$, such that for all x either $\overset{\vee}{x} = \top$ or $\hat{x} = \bot$. The final coalgebra we seek is the terminal object in the category whose objects are these structures.[5] To be formal, begin with the category whose objects are quintuples $\langle X, \bot, \top, {}^\wedge, {}^\vee\rangle$ where $\bot, \top \in X$, and ${}^\wedge, {}^\vee$ signify self-maps on X. The maps of the category are the functions that preserve the two constants and the two self-maps. Then cut down to the full subcategory of objects that satisfy the conditions:

$$\overset{\wedge}{\top} = \top = \overset{\vee}{\top}$$

$$\overset{\wedge}{\bot} = \bot = \overset{\vee}{\bot}$$

$$\forall_x \, [\hat{x} = \bot \quad \text{or} \quad \overset{\vee}{x} = \top]$$

$$\bot \neq \top$$ [6]

[3]First announced in a note I posted on 22 December, 1999, http://www.mta.ca/∼cat-dist/1999/99-12

[4]The word "wedge" and its notation are borrowed from algebraic topology where $X \vee Y$ is the result of joining the (single) base-point of X to that of Y.

[5]This is the specialization of the general notion: given an endofunctor T, a "T-coalgebra" is just a map $X \to TX$.

[6]I did not say "with a pair of distinguished points" above. What I said was "with *two* distinguished points"

We will call such a structure an **interval coalgebra**.[7]

I said that we will eventually construct the reals from $\mathtt{I}$. But if one already has the reals then one may chose $\bot < \top$ and define a coalgebra structure on $[\bot, \top]$ as

$$\overset{\vee}{x} = \min(2x - \bot, \top)$$

$$\hat{x} = \max(2x - \top, \bot)$$

Note that each of the two self-maps evenly expands a half interval to fill the entire interval—one the bottom half the other the top half. We will call them **zoom operators**. (By convention we will not say "zoom in" or "zoom out." All zooming herein is expansive, not contractive.)

The general definition of "final coalgebra" reduces—in this case—to the characterization of such a closed interval, $\mathtt{I}$, as the terminal object in this category.[8]

The general notion of "co-induction" reduces—in this case—to the fact that given any quintuple $\langle X, \bot, \top, {}^\wedge, {}^\vee \rangle$ satisfying the above-displayed conditions there is a unique $X \xrightarrow{f} \mathtt{I}$ such that $f(\bot) = \bot$, $f(\top) = \top$, $f(\overset{\vee}{x}) = (fx)^\vee$ and $f(\hat{x}) = (fx)^\wedge$. If $\mathtt{I}$ is taken as the unit interval, $[0, 1]$, in the classical setting (and if one pays no attention to computational feasibility) a quick and dirty construction of f is to define the binary expansion of $f(x) \in \mathtt{I}$ by iterating (forever) the following procedure:

If $\overset{\vee}{x} = \top$ then
        emit 1 and replace x with $\hat{x}$
    else
        emit 0 and replace x with $\overset{\vee}{x}$.

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[7] The modal operations $\Diamond$ for *possibly* and $\Box$ for *necessarily* have received many formalizations but it is safe to say that no one allows simultaneously both $\Diamond\Phi \neq \top$ and $\Box\Phi \neq \bot$: "less than possible but somewhat necessary." (The coalgebra condition can be viewed as a much weakened excluded middle: when the two unary operations are trivialized—that is, both taken to be the identity operation—then $\Diamond\Phi = \top$ *or* $\Box\Phi = \bot$ becomes just standard excluded middle.)

If we assume, for the moment, that $\top$ and $\bot$ are fixed points for $\Diamond$ and $\Box$ then we have an example of an interval coalgebra where $\Box\Phi$ is $\overset{\wedge}{\Phi}$ and $\Diamond\Phi$ is $\overset{\vee}{\Phi}$. The finality of $\mathtt{I}$ yields what may be considered truth values for sentences (e.g. the truth value of $\odot = \bot|\top$ translates to "entirely possible but totally unnecessary" and a truth value greater than $\top|\odot$ means "necessarily entirely possible").

The fixed-point conditions are not, in fact, appropriate—true does not imply necessarily true nor does possibly false imply false—but, fortunately, they're not needed: an easy corollary of the finality of $\mathtt{I}$ says that it suffices to assume the disjointness of the orbits of $\top$ and $\bot$ under the action of the two operators. If we work in a context in which the modal operations are monotonic (that is, when Φ implies Ψ it is the case that $\Box\Phi$ implies $\Box\Psi$ and $\Diamond\Phi$ implies $\Diamond\Psi$) it suffices to assume that $\Box\Phi$ implies Φ, that Φ implies $\Diamond\Phi$ and that $\Box^n\top$ never implies $\Diamond^n\bot$. If this last condition has never previously been formalized it's only because no one ever thought of it.

The same treatment of modal operators holds when $\Diamond$ is interpreted as *tenable* and $\Box$ as *certain*; or $\Diamond$ as *conceivable* and $\Box$ as *known*.

This topic will be much better discussed in the intuitionistic foundations considered in Section 30.

[8] If the case with $\bot = \top$ were allowed then the terminal object would be just the one-point set. (In some sense, then, the separation of $\top$ and $\bot$ requires no less than an entire continuum.)

Note that the symmetry on $\mathtt{I}$ is forced by its finality: if $\top$ and $\bot$ are interchanged and if $\top$- and $\bot$-zooming are interchanged the definition of interval coalgebra is maintained, hence there is a unique map from $\langle \mathtt{I}, \bot, \top, ^\wedge, ^\vee \rangle$ to $\langle \mathtt{I}, \top, \bot, ^\vee, ^\wedge \rangle$ that effects those interchanges and it is necessarily an involution. It is the symmetry being sought.

The $\leq$ relation on $\mathtt{I}$ may be defined as the most inclusive binary relation preserved by $^\wedge$ and $^\vee$ that avoids $\top \leq \bot$. We will delay the (more difficult) proof that the characterization yields a construction of the midpoint operator that figures so prominently in the opening (and throughout this work).

The assertion that the final coalgebra may be taken as the standard interval[9] needs a full proof—actually several proofs depending on the extent of constructive meaning one desires in his notion of the standard interval (see Sections 30–32). But we move now from the coalgebraic theory with its disjunctive condition to an algebraic theory in the usual purely equational sense.

2. The Equational Theory of Scales

The theory of **scales** is given by:

a constant **top**

denoted $\top$;

a unary operation **dotting**

whose values are denoted $\dot{\boldsymbol{x}}$;

a unary operation $\top$-**zooming**

whose values are denoted $\overset{\wedge}{\boldsymbol{x}}$ and;

a binary operation **midpointing**

whose values are denoted $\boldsymbol{x}|\boldsymbol{y}$.

Define:

the constant **bottom**

by $\bot = \dot{\top}$;

the constant **center**

by $\odot = \bot|\top$ and;

unary operation $\bot$-**zooming**

by $\overset{\vee}{\boldsymbol{x}} = \dot{\overset{\wedge}{\dot{x}}}$.

[9] $\mathtt{I}$ is also the final coalgebra of any finite iteration of ordered wedges. If we take the n-fold iteration, $X \vee X \vee \cdots \vee X$, as the set of n-tuples of the form $\langle x_0, x_1, \ldots, x_{n-1} \rangle$ such that either $x_i = \top$ or $x_{i+1} = \bot$ for $i = 0, 1, \ldots, n-2$ then the coalgebra structure is a sequence of functions $z_0, z_1, \ldots, z_{n-1}$ such that either $z_i(x) = \top$ or $z_{i+1}(x) = \bot$ for $i = 0, 1, \ldots, n-2$. The coalgebra structure on the unit interval is given by $z_i(x) = \max(0, \min(1, nx - i))$. Given $x \in X$ obtain the base-n expansion for its corresponding element in $\mathtt{I}$ by iterating (forever) the following procedure:

```
Let i = 0;
While z_i(x) = ⊤ and i < n−1 replace i with i+1;
Emit i;
Replace x with z_i(x).
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We will, when convenient, denote $(x|y)^{\wedge}$ as $\boldsymbol{x}\overset{\wedge}{|}\boldsymbol{y}$ or $\widehat{\boldsymbol{x}|\boldsymbol{y}}$ and $(x|y)^{\vee}$ as $\boldsymbol{x}\overset{\vee}{|}\boldsymbol{y}$.

The equations:

IDEMPOTENT:

$$x|x \;=\; x$$

COMMUTATIVE: [10]

$$x|y \;=\; y|x$$

MEDIAL (sometimes called "middle-two interchange"): [11]

$$(v|w) \;\big|\; (x|y) \;=\; (v|x) \;\big|\; (w|y)$$

CONSTANT: [12]

$$\dot{x}|x \;=\; \odot$$

UNITAL: [13]

$$\top\overset{\wedge}{|}x \;=\; x \;=\; \bot\overset{\vee}{|}x$$

And, finally, THE SCALE IDENTITY:

$$\widehat{u|v} \;=\; \widehat{\overset{\vee}{u}|\hat{v}} \;\big|\; \widehat{\hat{u}|\overset{\vee}{v}}$$

The **standard model** is the closed real interval $\mathbf{I}$ of all real numbers from -1 through $+1$. More generally, let $\mathbb{D}$ be the ring of dyadic rationals (those with denominator a power of 2). In $\mathbb{D}$-modules, or as we will call them, **dy-modules**, with total orderings we may choose elements $\bot < \top$, and define a scale as the set of all elements from $\bot$ through $\top$ with $x|y = (x+y)/2$, $\dot{x} = \bot + \top - x$, $\hat{x} = \max(2x - \top, \bot)$ (hence $\overset{\vee}{x} = \min(2x - \bot, \top)$). [14] The **standard interval** in $\mathbb{D}$, that is, the interval from -1 through $+1$, will be shown in

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[10] This axiom can be replaced with a single instance: $\bot|\top = \top|\bot$. See footnote below.

[11] The medial law has a geometric interpretation: it says that the midpoints of a cycle of four edges on a tetrahedron are the vertices of a parallelogram. That is, view four points $\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}$ in general position in $\mathbb{R}^3$. Consider the closed path from A to B to D to C back to A and note that the four successive midpoints $\mathtt{A|B}$, $\mathtt{B|D}$, $\mathtt{D|C}$, $\mathtt{C|A}$ appear in the medial law $(\mathtt{A|B})\big|(\mathtt{C|D}) \;=\; (\mathtt{A|C})\big|(\mathtt{B|D})$. This equation says, among other things, that the two line segments, the one from $\mathtt{A|B}$ to $\mathtt{C|D}$ and the one from $\mathtt{A|C}$ to $\mathtt{B|D}$, having a point in common, are coplanar, forcing the four midpoints, $\mathtt{A|B}$, $\mathtt{B|D}$, $\mathtt{D|C}$, $\mathtt{C|A}$ to be coplanar. The medial law says, further, that these two coplanar line segments have their midpoints in common, And that says—indeed, is equivalent with— $\mathtt{A|B}$, $\mathtt{B|D}$, $\mathtt{D|C}$, $\mathtt{C|A}$ being the vertices of a parallelogram. (A traditional proof is obtainable from the observation that two of the line segments, the one from $\mathtt{A|B}$ to $\mathtt{A|C}$ and the other from $\mathtt{D|B}$ to $\mathtt{D|C}$, are both parallel to the line segment from B to C, hence are themselves parallel.)

[12] A technically simpler equation is the two-variable $\dot{u}|u = \dot{v}|v$.

[13] The commutative axiom can be removed entirely if the first (left) unital law is replaced with $\dot{\bot}\overset{\wedge}{|}\,x = x$. See Section 29.

[14] In Sections 8 and 14 we will see that every scale has a faithful representation into a product of scales that arise in this way.

the next section to be isomorphic to the **initial scale** (the scale freely generated by its constants). It will be denoted as $\mathbb{I}$. [15]

The verification of all but the last of the defining equations on the standard interval is entirely routine. It will take a while before the scale identity reveals its secrets: how it first became known; how it can be best viewed; why it is true for the standard models.[16]

An *ad hoc* verification of the scale identity on the standard model may be obtained by noting first that:

$$\hat{x} = \begin{cases} -1 & \text{if } x \le 0 \\ ,2x-1 & \text{if } x \ge 0 \end{cases}$$

$$\overset{\vee}{x} = \begin{cases} 2x+1 & \text{if } x \le 0 \\ +1 & \text{if } x \ge 0 \end{cases}$$

The scale identity then separates into four cases depending on the signatures of the two variables. When both are positive the two sides of the identity quickly reduce to $x+y-1$ and, when both are negative, to -1. In the mixed case (because of commutativity we know in advance that the two mixed cases are equivalent) where x is positive and y is negative, the left side is, of course, $\widehat{(\frac{x+y}{2})}$ and the right side reduces to $(-1)|\widehat{(x+y)}$. The verification is completed therefore with the verification of $\widehat{z/2} = (-1)|\hat{z}$ which is, in turn, quickly verified by considering, once again, the two possible signatures of z. [17]

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[15]The free scale on one generator will be shown in Section 20 to be isomorphic to the scale of continuous piecewise affine functions (usually called piecewise linear) from $\mathtt{I}$ to $\mathtt{I}$ where each affine piece is given using dyadic rationals. We will give a generalization of the notion of piecewise affine so that the result generalizes: a free scale on n generators is isomorphic to the scale of all functions from $\mathtt{I}^n$ to $\mathtt{I}$ that are continuous piecewise affine with each piece given by dyadic rationals. The result further generalizes: essentially for every finitely presented scale there is a finite simplicial complex such that the scale is isomorphic to the scale of continuous piecewise affine maps with dyadic coefficients from the complex to $\mathtt{I}$. Their full subcategory can then be described in a piecewise affine manner. See Section 22.

[16]As forbidding as the scale identity appears, this writer, at least, finds comfort in the fact that the Jacobi identity for Lie algebras looks at first sight no less forbidding. Indeed, the scale identity has 2 variables and the standard Jacobi identity has 3 variables (with each variable appearing three times in each identity); the scale identity has 1 binary operation and it appears 4 times, the Jacobi has 2 binary operations, one of which appears twice and the other 6 times. (Its more efficient—and meaningful—form is a bit simpler: $[[x,y],z] = [x,[y.z]] - [y,[x,z]]$.) By these counts even the high-school distributivity law is worse than the scale identity (it has 3 variables that appear a total of 7 times and 2 binary operations that appear a total of five times). It is only when the unary operations are counted that the scale identity looks worse.

[17]In the next section we will show that the defining equations for scales are complete, that is, any new equation involving only the operators under discussion is either a consequence of the given equations or is inconsistent with them. Put another way: any equation not a consequence of these axioms fails in every non-degenerate scale. In particular, it fails in the initial scale. Put still another way: any equation true for any non-degenerate scale is true in all scales. A consequence is that the equational theory is decidable. With it will be shown in Section 28 that the problem is NP-complete.

It may be noted that the previously stated faithful representation of free scales as scales of functions makes the equational completeness clear: if an equation on n variables fails anywhere it fails in the free scale on n generators; but if the two sides of the failed equation are not equal when represented as functions from the n-cube we may apply the evaluation operator at a dyadic-rational point where the

For a fixed element a we will use $\boldsymbol{a}|$ to denote the **contraction** at a, the unary operation that sends x to $a|x$.

We will use freely:

2.1. LEMMA. SELF-DISTRIBUTIVITY:

$$a\big|(x|y) \;=\; (a|x)\big|(a|y)$$

an immediate consequence of idempotence and mediation: $a|(x|y) \;=\; (a|a)\big|(x|y) \;=\; (a|x)\big|(a|y)$. This is equivalent, of course, with contractions being midpoint homomorphisms.[18]

Define $\boldsymbol{a}\triangleleft$, the **dilatation** at a, by:

$$a\triangleleft x \;=\; \widehat{(\dot{a}\,|\bot)\overset{\vee}{|}x} \quad [19]$$

2.2. LEMMA. *Dilatation undoes contraction:*

$$a\triangleleft(a|x) \;=\; x$$

because using the medial, constant, self-distributive and both unital laws: $a\triangleleft(a|x) = \widehat{(\dot{a}\,|\bot)\overset{\vee}{|}(a|x)} = \widehat{(\dot{a}\,|a)\overset{\vee}{|}(\bot|x)} = \widehat{(\bot|\top)\overset{\vee}{|}(\bot|x)} = \widehat{\bot\overset{\vee}{|}(\top|x)} = \widehat{\top|x} = x.$ [20]

We immediately obtain:

2.3. LEMMA. THE CANCELLATION LAW:

$$\textit{If}\quad a|x \;=\; a|y \;\;\textit{then}\;\; x = y$$

Two important equations for dotting:

2.4. LEMMA. THE INVOLUTORY LAW:

$$\dot{\dot{x}} \;=\; x$$

and

functions disagree to obtain two distinct points in the initial scale, $\mathbb{I}$. Because the evaluation operator is a homomorphism of scales we thus obtain a counterexample in $\mathbb{I}$. Hence the set of equations that hold in all scales is the same as the set of equations that hold in $\mathbb{I}$, necessarily a complete equational theory. (Alas, the proof of the faithfulness of the representation in question requires the equational completeness.)

[18]It's worth finding the high-school–geometry proof for self-distributivity in the case that a, x and y are points in $\mathbb{R}^2$.

[19]For one way of finding this formula for dilatation see the last footnote in Section 4.

[20]The zooming operations may be viewed as special cases of dilatations. One can easily verify that $\top\triangleleft x = \hat{x}$ and we will prove below that $\bot\triangleleft x = \overset{\vee}{x}$. And for those looking for a Mal′cev operator, $txyz$, note that $y\triangleleft(x|z)$ is exactly that.

2.5. LEMMA. DOT-DISTRIBUTIVITY:

$$(u|v)^{\bullet} \;=\; \dot{u}\,|\,\dot{v}$$

Both can be quickly verified using cancellation: $\dot{x}\,|\,\dot{\dot{x}} \;=\; \dot{\dot{x}}\,|\,\dot{x} \;=\; \dot{x}\,|x$ and $(u|v)\big|(u|v)^{\bullet} = (u|v)^{\bullet}\,\big|(u|v) = \dot{u}\,|u = (\dot{u}\,|u)\big|(\dot{u}\,|u) = (\dot{u}\,|u)\big|(\dot{v}\,|v) = (u|\,\dot{u})\big|(v|\,\dot{v}) = (u|v)\big|(\dot{u}\,|\,\dot{v})$. [21]

Given a term $txy\ldots z$ involving $\top$, $\bot$, midpointing, dotting, $\top$- and $\bot$-zooming, the **dual term** is the result of fully applying the distributivity and involutory laws to $(t\dot{x}\cdots\dot{z})^{\bullet}$. It has the effect of swapping $^\wedge$ with $^\vee$ and $\top$ with $\bot$. If we replace both sides of an equation with their dual terms we obtain the **dual equation**.

We have already seen one pair of dual equations, to wit, the unital laws. The dual equation of the scale identity is:

$$u \overset{\vee}{|} v \;=\; (\hat{u} \overset{\vee}{|} \check{v}) \;\big|\; (\check{u} \overset{\vee}{|} \hat{v})$$

Note that we have not yet allowed dilatations in the terms to be dualized.[22]

As a direct consequence of the idempotent and unital laws we have that $\top$ is a fixed point for $\top$-zooming: $\hat{\top} = \widehat{\top|\top} = \top$. Dually, $\check{\bot} = \bot$. $\top$ is a also a fixed point for $\bot$-zooming using the unital law, scale identity (for the first time), idempotent, commutative and unital laws: $\top = \widehat{\top|\top} = \widehat{\check{\top}|\hat{\top}} \;\big|\; \widehat{\hat{\top}|\check{\top}} = \widehat{\check{\top}|\top} \;\big|\; \widehat{\top|\check{\top}} = \check{\top}|\check{\top} = \check{\top}$. Dually $\hat{\bot} = \bot$. That is:

2.6. LEMMA. *Both $\top$ and $\bot$ are fixed points for both $^\wedge$ and $^\vee$.*

In our second direct use of the scale identity we replace its second variable with $\top$ and use the unital law to obtain:

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[21] To see how the axiom $\bot|\top = \top|\bot$ suffices for commutativity, first note that commutativity was not used to obtain the left cancellation law ($a|x = a|y$ implies $x = y$). One consequence is that $\dot{x} = \dot{v}$ implies $x = v$ (use left cancellation on $\dot{x}\,|x = \dot{x}\,|v$). Besides being monic, dotting is epic because the second unital law, $\bot \overset{\vee}{|} x = x$, when written in full says $(((\bot|x)^{\bullet})^{\wedge})^{\bullet} = x$, hence for all x there is v such that $\dot{v} = x$ (to wit, $((\bot|x)^{\bullet})^{\wedge}$). Hence dotting is an invertible operation.

If $y|x = \odot$ then $y = \dot{x}$ because if we let z be such that $\dot{z} = y$ then $y|x = y|z$ and we use cancellation to obtain $x = z$ and, hence, $y = \dot{x}$. A consequence is $(u|v)^{\bullet} = \dot{u}\,|\,\dot{v}$ because it suffices to show $(\dot{u}\,|\,\dot{v}) \;\big|\; (u|v) = \odot$ which follows easily using the medial, constant and idempotence laws.

The commutativity of $\top$ and $\bot$ says $\top|\bot = \odot$ hence $\top = \dot{\bot}$ and that equation when combined with dot-distributivity and the second unital law yields $x = \bot \overset{\vee}{|} x = (((\bot|x)^{\bullet})^{\wedge})^{\bullet} = ((\dot{\bot}\,|\,\dot{x})^{\wedge})^{\bullet} = ((\top|\,\dot{x})^{\wedge})^{\bullet} = x|\,\dot{\dot{x}}$, hence $x = \dot{\dot{x}}$ (since $x|x = x|\dot{\dot{x}}$) quite enough to establish that the center is central: $x|\odot = (x|x)\big|(\dot{x}\,|x) = (x|\,\dot{x})\big|(x|x) = (\dot{\dot{x}}\,|\,\dot{x})\big|x = \odot|x$. Finally, $\odot\big|(x|y) = (\odot|x)\big|(\odot|y) = (\odot|x)\big|(y|\odot) = (\odot|y)\big|(x|\odot) = (\odot|y)\big|(\odot|x) = \odot\big|(y|x)$ and cancellation yields $x|y = y|x$.

[22] In time we will be able to do so. We will show that dilatations are self-dual just as are midpointing and dotting. That is, we will show $(((\dot{a}\,|\bot)|x)^{\vee})^{\wedge} = (((\dot{a}\,|\top)|x)^{\wedge})^{\vee}$.

2.7. LEMMA. THE LAW OF COMPENSATION:

$$x = \overset{\vee}{x} \,|\, \hat{x}$$

because $x = \widehat{x|\top} = \widehat{\overset{\vee}{x}|\overset{\wedge}{\top}} \,\big|\, \widehat{\hat{x}|\overset{\vee}{\top}} = \widehat{\overset{\vee}{x}|\top} \,\big|\, \widehat{\hat{x}|\top} = \overset{\vee}{x}|\hat{x}$.

A consequence:

2.8. LEMMA. THE ABSORBING LAWS*:*

$$x\overset{\wedge}{|}\bot = \bot \quad \textit{and} \quad x\overset{\vee}{|}\top = \top$$

because we can use cancellation and the law of compensation on $x|\bot = (x\overset{\vee}{|}\bot)\,\big|\,(x\overset{\wedge}{|}\bot) = x\big|(x\overset{\wedge}{|}\bot)$. [23]

2.9. LEMMA. *A scale is trivial iff* $\top = \bot$.

Because if $\top = \bot$ then $x = \widehat{x|\top} = \widehat{x|\bot} = \bot$ for all x.

The unital laws (or, for that matter, the absorbing laws) easily yield:

2.10. LEMMA.

$$\overset{\wedge}{\odot} = \bot \quad \textit{and} \quad \overset{\vee}{\odot} = \top$$

The center is the only self-dual element, $\dot{\odot} = \odot$. (If $\dot{x} = x$ then apply $x|$ to both sides to obtain $x|\dot{x} = x|x$, that is, $\odot = x$.)

If the second variable is replaced with $\odot$ in the scale identity (this is its third direct use [24]), the equations $\overset{\wedge}{\odot} = \bot$ and $\overset{\vee}{\odot} = \top$ yield a special case of the (not correct-in-general) distributive laws for $\top$- and $\bot$-zooming:

2.11. LEMMA. THE CENTRAL DISTRIBUTIVITY LAWS:

$$x\overset{\wedge}{|}\odot = \hat{x}|\bot \quad \textit{and} \quad x\overset{\vee}{|}\odot = \overset{\vee}{x}|\top$$

because $\widehat{x|\odot} = \widehat{\overset{\vee}{x}|\overset{\wedge}{\odot}} \,\big|\, \widehat{\hat{x}|\overset{\vee}{\odot}} = \widehat{\overset{\vee}{x}|\bot} \,\big|\, \widehat{\hat{x}|\top} = \bot\big|\hat{x}$. [25]

We will need:

2.12. LEMMA.

$$x = y \quad \textit{iff} \quad \dot{x}|y = \odot$$

because if $\dot{x}|y = \odot$ we can use cancellation on $\dot{x}|y = \dot{x}|x$.

A consequence is what is called "**swap-and-dot**:" given $w|x = v|z$ swap-and-dot any pair of variables from opposite sides to obtain equations such as $w|\dot{v} = \dot{x}|z$. (From $w|x = v|z$ infer $\odot = (w|x)\big|\dot{(v|z)} = (w|x)\big|(\dot{v}|\dot{z}) = (w|\dot{v})\big|(x|\dot{z}) = (w|\dot{v})\big|\dot{(\dot{x}|z)}$.)

Note that the commutative and medial laws say that $(w|x)\big|(y|z)$ is invariant under all 24 permutations of the variables (as, of course, are $(w|x)\overset{\vee}{|}(y|z)$ and $(w|x)\overset{\wedge}{|}(y|z)$).

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23 $\overset{\vee}{\top} = \top$ can now be viewed as a special case of the absorbing law: $\overset{\vee}{\top} = \top\overset{\vee}{|}\top = \top$.

24 It will be some time before we again invoke the scale identity.

25 As promised, we can now easily prove $\bot \triangleleft x = \widehat{(\dot{\bot}|\bot)\overset{\vee}{|}x} = \widehat{\odot\overset{\vee}{|}x} = \widehat{\top|\overset{\vee}{x}} = \overset{\vee}{x}$.

3. The Initial Scale

3.1. THEOREM. *The standard interval of dyadic rationals,* $\mathbb{I}$*, is isomorphic to the initial scale and it is simple.*

(Recall that a for any equational theory "simple" means no proper non-trivial quotient structures.) When coupled with the previous observation that $\perp \neq \top$ in all non-trivial scales we thus obtain:

3.2. THEOREM. $\mathbb{I}$ *appears uniquely as a subscale of every non-trivial scale.*

The proof is on the computational side as, apparently, it must be. It turns out that not all of the axioms are needed for the proof and that leads to another theorem of interest.

Let the theory of **minor scales** be the result of removing the scale identity but adding the absorbing laws (either one, by itself, would suffice).[26]

3.3. THEOREM. *The theory of minor scales has a unique equational completion, to wit, the theory of scales.*

There are several ways of restating this fact: equations consistent with the theory are consistent with each other; an equation is true for all scales iff it holds for any non-trivial minor scale; an equation is true for all scales iff it is consistent with the theory of minor scales; using the completeness of the theory of scales, every equation is either inconsistent with the theory of minor scales or is a consequence of the scale identity.[27]

The proof is obtained by showing that the initial minor scale is $\mathbb{I}$ and is simple. (Thus every consistent extension of the theory of minor scales, having a non-degenerate model, must hold for every subalgebra, hence must hold for the initial model. The complete equational theory of the initial model is thus the unique consistent extension of the theory of minor scales.)[28]

[26]For an example of a minor scale that is not a scale see Section 29.

[27]The same relationship holds between the theory of lattices and the theory of distributive lattices, and between the theories of Heyting and Boolean algebras. A less well-known example: for any prime p, the unique equationally consistent extension of the theory of characteristic-p unital rings is the theory of characteristic-p unital rings satisfying the further equation $x^p = x$. This almost remains true when the unit is dropped: given a maximal consistent extension of the theory of rings there is a prime p such that the theory is either the theory of characteristic-p rings that satisfy the same equation as above ($x^p = x$), or the theory of elementary p-groups with trivial multiplication ($xy = 0$).

A telling pair of examples: the equational theory of lattice-ordered groups and the equational theory of lattice-ordered unital rings. In each case the unique maximal consistent equational extension is the set of equations that hold for the integers. The first case is decidable (and all one needs to add to obtain a complete set of axioms is the commutativity of the group operation—see Section 27). The second case is undecidable: the non-solvability of any Diophantine equation, $P = 0$, is equivalent to the consistency of the equation $1 \wedge P^2 = 1$ (conversely, one may show that the consistency of any equation is equivalent to the non-solvability of some Diophantine equation).

[28]We can not only drop axioms but structure: $\mathbb{I}$ is the free **midpoint algebra** on two generators and the free **symmetric midpoint algebra** on one generator where we understand the first three scale equations (idempotent, commutative and medial) to define midpoint algebras and the first four (add the constant law) together with the involutory and distributive laws for dotting to define symmetric

We first construct the initial minor scale via a "canonical form" theorem and show that it is simple. Define a term in the signature of scales to be of **type 0** if it is either $\top$ or $\bot$, of **type 1** if it is $\odot$, and of **type n+1** if it is either $\top|A$ or $\bot|A$ where A is of type n with $n > 0$. We need to show that the elements named by typed terms form a subscale. Closure under the unary operations—dotting, $\top$- and $\bot$-zooming—is straightforward (but note that the absorbing laws are needed). For closure under midpointing we need an inductive proof. We consider $A|B$ where A is of type a and B is of type b. Because of commutativity we may assume that $a \leq b$. The induction is first on a. The case $a = 0$ presents no difficulties. For $a = 1$ we must consider two sub-cases, to wit when $b = 1$ and when $b > 1$. If $b = 1$ then $A|B = \odot$. When $b > 1$ we may, without loss of generality, assume that $B = \top|B'$ for B' of type b−1. But then $A|B = \odot\big|(\top|B') = (\top|\bot)\ \big|\ (\top|B') = \top\big|(\bot|B')$ which is of type b+1. If $a > 1$ there are, officially, four sub-cases to consider, but, without loss of generality, we may assume that $A = \top|A'$ and either $B = \top|B'$ or $B = \bot|B'$ where A' is of type $a-1$ and B' is of type $b-1$. In the homogeneous sub-case we have that $A|B = (\top|A')\ \big|\ (\top|B') = \top|(A'|B')$ and by inductive hypothesis we know that $A'|B'$ is named by a typed term, hence so is $A|B$. In the heterogeneous sub-case we have that $A|B = (\top|A')\ \big|\ (\bot|B') = (\top|\bot)\ \big|\ (A'|B') = \odot\big|(A'|B')$ and by inductive hypothesis we know that $A'|B'$ is named by a typed term and we then finish by invoking again the case $a = 1$.

The simplicity of the initial scale—and the uniqueness of typed terms—also requires induction. Suppose that A and B are distinct typed terms and that $\equiv$ is a congruence such that $A \equiv B$. Again we may assume that $a \leq b$. In the case $a = 0$ we may assume without loss of generality that $A = \top$. The sub-case $b = 0$ is, of course, the prototypical case ($B = \bot$ else $A = B$). For $b = 1$ we infer from $\top \equiv \odot$ that $\overset{\wedge}{\top} \equiv \overset{\wedge}{\odot}$, hence $\top \equiv \bot$, returning to the sub-case $b = 0$. For $b > 1$ we must consider the two sub-cases, $B = \top|B'$ and $B = \bot|B'$ where B' is of type $b-1$. From $\top \equiv \top|B'$ we may infer $\overset{\wedge}{\top} \equiv \widehat{\top|B'}$, hence $\top \equiv B'$ and thus reduce to the earlier sub-case b–1. From $\top \equiv \bot|B'$ we may infer $\overset{\wedge}{\top} \equiv \widehat{\bot|B'}$

midpoint algebras. In the opening section I talked about **closed midpoint algebras** with reference to the structure embodied by top, bottom and midpointing, with the remark that the axioms were not needed in the material of that section. Let me now legislate that the axioms are the first three scale equations together with the non-equational Horn sentence of cancellation for midpointing. Since $\mathbb{I}$ is such, it will perforce be the case that $\mathbb{I}$ is the initial closed midpoint algebra. (The set $\{\bot, \top\}$ is a two-element midpoint algebra but not a closed midpoint algebra when we take $\bot|\top = \bot$ and it is a symmetric midpoint algebra when we take $\dot{\top} = \bot$.) For a **symmetric closed midpoint algebra** add dotting and the constant law (the involutory and distributive laws are consequences of cancellation). $\mathbb{I}$ is also the initial symmetric closed midpoint algebra.

It should be noted, however, that there are closed midpoint algebras, even symmetric closed midpoint algebras, that challenge the word "midpoint." Choose an odd number of evenly spaced points on a circle and define the midpoint of any two of them to be the unique equidistant point in the collection. Chose any two points for $\top$ and $\bot$ and define $\dot{x}$ to be the unique element such that $\dot{x}\,|x = \bot|\top$. If one chooses $\bot, \bot|\top, \top$ to be adjacent then the induced map from $\mathbb{I}$ is guaranteed to be onto. $\mathbb{I}$ thus has an infinite number of closed midpoint quotients and it is far from simple (the simple algebras are precisely the cyclic examples of prime order).

immediately reducing to the sub-case $b = 0$. For the case $a = 1$ we know from $a \le b$ and $A \ne B$ that $b > 1$ and we may assume without loss of generality that $B = \top|B'$ where B' is of type $b-1$. But $\odot \equiv B$ then says that $\overset{\wedge}{\odot} \equiv \hat{B}$, hence $\top \equiv B'$ and we reduce to the case $a = 0$. For the case $a > 1$ we again come down to two sub-cases. In the homogeneous sub-case $A = \top|A'$ and $B = \top|B'$ we infer from $A \equiv B$ that $\widehat{\top|A'} \equiv \widehat{\top|B'}$ hence that $A' \equiv B'$. Since $A \ne B$ we have that $A' \ne B'$ and we reduce to the case a–1. Finally, in the heterogeneous sub-case $A = \top|A'$ and $B = \bot|B'$ we infer from $A \equiv B$ that $\top \overset{\vee}{|} A' \equiv \bot \overset{\vee}{|} B'$ hence that $\top \equiv B'$. Since the type of B' is positive such reduces to the case $a = 0$.

When we know that every non-trivial scale contains a minimal scale isomorphic to the initial scale, then perforce we know that there is, up to isomorphism, only one non-trivial minimal scale. Hence, to see that the initial scale is isomorphic to $\mathbb{I}$ it suffices to show that $\mathbb{I}$ is without proper subscales, or to put it more constructively, that every element in $\mathbb{I}$ can be accounted for starting with $\top$. By definition $\bot = \dot{\top}$ and $\odot = \top|\bot$. Switching to $\mathbb{D}$-notation, we know that every element different from $\top, \odot$ and $\bot$ is of the form $n2^{-m}$ where n and m are integers, $m > 0$, n odd and $-2^m < n < 2^m$. Inductively,

$$n2^{-(m+1)} = \begin{cases} \top|(n - 2^m)2^{-m} & \text{if } n > 0 \\ \bot|(n + 2^m)2^{-m} & \text{if } n < 0 \end{cases}$$

4. Lattice Structure

The most primitive way of defining the natural partial order on a scale is to define $u \le v$ iff there is an element w such that $u|\top = v|w$. From this definition it is clear that any map that preserves midpointing and $\top$ must preserve order (which together with von Neumann is quite enough to prove the opening assertion of this work).[29]

But in the presence of zooming we may remove the existential. First note that

$$\exists_w\ z = \top|w \quad \text{iff} \quad \overset{\vee}{z} = \top$$

because if $z = \top|w$ then the absorbing law says $\overset{\vee}{z} = \top\overset{\vee}{|}w = \top$. Conversely, if $\overset{\vee}{z} = \top$ then we may take $w = \hat{z}$ (the law of compensation gives us $z = \overset{\vee}{z} \mid \hat{z} = \top| \hat{z}$).

If we use swap-and-dot and the involutory law to rewrite the existential condition for $u \le v$ as $\exists_w\ \dot{u}|v = \top|w$ we are led to define a new binary operation $\boldsymbol{u} \multimap \boldsymbol{v} = \dot{u}\overset{\vee}{|}v$ and we see by the absorbing law that

$$u \le v \quad \text{iff} \quad u \multimap v \ = \ \top.$$

We make this our official definition ($u \multimap v$ may be read as "the extent to which u is less than v" where $\top$ is taken as "true" [30]).

A neat way to encapsulate this material is with:

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[29] Left as an easy exercise: a map that preserves midpointing and $\bot$ also preserves order.

[30] See the next section on Łukasiewicz and Girard.

4.1. LEMMA. THE LAW OF BALANCE:

$$u \,\big|\, (u \multimap v) \;\;=\;\; v \,\big|\, (v \multimap u)$$

(One can see at once that $u \multimap v = \top$ implies that $u|\top = v|w$ is solvable.) To prove the law of balance note that the law of compensation yields $u|\dot{v} = (u \overset{\wedge}{|} \dot{v})\big|(u \overset{\vee}{|} \dot{v})$ and a swap-and-dot yields $x\big|\dot{(u \overset{\wedge}{|} \dot{v})} = v\big|(u \overset{\vee}{|} \dot{v})$; the left side rewrites as $u\big|(\dot{u} \overset{\vee}{|} v) = u\big|(u \multimap v)$ and the right as $v\big|(\dot{w} \overset{\vee}{|} u) = v\big|(v \multimap u)$.

We verify that $\leq$ is a partial order as follows:

Reflexivity is immediate: $x \multimap x \;= \dot{x} \overset{\vee}{|} x \;= \overset{\vee}{\odot} = \top$.

For antisymmetry, given $x \multimap y = y \multimap x = \top$ just apply the unital law to both sides of the law of balance.

Transitivity is not so immediate. We will need that $\overset{\vee}{u} = \top = \overset{\vee}{v}$ implies $u \overset{\vee}{|} v = \top$ (true because, using the law of compensation, $\overset{\vee}{u} = \top = \overset{\vee}{v}$ says $u \overset{\vee}{|} v \;= (\overset{\vee}{u} \,|\, \hat{u}) \overset{\vee}{|} (\overset{\vee}{v} \,|\, \hat{v}) = (\top \,|\hat{u}) \overset{\vee}{|} (\top \,|\hat{v}) = \top \overset{\vee}{|} (\hat{u} \,|\, \hat{v}) = \top$). Hence if $u \multimap v$ and $v \multimap w$ are both $\top$ then so is $(\dot{u}|v) \overset{\vee}{|} (\dot{v}|w)$. But this last term is equal (using the commutative and medial laws) to $\odot \overset{\vee}{|} (\dot{u}\,|w)$ which by the central distributivity law is $\top\big|(\dot{u} \overset{\vee}{|} w)$. Hence if $u \leq v$ and $v \leq w$ we have that $\top\big|(u \multimap w) = \top$ and when both sides are $\top$-zoomed we obtain $u \multimap w = \top$.

Covariance of $z|$ follows from central distributivity:

$$(z|x) \multimap (z|y) = \dot{(z|x)} \overset{\vee}{|} (z|y) = (\dot{z}\,|\,\dot{x}) \overset{\vee}{|} (z|y) = (\dot{z}\,|z) \overset{\vee}{|} (\dot{x}\,|y) = \odot \overset{\vee}{|} (\dot{x}\,|y) = \top\big|(x \multimap y)$$

. Hence $x \leq y$ implies $z|x \leq z|y$. Not only does $z|$ preserve order, it also reflects it.

Contravariance of dotting is immediate:

$$\dot{u} \multimap \dot{v} = \; v \multimap u.$$

Given an inequality we obtain the **dual inequality** by replacing the terms with their duals and reversing the inequality.

A few more formulas worth noting are:

$$\begin{aligned}
\top \multimap x \;&=\; x \\
x \multimap \bot \;&=\; \dot{x} \\
\dot{x} \multimap x \;&=\; \overset{\vee}{x} \\
x \multimap \dot{x} \;&=\; \dot{\hat{x}}
\end{aligned}$$

Note that $\odot \multimap x = \top|\overset{\vee}{x}$ hence $\overset{\vee}{x} = \top$ iff $\odot \leq x$. The lemma we needed (and proved) for transitivity, that $\overset{\vee}{u} = \top = \overset{\vee}{v}$ implies $u \overset{\vee}{|} v = \top$, is now an easy consequence of the covariance of midpointing.

It is immediate from the definition and absorbing laws that $\bot \leq x \leq \top$ all x.

We obtain a swap-and-dot lemma for inequalities:

4.2. LEMMA.

$$u|v \;\le\; w|x \quad \textit{iff} \quad u|\dot{w} \le \dot{v}\,|x$$

Because $\top = (u|v) \multimap (w|x) = (u|v)^{\bullet}\overset{\vee}{|}(w|x) = \dot{u}\,|\,\dot{v}\,\overset{\vee}{|}\,w|x = \dot{u}\,|w\,\overset{\vee}{|}\,\dot{v}\,|v = (u|\,\dot{w})^{\bullet}\overset{\vee}{|}\,\dot{v}\,|x =$ $(u|\,\dot{w}) \multimap (\dot{v}\,|x)$.

An important fact: $\top$ is an **extreme point** in the convex-set sense, that is, it is not the midpoint of other points:

4.3. LEMMA. $x|y = \top$ *iff* $x = \top = y$.

Because $x|y \le \top|y$ (without any hypothesis), hence $x|y = \top$ implies $\top = \; x|y \;\le\; \top|y \;\le\; \top$ forcing $\top|y = \top$, hence $y = \widehat{\top|y} = \widehat{\top} = \top$.

The covariance of $\top$-zooming requires work. In constructing this theory an equational condition was needed that would yield the Horn condition that $u \le v$ implies $\hat{u} \le \hat{v}$. (The equation, for example, $(u \multimap v) \multimap (\hat{u} \multimap \hat{v}) = \top$ would certainly suffice. Alas, this equation is inconsistent with even the axioms of minor scales: if we replace u with $\top$ and v with $\odot$ it becomes the assertion that $\odot \le \bot$.)

The fact that $\top$ is an extreme point says that it would suffice to have:

$$u \multimap v \;\;=\;\; (\hat{u} \;\multimap\; \hat{v}) \;\big|\; (\overset{\vee}{u} \;\multimap\; \overset{\vee}{v}).$$

Indeed, this equation implies that the two zooming operations collectively preserve and *reflect* the order.

Finding a condition strong enough is, as noted, easy. To check that it is not *too* strong, that is to check the equation on the standard model, it helps to translate back to more primitive terms:

$$\dot{u}\,\overset{\vee}{|}\,v \;\;=\;\; (\dot{\hat{u}}\,\overset{\vee}{|}\,\hat{v}) \;\big|\; (\dot{\overset{\vee}{u}}\,\overset{\vee}{|}\,\overset{\vee}{v}) \;\;=\;\; (\overset{\vee}{\dot{u}}\,\overset{\vee}{|}\,\hat{v}) \;\big|\; (\overset{\wedge}{\dot{u}}\,\overset{\vee}{|}\,\overset{\vee}{v}).$$

Since dotting is involutory this is equivalent to the outer equation but without the dots:

$$u\overset{\vee}{|}v \;\;=\;\; (\overset{\vee}{u}\,\overset{\vee}{|}\,\hat{v}) \;\big|\; (\hat{u}\,\overset{\vee}{|}\,\overset{\vee}{v}),$$

to wit, the dual of the scale identity. And it was this that was the first appearance of the scale identity (and its first serious use—the three previous direct applications that have appeared here served only to replace what had, in fact, once been axioms, to wit, the variable-free equation $\overset{\vee}{\bot} = \bot$ and the two one-variable equations, $\overset{\vee}{x} \mid \hat{x} = x$ and $\widehat{\odot|x} = \bot|\,\hat{x}$, which three laws are much more apparent than the scale identity).

Among the corollaries are the covariance of the binary operations $\overset{\wedge}{|}$, $\overset{\vee}{|}$ and the important inequalities:

4.4. LEMMA.

$$\hat{x} \;\le\; x \;\le\; \overset{\vee}{x}$$

Because $\hat{x} = \; x\overset{\wedge}{|}x \;\le\; \top\overset{\wedge}{|}x \;=\; x \;=\; x\overset{\vee}{|}\bot \;\le\; x\overset{\vee}{|}x \;= \overset{\vee}{x}$.

Further corollaries: $x \multimap y$ is covariant in y and contravariant in x. (If $a \triangleleft x$ is viewed as a binary operation then it is covariant in x and contravariant in a.)

4.5. Lemma. the convexity of $\top$-zooming:

$$\widehat{u|v} \;\le\; \hat{u} \,|\, \hat{v}$$

Because $u \overset{\wedge}{|} v = (\overset{\vee}{u}|\hat{u}) \overset{\wedge}{|} (\overset{\vee}{v}|\hat{v}) \le (\top|\hat{u}) \overset{\wedge}{|} (\top|\hat{v}) = \top \overset{\wedge}{|} (\hat{u}\,|\,\hat{v}) = \hat{u}\,|\,\hat{v}$.

The dual inequality:

4.6. Lemma.

$$u \overset{\vee}{|} v \;\ge\; \overset{\vee}{u} \,|\, \overset{\vee}{v}$$

Define a binary operation, temporarily denoted $\diamond y$, as $x \overset{\wedge}{|} (x \multimap y)$. The law of balance says, in particular, that the $\diamond$-operation is commutative and consequently covariant not just in y but in both variables. Note that if $x \le y$ then $x \diamond y = x \overset{\wedge}{|} (x \multimap y) = x \overset{\wedge}{|} \top = x$, which together with commutativity says that whenever x and y are comparable, $x \diamond y$ is the smaller of the two. As special cases we obtain the three equations: $x \diamond \top = \top \diamond x = x \diamond x = x$. These three equations together with the covariance are, in turn, enough to imply that $x \diamond y$ is the greatest lower bound of x and y: from the covariance and $x \diamond \top = x$ we may infer that $x \diamond y \le x \diamond \top = x$ and, similarly, $x \diamond y \le y$; from covariance and $z \diamond z = z$ we may infer that $x \diamond y$ is the greatest lower bound (because $z \le x$ and $z \le y$ imply $z = z \diamond z \le x \diamond y$).

All of which gives us the lattice operations (using duality for $x \vee y$):

4.7. Lemma.

$$x \wedge y \;=\; x \overset{\wedge}{|} (x \multimap y) \;=\; x \overset{\wedge}{|} (\dot{x} \overset{\vee}{|} y) \quad [31]$$

$$x \vee y \;=\; x \overset{\vee}{|} \widehat{\dot{x}|y}$$

We will extend the notion of duality to include the lattice structure. (But note that we do not have a symbol for the dual of $\multimap$.)

Direct computation now yields what we will see must be known by an oxymoronic name (it is the "internalization" of the—external—disjunction $\overset{\vee}{x} = \top$ *or* $\hat{x} = \bot$).

4.8. Lemma. the coalgebra equation:

$$\overset{\vee}{x} \vee \dot{\hat{x}} \;=\; \top$$

because: $\overset{\vee}{x} \overset{\vee}{|} \widehat{\dot{\overset{\vee}{x}} \,|\, \dot{\hat{x}}} \;=\; \overset{\vee}{x} \overset{\vee}{|} ((\overset{\vee}{x}|\hat{x})^{\vee})^{\bullet} \;=\; \overset{\vee}{x} \overset{\vee}{|} \dot{\overset{\vee}{x}} \;=\; \overset{\vee}{\odot} \;=\; \top$.

If we replace x with $\dot{x}\,|y$ we obtain the internalization of the disjunction $(x \le y)$ *or* $(y \le x)$):

[31] It behooves us to figure out just what the term $x\Big|(x \multimap y)$ is before it is $\top$-zoomed. We know that it is commutative and covariant in both variables. The law of compensation says that it is equal to $(x \overset{\vee}{|} (x \multimap y)) \Big| (x \overset{\wedge}{|} (x \multimap y))$. We know now what the right-hand term, $x \overset{\wedge}{|} (x \multimap y)$, is. For the left-hand term, $x \overset{\vee}{|} (x \multimap y)$, note that its covariance implies that it is always at least $\bot \overset{\vee}{|} (\bot \multimap \bot) = \bot \overset{\vee}{|} \top = \top$. Hence $x\Big|(x \multimap y) = \top\Big|(x \wedge y)$.

4.9. Lemma. the equation of linearity:

$$(x \multimap y) \vee (y \multimap x) \quad = \quad \top \ ^{[32]}$$

Indeed it says that what logicians call the **disjunction property** (the principle that a disjunction equals $\top$ only if one of the terms equals $\top$) is equivalent with linearity. The following are equivalent for scales: linearity, the disjunction property, the coalgebra condition.

We will need:

4.10. Lemma. the adjointness lemma:

$$u \ \le \ v \multimap w \qquad \textit{iff} \qquad u \overset{\wedge}{|} v \ \le \ w$$

Because if $u \le v \multimap w$ then $v|u \le v\big|(v \multimap w) = w\big|(w \multimap v) \le w|\top$ and we may $\top$-zoom the two ends to obtain $u \overset{\wedge}{|} v \ \le w$. And if $u \overset{\wedge}{|} v \ \le w$ then $v \multimap \widehat{u|v} \le v \multimap w$. But $v \multimap \widehat{u|v} = \dot{v} \overset{\vee}{|} \widehat{u|v} = u \vee \dot{v}$ hence we have $u \ \le \ u \vee \dot{v} \le \ v \multimap w$. [33]

We close this section with a few **interval isomorphisms**. For any $b < t$ the interval $[b, t|$ is order-isomorphic with an interval whose top end-point is $\top$, to wit, the interval $[t \multimap b, \top]$. The isomorphism is $t \multimap (-)$. Its inverse is $t \overset{\wedge}{|} (-)$. (The fact that the composition $t \overset{\wedge}{|} (t \multimap x) = x$ for all $x \in [b, t]$ is just the fact that $t \overset{\wedge}{|} (t \multimap x) = t \wedge x$. The fact that the composition $t \multimap (t \overset{\wedge}{|} x) = x$ for all $x \in [t \multimap b, \top]$ is just the fact that $\dot{t} \overset{\vee}{|} (t \overset{\wedge}{|} x) = \dot{t} \vee x$ and the fact that $x \ge (t \multimap b)$ implies $x \ge (t \multimap b) \ge (t \multimap \bot) = \dot{t}$.) These are not just order-isomorphisms. With the forthcoming linear representation theorem (Section 8) it will be easy to prove that they preserve midpointing and when—in Section 6—we note that all closed intervals have intrinsic scale structures it will be easy to see that they are scale isomorphisms.[34]

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[32]The coalgebra equation is obtainable, in turn, from the equation of linearity by replacing y with $\dot{x}$.

[33]Using the linear representation theorem below one can show the rather surprising fact that the binary operation $\overset{\wedge}{|}$ is associative. Any poset may be viewed as a category and this associativity together with the adjointness lemma allows us to view a scale as a "symmetric monoidal closed category" with $\overset{\wedge}{|}$ as the monoidal product and $\multimap$ as the closed structure. The monoidal unit is $\top$. A scale is, in fact, a "$\star$-autonomous category": $\bot$ is its "dualizing object."

A straightforward verification of the associativity of $\overset{\wedge}{|}$ on a linear scale entails a lot of case analysis. Perhaps it is best to use the equational completion that will be proved. It then suffices to verify it on just one non-trivial example. The easiest we have found is to take the unit interval—not the standard interval—and to verify the dual equation, the associativity of $\overset{\vee}{|}$. On the unit interval $x \overset{\vee}{|} y$ is addition truncated at 1, quite easily seen to be associative.

[34]The construction of the dilatation operator can be motivated by this material. For any a, the function $(a|\top) \multimap (-)$ sends the image of $a|-$ to the interval $[\odot, \top]$, quite enough to suggest that $(a|\top) \multimap (a|-)$ is the same as $\top|-$, hence (using the unital law) that $\widehat{(a|\top) \multimap (a|x)} = x$. The function $\widehat{(a|\top) \multimap (-)}$ is $a \triangleleft -$.

5. Diversion: Łukasiewicz vs. Girard

On the unit interval the formula for $\multimap$ has a prior history as the Łukasiewicz notion of many-valued logical implication. A traditional interpretation of $\Phi \le \Psi$ is "Ψ is at least as likely as Φ." Then $\Phi \multimap \Psi$ becomes the "likelihood of Ψ being at least as likely as Φ." [35]

The unit interval when viewed as a $\star$-autonomous category (resulting from its scale-algebra structure) reveals Łukasiewicz inference as a special case of Girard's linear logic. We will write $\overset{\wedge}{|}$ as $\otimes$ and its "de Morgan dual" $\overset{\vee}{|}$ as $⅋$ ("par"). (The midpoint operation is an example of a "seq" operation—it lies between $\otimes$ and $⅋$.)

If we interpret the truth-values as frequencies (or probabilities) we can not infer, of course, the frequency of a conjunction from the individual frequencies. But we can infer the *range* of possible frequencies. If Φ and Ψ are the individual frequencies then the maximum possible frequency for their disjunction occurs when they are maximally exclusive: if their frequencies add to 1 or less and if they never occur together then the maximal possible frequency of the disjunction is their sum; if their frequencies add to more than 1 then the maximal possible frequency of the disjunction is, of course, 1. That is, the maximum possible frequency for their disjunction is $\Phi ⅋ \Psi$. The minimal possible frequency for their conjunction likewise occurs when they are maximally exclusive and similar consideration yields $\Phi \otimes \Psi$. This works best if we understand that separate observations are made, one for Φ and one for Ψ (hence $\Phi \otimes \Phi$ is the minimal possible frequency that Φ occurs in both observations, $\Phi ⅋ \Phi$ the maximal possible frequency that Φ occurs in at least one of the two observations).

The "additive" connectives, likewise, have such an interpretation. The minimal possible frequency for their disjunction occurs when they are minimally exclusive, that is, when the less probable event occurs only when the more probable event occurs, hence the minimal possible frequency for the disjunction is $\Phi \vee \Psi$. Similar computation yields that the maximal possible frequency of their conjunction is $\Phi \wedge \Psi$. The midpoint of Φ and Ψ is also the midpoint of $\Phi \otimes \Psi$ and $\Phi ⅋ \Psi$ (using the law of compensation) and the midpoint of $\Phi \wedge \Psi$ and $\Phi \vee \Psi$ (using the forthcoming linear representation theorem of Section 8). Note that we have in descending order:

$$\begin{array}{c} 1 \\ \Phi ⅋ \Psi \\ \Phi \vee \Psi \\ \Phi \mid \Psi \\ \Phi \wedge \Psi \\ \Phi \otimes \Psi \\ 0 \end{array}$$

[35] We may interpret $\perp$-zooming using the equation $\dot{\Phi} \multimap \Phi = \overset{\vee}{\Phi}$: given a sentence Φ it says that $\overset{\vee}{\Phi}$ is the likelihood that Φ is at least as likely as not. Using the companion equation $\Phi \multimap \dot{\Phi} = \dot{\overset{\wedge}{\Phi}}$ we see that in the Łukasiewicz interpretation the coalgebra condition ($\overset{\vee}{\Phi} = \top$ *or* $\overset{\wedge}{\Phi} = \perp$) says that for any statement either it or its negation is at least as likely as not.

$\Phi \multimap \Psi = 1$ means that it is possible (just knowing the frequencies of Φ and Ψ) that whenever Φ occurs Ψ will occur. In general $\Phi \multimap \Psi$ gives the maximal possible probability that a single pair of observations will fail to falsify the hypothesis "if Φ then Ψ." The adjointness lemma, $\Phi \leq \Psi \multimap \Lambda \quad \Leftrightarrow \quad \Phi \otimes \Psi \leq \Lambda$, then says that Φ is possibly less frequent than Ψ appearing to imply Λ iff the frequency of the conjunction of Φ and Ψ is possibly less than the frequency of Λ. [36] We constructed the meet operation as $\Phi \otimes (\Phi \multimap \Psi)$. That is, the maximal possible frequency for the conjuction of two events is equal to the minimal possible frequency of the conjunction of another pair of events, the first of which remains the same and the second is the maximal possible frequency of failing to refute the hypothesis that the first implies the second. (Surely someone previously must have observed this.)

When the coalgebra condition is interpreted we obtain the **interval rule**:

$$\Phi \parr \Psi = 1 \quad \text{or} \quad \Phi \otimes \Psi = 0$$

(Either it is possible for one to succeed or it is possible that both fail.) Alternatively we may replace Φ with $\dot{\Phi}$ so that the coalgebra condition becomes

$$\Phi \leq \Psi \quad \text{or} \quad \Psi \leq \Phi$$

delivering a theory of linear linear logic.[37]

Missing above are Girard's modal unary operations, *of-course* and *why-not*, which he denoted with a **!** and a **?**. [38] In Section 19, below, on "chromatic scales" we introduce the (discontinuous) "support" operations on scales. Using chromatic-scale notation one may argue that $!\Phi = \underline{\Phi}$ and $?\Phi = \overline{\Phi}$.

6. Diversion: The Final Interval Coalgebra is a Scale

The final interval coalgebra, $\mathtt{I}$, comes equipped, of course, with the two constants $\top$ and $\bot$, and the two zooming operations $\hat{x}$ and $\overset{\vee}{x}$. We may define $\dot{x}$ via the unique coalgebra map $\dot{\mathtt{I}} \to \mathtt{I}$ where $\dot{\mathtt{I}}$ is the coalgebra obtained by swapping the two constants and the two zooming operations. The order on $\mathtt{I}$ is definable via the observation that $x < y$ iff there is a sequence of zooming operations $({}^{\wedge}, {}^{\vee})$ that carries x to $\bot$ and y to $\top$.

There is, indeed, a useful interval coalgebra structure on $\mathtt{I} \times \mathtt{I}$ so that its unique coalgebra map to $\mathtt{I}$ is the midpoint operation,[39] but, alas, this coalgebra structure on

[36] This would not, of course, be heard as an acceptable sentence in ordinary language. But few translations from the mathematical notation to ordinary language yield acceptable sentences—else who would need the math?

[37] That which is linear^2 is planar.

[38] Hollow men pronounce these as *bang* and *whimper*.

[39] The word "useful" is important here. Given any functor T with a final coalgebra $F \to TF$ then for any retraction $F \xrightarrow{x} A \xrightarrow{y} F = 1_F$ there is a coalgebra structure on A that makes y a coalgebra map, to wit, $A \xrightarrow{y} F \to TF \xrightarrow{Tx} TA$.

$\mathtt{I}\times\mathtt{I}$ requires the midpoint operation for the construction of its two zooming operations: $\langle u,v\rangle$ is sent by $\top$-zooming to $\langle\, \overset{\vee}{u}\overset{\wedge}{|}\hat{v}\,,\ \hat{u}\overset{\wedge}{|}\overset{\vee}{v}\,\rangle$ and by $\bot$-zooming to $\langle\, \overset{\vee}{u}\overset{\vee}{|}\hat{v}\,,\ \hat{u}\overset{\vee}{|}\overset{\vee}{v}\,\rangle$.[40]

We must eventually come to grips with the notion of *co-recursion* but will settle now for a quick and dirty proof that for $u,v\in[0,1]$ the binary expansion of $u|v$ is forced by the scale axioms. Recall the earlier quick and dirty proof. In this case it says that we should iterate (forever) a procedure equivalent to: `If` $u\overset{\wedge}{|}v\ =\top$ `then emit 1 and replace` $u|v$ `with` $u\overset{\wedge}{|}v$ `else emit 0 and replace` $u|v$ `with` $u\overset{\vee}{|}v$. We need, obviously, to expand.

We will use that $u\overset{\vee}{|}v\ =\top$ iff $\dot{u}\le v$ and we will attack the computation of $u\overset{\wedge}{|}v$ and $u\overset{\vee}{|}v$ by using the scale identity. We need a single procedure for the three cases $u|v,\ u\overset{\wedge}{|}v,\ u\overset{\vee}{|}v$. Hence we iterate (forever) a procedure that takes an ordered triple $\langle u,s,v\rangle$ as input where u and v are elements of $[0,1]$ and s is an element of the set of three symbols $\{\,\overset{\vee}{|},|,\overset{\wedge}{|}\,\}$.

`If` $s=\big|$ `then`
  `if` $\dot{u}\le v$ `then`
    `emit 1; replace` $\langle u,|,v\rangle$ `with` $\langle u,\overset{\wedge}{|},v\rangle$.
  `else`
    `emit 0; replace` $\langle u,|,v\rangle$ `with` $\langle u,\overset{\vee}{|},v\rangle$.
`else if` $s=\ \overset{\wedge}{|}$ `then`
  `if` $\hat{u}\ =\bot$ `then`
    `emit 0; replace` $\langle u,\overset{\wedge}{|},v\rangle$ `with` $\langle\overset{\vee}{u},\overset{\wedge}{|},\hat{v}\rangle$.
  `else if` $\hat{v}=\bot$ `then`
    `emit 0; replace` $\langle u,\overset{\wedge}{|},v\rangle$ `with` $\langle\hat{u},\overset{\wedge}{|},\overset{\vee}{v}\rangle$.
  `else`
    `replace` $\langle u,\overset{\wedge}{|},v\rangle$ `with` $\langle\hat{u},|,\hat{v}\rangle$.
`else`
  `if` $\overset{\vee}{u}=\top$ `then`
    `emit 1; replace` $\langle u,\overset{\vee}{|},v\rangle$ `with` $\langle\hat{u},\overset{\vee}{|},\overset{\vee}{v}\rangle$.
  `else if` $\overset{\vee}{v}=\top$ `then`
    `emit 1; replace` $\langle u,\overset{\vee}{|},v\rangle$ `with` $\langle\overset{\vee}{u},\overset{\vee}{|},\hat{v}\rangle$.
  `else`
    `replace` $\langle u,\overset{\vee}{|},v\rangle$ `with` $\langle\overset{\vee}{u},|,\overset{\vee}{v}\rangle$.

For a proof that this is forced by the axioms for midpointing note first that $\dot{u}\le v$

[40]Imagine stumbling across this use of the scale identity, the initial discovery of which was in answer to a very different question.

implies $\odot \leq u|v$, hence $u \overset{\vee}{|} v = \top$ which means that the first digit is 1 and the remaining digits are determined by $u \overset{\wedge}{|} v$. For $u \overset{\wedge}{|} v$ we use the scale identity:

$$u \overset{\wedge}{|} v = (\overset{\vee}{u} \overset{\wedge}{|} \hat{v}) \mid (\hat{u} \overset{\wedge}{|} \overset{\vee}{v})$$

When $\hat{u} = \bot$ this becomes:

$$u \overset{\wedge}{|} v = (\overset{\vee}{u} \overset{\wedge}{|} \hat{v}) \mid (\bot \overset{\wedge}{|} \overset{\vee}{v}) = (\overset{\vee}{u} \overset{\wedge}{|} \hat{v})|\bot$$

hence, by the absorbing law, $(u \overset{\wedge}{|} v)^{\wedge} = \bot$ which means that the first digit is 0 and the remaining digits are determined by $(u \overset{\wedge}{|} v)^{\vee} = ((\overset{\vee}{u} \overset{\wedge}{|} \hat{v}) \mid \bot)^{\vee}$ which by the unital law is $\overset{\vee}{u} \overset{\wedge}{|} \hat{v}$. A similar argument holds for the case $\hat{v} = \bot$. If neither $\hat{u}$ nor $\hat{v}$ are $\bot$ we have $\overset{\vee}{u} = \overset{\vee}{v} = \top$ and the scale identity and unital law yield

$$u \overset{\wedge}{|} v = (\top \overset{\wedge}{|} \hat{v}) \mid (\hat{u} \overset{\wedge}{|} \top) = \hat{v} \mid \hat{u}$$

which returns us to the case $s = |$. The dual argument holds for the case $s = \overset{\vee}{|}$.

7. Congruences, or: ⊤–Faces

One of our first aims is to prove that every scale can be embedded in a product of linear scales. Put another way: we wish to find, on any scale, a lot of quotient structures that are linearly ordered. And for that we must get an understanding of quotient structures.

As for any equational theory, the quotient structures of a particular algebra correspond to the "congruences" on that structure, that is, the equivalence relations that are compatible with the operators that define the structure. For some well-endowed theories the congruences correspond, in turn, to certain subsets. Such is the case for scales.[41]

Given a congruence $\equiv$ define its **kernel**, denoted **ker(≡)**, to be the set of elements congruent to $\top$. Clearly, $\equiv$ can be recovered from ker($\equiv$) (because $x \equiv y$ iff both $x \multimap y$ and $y \multimap x$ are in ker($\equiv$)). We need to characterize the subsets that appear as kernels.

Borrowing again from convex-set terminology, we say that a subset is a "face" if it is not just closed under midpointing but has the property that it includes any two elements whenever it includes their midpoint. (Saying that an element is an extreme point, therefore, is the same as saying that it forms a one-element face.) We will be interested particularly in those faces that include $\top$. Thus we define a subset, $\mathcal{F}$, to be a **⊤-face**, "top-face," if:

$$\top \in \mathcal{F}$$
$$x|y \in \mathcal{F} \quad \text{iff} \quad x \in \mathcal{F} \text{ and } y \in \mathcal{F}$$

[41] Almost all well-endowed theories in nature contain the theory of groups. Two exceptions (besides scales): the theory of Heyting algebras and (its generalization) the theory of division allegories.

Because inverse homomorphic images of faces are faces and because $\{\top\}$ is a face it is clear that $\ker(\equiv)$ is a $\top$-face for any congruence. We need to show that all $\top$-faces so arise.

Given a $\top$-face, $\mathcal{F}$, define $\boldsymbol{x \preceq y}\,(\mathbf{mod}\ \boldsymbol{\mathcal{F}})$ to mean $x \multimap y \in \mathcal{F}$ and define $\boldsymbol{x \equiv y}\,(\mathbf{mod}\ \boldsymbol{\mathcal{F}})$ as the "symmetric part" of $\preceq$, that is, $x \equiv y$ iff $x \preceq y$ and $y \preceq x$. It is routine that $x \equiv \top$ iff $x \in \mathcal{F}$.

Clearly $\equiv$ is reflexive (because $x \multimap x = \top$) and it is symmetric by fiat. Transitivity requires a little more. First note that a $\top$-face is an updeal, that is, $x \in \mathcal{F}$ and $x \le y$ imply $y \in \mathcal{F}$ (immediate from the law of balance). Second, in the dual of the convexity of $\top$-zooming, $\overset{\vee}{u}|\overset{\vee}{v} \le u \overset{\vee}{|} v$, replace u with $\dot{w}|x$ and v with $\dot{x}|y$ to obtain:

$$(w \multimap x) \;\big|\; (x \multimap y) \le \top\big|(w \multimap y) \quad [42]$$

because $(w \multimap x)\big|(x \multimap y) = (\dot{w} \overset{\vee}{|} x)\big|(\dot{x} \overset{\vee}{|} y) \le (\dot{w}|x) \overset{\vee}{|} (\dot{x}|y) = (\dot{x}|x) \overset{\vee}{|} (\dot{w}|y) = \odot \overset{\vee}{|} (\dot{w}|y) = \top\big|(\dot{w} \overset{\vee}{|} y) = \top\big|(w \multimap y)$. Hence if $w \preceq x$ and $x \preceq y$ then $(w \multimap x)\big|(x \multimap y) \in \mathcal{F}$ forcing $\top\big|(w \multimap y) \in \mathcal{F}$ and, finally, $(w \multimap y) \in \mathcal{F}$.

Thus $\equiv$ is an equivalence relation. It is a congruence with respect to dotting because $w \multimap x = \dot{x} \multimap \dot{w}$, hence $\dot{w} \preceq \dot{x}$ iff $x \preceq w$. In the verification that $y|$ is covariant we used the equation $(y|w) \multimap (y|x) = \top\big|(w \multimap x)$ which quite suffices to show that $w \preceq x$ iff $y|w \preceq y|x$ and consequently that $\equiv$ is a congruence with respect to midpointing. Finally, to see that $\equiv$ is a congruence with respect to zooming it suffices to show that $w \preceq x$ implies $\hat{w} \preceq \hat{x}$. We may as well show that it implies $\overset{\vee}{w} \preceq \overset{\vee}{x}$ at the same time. The scale identity, in the form $w \multimap x = (\hat{w} \multimap \hat{x}) \,\big|\, (\overset{\vee}{w} \multimap \overset{\vee}{x})$ does just that.

Given an element, a, in a scale we will need to see how to construct $(\!(\boldsymbol{a})\!)$ the **principal $\top$-face** it generates, that is, the smallest $\top$-face containing a.

7.1. Lemma. *The principal $\top$-face, $(\!(a)\!)$, is the set of all x such that $a \le (\top|)^n x$ for all large n.*

($(\top|)^n$ is the nth iterate of the contraction at $\top$.) Clearly this set includes $\top$ and is closed under midpointing; for the other direction, suppose it includes $x|y$; then from $a \le (\top|)^n(x|y)$ we may infer $a \le (\top|)^n(x|y) \le (\top|)^n(\top|y) = (\top|)^{n+1}y$ for sufficiently large n and y is clearly in the set. For the other direction note that in any quotient where a becomes $\top$ the inequality $a \le (\top|)^n x$ clearly forces $(\top|)^n x$ to become $\top$, after which n

[42] If both sides of this inequality are $\top$-zoomed we obtain

$$(w \multimap x) \overset{\wedge}{|} (x \multimap y) \le (w \multimap y).$$

When this inequality is viewed as a map in a monoidal closed category:

$$(w \multimap x) \otimes (x \multimap y) \to (w \multimap y)$$

its name is the "composition map."

applications of $\top$-zooming will force x itself to be $\top$. That is, if $a \leq (\top|)^n x$ for any n, then x must be in any $\top$-face that contains a.

8. The Linear Representation Theorem

We wish to prove:

8.1. THEOREM. *Every scale can be embedded in a product of linear scales.*

An algebra (for any equational theory) is said to be **sub-directly irreducible**, or **SDI** for short, if whenever it is embedded into a product of algebras one of the coordinate maps is itself an embedding. Every algebra (for any equational theory) is embedded in the product of all of its SDI quotients (we will repeat the proof for this case). But first:

8.2. LEMMA. *If a scale is an* SDI *then it is linearly ordered.*

A homomorphism of scales is an embedding iff its kernel is trivial. A scale is an SDI iff the map into the product of all of its proper quotient scales fails to be an embedding. Hence it is an SDI iff the intersection of all non-trivial $\top$-faces is non-trivial. Let $s < \top$ be an element in that minimal non-trivial $\top$-face. Then for every element $a < \top$, its principal $\top$-face, $(\!(a)\!)$, must contain s. Thus an SDI scale has an element $s < \top$ such that for all $a < \top$ it is the case that $a < (\top|)^n s$ for almost all n. (This may be rephrased: a scale is an SDI iff there is a sequence of the form $\{(\top|)^n s\}_n$ cofinal among elements below $\top$.) If x and y are both below $\top$ then clearly $x \vee y < (\top|)^n s$ almost all n, in particular $x \vee y$ is below top. That is, SDI scales satisfy the disjunction property which, as has already been observed, implies linearity via the equation of linearity, $(x \multimap y) \vee (y \multimap x) \;=\; \top$.

The fact that all scales can be embedded in a product of linear scales is now easily obtainable: for each element $s < \top$ use the axiom of choice to obtain a $\top$-face, $\mathcal{F}_s$, maximal among $\top$-faces that exclude s; it is routine that in the corresponding quotient scale the element in the image of s becomes equal to $\top$ in every proper quotient thereof hence is, as just argued, linearly ordered. The intersection of all the $\top$-faces of the form $\mathcal{F}_s$ is clearly trivial. (Note that the structure of this proof of the linear representation theorem is forced: if the result is true then necessarily every SDI is linear and the theorem is equivalent to SDIs being linear.)

An immediate corollary:

8.3. COROLLARY. *Every equation, indeed every universal Horn sentence, true for all linear scales is true for all scales.*

It should be noted that the axiom of choice is avoidable for purposes of this corollary. Given a Horn sentence,

$$(s_1 = t_1) \,\&\, \cdots \,\&\, (s_n = t_n) \quad \Rightarrow \quad (u = v)$$

suppose there were a counterexample in some scale, A. The elements used for the counterexample generate a countable subscale, A'. The term $(u \multimap v) \wedge (v \multimap u)$ evaluates to an element $b < \top$. We can construct a $\top$-face, $\mathcal{F}$, in A' maximal among those that exclude b without using choice since A' is countable. The image of the counterexample in the linear scale $A'/\mathcal{F}$ remains a counterexample.

When working with a linearly ordered set it is completely trivial that covariant functions automatically distribute with the lattice operations. Hence for all scales we have:

$$\begin{aligned}
x\big|(y \wedge z) &= (x|y) \wedge (x|z) \\
x\big|(y \vee z) &= (x|y) \vee (x|z) \\
\widehat{x \wedge z} &= \hat{x} \wedge \hat{y} \\
\widehat{x \vee z} &= \hat{x} \vee \hat{y} \\
(x \wedge z)^{\vee} &= \overset{\vee}{x} \wedge \overset{\vee}{y} \\
(x \vee z)^{\vee} &= \overset{\vee}{x} \vee \overset{\vee}{y} \\
x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\
x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z)
\end{aligned}$$

(The last two equations—the definition of distributive lattices—are, of course, equivalent in any lattice.)

It is easy to check that if f is any binary operation on a linear scale satisfying the "dilatation equation," $f\langle a, a|x\rangle = x$, and if, further, for any fixed a it is covariant in x, then $f\langle a, x\rangle = a \triangleleft x$, hence dilatations are self-dual.[43]

Using the linear representation theorem we obtain a proof for a lemma that we will need later:

8.4. Lemma. *The image of the central contraction, $\odot|$, is the sub-interval* $[\bot|\odot, \odot|\top]$.

We need to show that if $\odot|\bot \le x \le \odot|\top$ then we can solve for $x = \odot|y$. We remove the existential to obtain a Horn sentence by setting $y = \odot \triangleleft x$. Thus we need to show that in any linear scale $\odot|\bot \le x \le \odot|\top$ implies $x = \odot|(\odot \triangleleft x)$. Linearity allows us to reduce to the two cases $x \le \odot$ and $\odot \le x$. Symmetry allows us to concentrate on the case $\odot \le x$, hence we can assume $x = \overset{\vee}{x}\,|\,\hat{x} = \top|\,\hat{x}$. From $x \le \odot|\top$ we infer that $\hat{x} \le \widehat{\odot|\top} = \odot$ hence that $\hat{x} = \overset{\vee}{\hat{x}}\,|\,\overset{\wedge}{\hat{x}} = \overset{\vee}{\hat{x}}\,|\bot$ which combines to give $x = \top\big|(\bot|\,\overset{\vee}{\hat{x}}) = (\top|\bot)\big|(\top|\,\overset{\vee}{\hat{x}}) = \odot\big|(\top|\,\overset{\vee}{\hat{x}})$. (So $y = \top|\,\overset{\vee}{\hat{x}}$.) It is routine now that $\odot|(\odot \triangleleft x) = x$. [44]

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[43] As promised, we now have $(((\dot{a}\,|\bot)|x)^{\vee})^{\wedge} = (((\dot{a}\,|\top)|x)^{\wedge})^{\vee}$. There are two other corollaries of interest. First, *any dilatation is definable using just central dilatation*: $a \triangleleft x = \odot \triangleleft (\odot \triangleleft ((\dot{a}\,|\odot)|x))$ because the one appearance of x is in a covariant position and $\odot \triangleleft (\odot \triangleleft ((\dot{a}\,|\odot)\big|(a|x))) = \odot \triangleleft (\odot \triangleleft ((\dot{a}\,|a)\big|(\odot|x))) = \odot \triangleleft (\odot|x) = x$. Second, *central dilatation is definable using (twice) any one dilatation*: $\odot \triangleleft x = a \triangleleft (a \triangleleft ((\odot|a)|\,\dot{x}))^{\bullet}$ because the one appearance of x is in a covariant position and $a \triangleleft (a \triangleleft ((\odot|a)\big|(\odot|x)^{\bullet}))^{\bullet} = a \triangleleft (a \triangleleft ((\odot|a)\big|(\odot|\,\dot{x})))^{\bullet} = a \triangleleft (a \triangleleft (\odot|(a|\,\dot{x})))^{\bullet} = a \triangleleft (a \triangleleft ((a|\,\dot{a}\,|(a|\,\dot{x}))))^{\bullet} = a \triangleleft (a \triangleleft (a|(\dot{a}\,|\,\dot{x})))^{\bullet} = a \triangleleft (\dot{a}\,|\,\dot{x})^{\bullet} = a \triangleleft (a|x) = x$. Hence any one dilatation can be used to construct any other dilatation.

[44] Let $\overline{x}$ denote the central dilatation $\odot \triangleleft x$. We could take $\overline{x}$ as primitive and define $\hat{x}$ as $\overline{\overline{(\bot|\odot)|x}}$. There

The fact that $\top$-zooming distributes with meet has an important application: the lattice of congruences is distributive. Recall, first, that in any lattice a set is called a "filter" if it is hereditary upwards and closed under finite meets.[45] By a **zoom-invariant filter** we mean a filter closed under the zooming operations. Since $\bot$-zooming is inflationary a filter is zoom-invariant iff it is closed under $\top$-zooming. An important lemma:

8.5. LEMMA. *A subset of a scale is a $\top$-face iff it is a zoom-invariant filter.*

Because: suppose that $\mathcal{F}$ is a filter invariant with respect to zooming; from $x \wedge y = (x \wedge y) \mid (x \wedge y) \leq x|y$ we know that $\mathcal{F}$ is closed under midpointing; that it is a face follows immediately from $\widehat{x|y} \leq \widehat{x|\top} = x$.

The other direction is an immediate consequence of the facts that zoom-invariant filters are preserved under inverse images of homomorphisms and that any $\top$-face is the inverse image of a one-element zoom-invariant filter, to wit, $\{\top\}$. (It is not hard to give a direct proof: we have already noted that the law of balance says that if x is an element of a $\top$-face, $\mathcal{F}$, and if $x \leq y$ then $y \in \mathcal{F}$; the law of compensation easily implies that $\hat{x} \in \mathcal{F}$; and if x and y are both in $\mathcal{F}$ we finish with $\widehat{x|y} \leq \widehat{x|\top} = x$ and similarly $\widehat{x|y} \leq y$ hence $\widehat{x|y} \leq x \wedge y$.)

8.6. THEOREM. *The congruence lattice of any scale is a spatial locale*

The pre-ordained name for the space in question is the **spectrum** of S, denoted **Spec($\boldsymbol{S}$)**.

First, the lattice of filters in any distributive lattice is itself a distributive lattice and the argument continues to work when we replace "filter" with "zoom-invariant fil-

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is something to be said for this choice. $\overline{x}$ (unlike $\hat{x}$ and $\overset{\vee}{x}$) appears as an innate operation on almost any graphic calculator. At first glance it looks like we could reduce by one the number of axioms. We would take the single $\overline{\odot|x} = x$ and use the previous footnote to obtain the two unital laws.

The important reason for not using this definition is that the origin of the notion of scales would be belied. But there is another: even when the scale identity is translated into this language the equations are not complete. They do not fix the primitive operation, $\overline{x}$. For a separating example take any scale and define $\overline{x}$ as the standard central dilatation with one exception: redefine $\overline{\top}$ any way one chooses. Thus—besides the translation of the scale identity—a further equation is needed to fix the primitive $\overline{x}$ operation as defined from $\hat{x}$. (Without such, note that there is no way of proving that the primitive operation $\overline{x}$ is covariant, hence no way of showing that $\top$-faces arise from congruences and no way of obtaining the linear representation theorem.)

One could redo the notion of minor scale using $\overline{x}$ as the primitive. After the constant law add the three equations $\overline{\odot|x} = x$, $\overline{\top|(\top|x)} = \top$, and $\overline{\bot|(\bot|x)} = \bot$. The proofs that the elements named by the typed terms are closed under dotting and midpointing remains unchanged. That they are closed under $\overline{x}$ one need only verify $\overline{\top} = \overline{\top|(\top|\top)} = \top$, $\overline{\bot} = \overline{\bot|(\bot|\bot)} = \bot$, $\overline{\top|(\bot|x)} = \overline{(\top|\bot)|(\top|x)} = \top|x$, and, similarly, $\overline{\bot|(\top|x)} = \bot|x$. The previous argument that the theory has a unique consistent equational completion still holds.

[45]It's worth noting—in the context of scales— an alternative definition: $\mathcal{F}$ is a filter if

$$\top \in \mathcal{F}$$
$$x \wedge y \in \mathcal{F} \quad \text{iff} \quad x \in \mathcal{F} \text{ and } y \in \mathcal{F}$$

ter." The main observation (for both proofs) is that the join of filters $\mathcal{F}$ and $\mathcal{G}$ is the set $\{\ x \wedge y\ :\ x \in \mathcal{F},\ y \in \mathcal{G}\ \}$. (It is clearly closed under meet and $\top$-zooming. If $x \wedge y \le z$ then, using distributivity of the lattice, $z = (x \vee z) \wedge (y \vee z)$ where, of course, $x \vee z \in \mathcal{F}$ and $y \vee z \in \mathcal{G}$.)

To see that $(\mathcal{F} \vee \mathcal{G}) \cap \mathcal{H} \subseteq (\mathcal{F} \cap \mathcal{H}) \vee (\mathcal{G} \cap \mathcal{H})$ (the reverse containment holds in any lattice) we note that an arbitrary element in the left-hand side is of the form $x \wedge y$ where $x \in \mathcal{F}$, $y \in \mathcal{G}$ and $x \wedge y \in \mathcal{H}$. But the last condition implies that both x and y are in $\mathcal{H}$. Hence, $x \in \mathcal{F} \cap \mathcal{H}$ and $y \in \mathcal{G} \cap \mathcal{H}$, thus $x \wedge y \in (\mathcal{F} \cap \mathcal{H}) \vee (\mathcal{G} \cap \mathcal{H})$.

Distributivity, recall, is quite enough to establish that a lattice of congruences is a locale, that is, finite meets distribute with arbitrary joins. It is always a spatial locale: the points are the "prime" congruences, that is, those that are not the intersection of two larger congruences. Translated to filters: $\mathcal{F}$ is a point if it has the property that whenever $x \vee y \in \mathcal{F}$ it is the case that either $x \in \mathcal{F}$ or $y \in \mathcal{F}$. Put another way, of course, the points of $\operatorname{Spec}(S)$ are the linearly ordered quotients of S. We will show that $\operatorname{Spec}(S)$ is compact normal (but not always Hausdorff). We obtain—just as in the ancestral subject of spectra for associative algebras—a representation of an arbitrary scale as the scale of global sections of a sheaf of linear scales.[46]

We pause to obtain a "pushout lemma" for scales:

8.7. LEMMA. *Let $A \to B$ be monic and $A \to C$ a quotient map. Then in the pushout*

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array}$$

the map $C \to D$ is monic (and, as in any category of algebras, $B \to D$ is a quotient map).

Because, if we view A as a subscale of B and take $\mathcal{F} = \ker(A \to C)$ then we obtain a $\top$-face of B, to wit, $\mathcal{F}^{\Uparrow} = \{\ b \in B\ :\ \exists_{a \in A}\ a \le b\ \}$. It is easy to check that $\mathcal{F}^{\Uparrow}$ is zoom-invariant and that $A \cap \mathcal{F}^{\Uparrow} = \mathcal{F}$. Define $D = B/\mathcal{F}^{\Uparrow}$. The map $A \to B \to D$ has the same kernel as $A \to C$ and we obtain an embedding $C \to D$. It is easily checked to yield a pushout diagram.[47]

[46]Recall that the space of points of a spatial locale is always "sober" (most easily defined as a space maximal among T_0 spaces with the given locale of open sets). There is often a minimal space, one with the fewest points (the pre-ordained name for this condition is "spaced-out"). For any distributive congruence lattice this minimal space does exist: its elements are the congruences of the SDI quotients. (Could this connection between universal algebra and Stone theory be new?)

[47]This pushout lemma fails in most equational theories. In the category of groups let $A \to B$ be the inclusion map of a subgroup and $A \to C$ be an epi whose kernel is A'. Then the kernel of $C \to D$ is the image in C of the intersection of A with the normal closure of A' in B. If one takes B to be the alternating group of order 12, A the unique subgroup of order 4 (the original Klein group), and A' one of its 2-element subgroups, then the kernel of $B \to D$ is all of A, and $C \to D$ is the trivial map from the 2-element cyclic group to the 3-element cyclic group. A more dramatic example is to enlarge B to the alternating group or order 60; D then collapses to a 1-element group.

9. Lipschitz Extensions and $\mathbb{I}$-Scales

Given an equational theory $\mathbb{T}$, we may say that an extension $\mathbb{T}'$ is "co-congruent" if congruences for the operations in $\mathbb{T}$ remain congruences for all the operations in $\mathbb{T}'$. If all new operations are constant then the extension is automatically co-congruent (e.g. the theory of monoids is a co-congruent extension of the theory of semigroups). A more interesting example is the next step: a monoid congruence is automatically a congruence with respect to the entire group structure (because $x \equiv y$ implies $x^{-1} = x^{-1}yy^{-1} \equiv x^{-1}xy^{-1} = y^{-1}$). [48]

An extension of the theory of scales is co-congruent iff the equivalence relation determined by any $\top$-face respects the new operations that appear in the extended theory. We will use $\boldsymbol{x \multimapboth y}$ to denote $(x \multimap y) \wedge (y \multimap x)$. A new unary operator f is co-congruent if every $\top$-face $\mathcal{F}$ that contains the element $x \multimapboth y$ also contains the element $fx \multimapboth fy$. If we take $\mathcal{F}$ to be the principal $\top$-face $\langle\!\langle x \multimapboth y \rangle\!\rangle$ then for co-congruence to hold we must have, for some integer n,

$$x \multimapboth y \leq (\top|)^n (fx \multimapboth fy)$$

When interpreted on the standard interval this becomes the assertion that f is Lipschitz continuous (with Lipschitz constant $\leq 2^n$):

$$|fx - fy| \leq 2^n |x - y| \quad [49]$$

If we move to the free algebra on two generators for the extended theory we see that co-congruence requires the existence of an n that works in all models. The argument for

[48] There are several similar examples that involve a unary involutory operation that delivers something like an inverse. A lattice-congruence on a Boolean algebra is automatically a congruence for negation: if $x \equiv y$ then $\neg x = \neg x \wedge (y \vee \neg y) \equiv \neg x \wedge (x \vee \neg y) = \neg x \wedge \neg y = (\neg x \vee y) \wedge \neg y \equiv (\neg x \vee x) \wedge \neg y = \neg y$. A ring-congruence on a von Neumann strongly regular ring is automatically a congruence for the "pseudo-inverse" (to wit, the unary operation that satisfies $x^2x^* = x = x^{**}$): if $x \equiv y$ then (using that $xx^* = x^*x$ is a consequence of the axioms) $x^* = x^{*2}x \equiv x^{*2}y = x^{*2}y^2y^* = x^{*2}y(y^2y^*)y^* = x^{*2}y^3y^{*2} \equiv x^{*2}x^3y^{*2} = x^*(x^*x^2)xy^{*2} = x^*x^2y^{*2} = xy^{*2} \equiv yy^{*2} = y^*$.

A congruence on a scale with respect to midpointing and the two zoom operations is automatically a congruence for dotting: if $u \equiv v$ then $\dot{u} = (\top|\,\dot{u})^{\wedge} = ((\bot\big|(\top|\,\dot{u}))^{\vee})^{\wedge} = (((\bot|\top)\big|(\bot|\,\dot{u}))^{\vee})^{\wedge} = (((\dot{u}\,|u)\big|(\bot|\,\dot{u}))^{\vee})^{\wedge} = (((\dot{u}\,|\bot)\big|(u|\,\dot{u}))^{\vee})^{\wedge} = (((\dot{u}\,|\bot)\big|(v|\,\dot{v}))^{\vee})^{\wedge} \equiv (((\dot{u}\,|\bot)\big|(u|\,\dot{v}))^{\vee})^{\wedge} = (((\dot{u}\,|u)\big|(\bot|\,\dot{v}))^{\vee})^{\wedge} = (((\bot|\top)\big|(\bot|\dot{v}))^{\vee})^{\wedge} = ((\bot\big|(\top|\dot{v}))^{\vee})^{\wedge} = (\top|\dot{v})^{\wedge} = \dot{v}$.

But these three examples are misleading. In each case the new operation is unique—when it exists—given the old structure. As we will see such is not the case for all the co-congruent extensions of the theory of scales. Nor is the uniqueness sufficient. A meet semi-lattice has at most one lattice structure but notice that on the four-element non-linear lattice the equivalence relation that smashes the three elements below the top to a single point is a meet- but not a lattice-congruence. A lattice has at most one Heyting-algebra structure but the only non-trivial variety of Heyting algebras in which lattice-congruences are automatically Heyting congruences is the variety of Boolean algebras: any non-Boolean Heyting algebra contains a three-element subalgebra and the equivalence relation on the three-element Heyting algebra that smashes the bottom two elements to a point is a lattice- but not a Heyting-algebra congruence.

[49] On the unit interval $x \mathbin{\dot{\multimapboth}} y$ (the dotting operation applied to $x \multimapboth y$) is $|x-y|$. The dual inequality of $x \multimapboth y \leq (\top|)^n (fx \multimapboth fy)$ is $(\bot|)^n (fx \mathbin{\dot{\multimapboth}} fy) \leq x \mathbin{\dot{\multimapboth}} y$.

unary operations easily extends to arbitrary arities. Hence for extensions of the theory of scales we will use the phrase **Lipschitz extension** instead of "co-congruent extension."

The first application is that Lipschitz extensions of the theory of scales all enjoy the linear representation theorem. (More important will be the consequence of the section to come: every consistent Lipschitz extension has an interpretation on the standard interval $\mathtt{I}$.)

If M is a monoid, we understand an ***M*-scale** to be a scale on which M acts. We will not require that the M-actions be endomorphisms of the entire scale structure or anything else in particular. In all the cases to be discussed, however, the actions will preserve midpointing.

We will be particularly interested in two cases: when $M = \mathtt{I}$ and when M is the submonoid of rationals in the standard interval. The **theory of I-scales** is obtained by adding a unary operator for every $r \in \mathtt{I}$ whose value at x is denoted rx and that satisfies the equations:

$$r\odot = \odot$$

$$r(x|y) = rx|ry$$

and for all $q \in \mathbb{I}$ with $q \geq r$ the equation [50]

$$q \geq r\top$$

and if $q \leq r$

$$q \leq r\top$$

There is obviously a unique interpretation of these operations on the standard interval $\mathtt{I}$. [51] The theory is not as it stands Lipschitz.[52] So we add the condition

$$x \circ\!\!-\!\!\circ\, y \quad \leq \quad rx \circ\!\!-\!\!\circ\, ry$$

9.1. Lemma. *Any scale is embedded in its* $\mathtt{I}$*-scale reflection.*

Because given a scale S consider the equational theory obtained by adding to the theory of scales a constant for each element of S and adding as equations all the variable-free equations in these constants that hold for S. Now add the $\mathtt{I}$-scale structure. As for its consistency we use the "compactness argument": every finite set of equations in this extended theory has a model (for each relevant $r \in \mathtt{I}$ we can find $q \in \mathbb{I}$ that will satisfy the finite number of equations that involve r) and such suffices for consistency.

It is a consequence of the results in the next section that this newly constructed $\mathtt{I}$-scale has a quotient isomorphic to $\mathtt{I}$ with its unique action.

[50]Bear in mind that in the presence of a lattice operation any inequality is equational: $x \leq y$ is equivalent with $x = x \wedge y$.

[51]It was discussed above in the last footnote of Section 6.

[52]Order the polynomial ring $\mathbb{R}[\varepsilon]$ by stipulating $P(\varepsilon) \geq 0$ iff $P(1/n) \geq 0$ for all sufficiently large n. Its standard interval is a scale and the map that sends $P(\varepsilon)$ to $P(2\varepsilon)$ is a scale-automorphism thereon, hence easily satisfies the equations (with r = 1). It is not Lipschitz.

Every scale may be viewed as an $\mathbb{I}$-scale: we may define qx for $q \in \mathbb{I}$ and x in an arbitrary scale using the inductive scheme introduced in the construction of $\mathbb{I}$: for type-0 terms take $\top x = x$ and $\bot x = \dot{x}$ and for all higher-typed term, q, take $(\top|q)x = x|(qx)$ and $(\bot|q)x = \dot{x}\,|(qx)$. [53]

10. Simple Scales

10.1. Theorem. *A scale is simple iff the sequence* $\bot, \top|\bot, \top|(\top|\bot), \ldots, (\top|)^n\bot, \ldots$ *is cofinal among all elements below* $\top$.

The cofinality clearly implies simplicity: if $\mathcal{F}$ is a non-trivial $\top$-face then necessarily there exists $x \in \mathcal{F}, x \neq \top$ and the cofinality says that $(\top|)^n(\bot) \in \mathcal{F}$ for some n, hence that $\bot \in \mathcal{F}$, which means—of course—that $\mathcal{F}$ is entire. Conversely, a simple algebra is necessarily an SDI, hence necessarily linear; but we have much more. We may take the element s used above in the characterization of SDIs to be the element $\bot$. Then, since $\bot$ is included in every non-trivial $\top$-face (because in a simple scale the entire set is the only non-trivial $\top$-face) we know that the sequence $\{(\top|)^n(\bot)\}$ is cofinal among all elements below $\top$.

A scale satisfying this condition is called, of course, **Archimedean**.[54]

10.2. Lemma. *A scale is simple iff between any two elements there is a constant (that is, an element from the initial subscale).*

One direction is immediate: if between $\top$ and any $x < \top$ there is some constant, then there must be one of the form $(\top|)^n(\bot)$. The other direction requires a little work. Given

[53] The set of operations that preserve midpointing and $\odot$ form a closed midpoint algebra and we can see this action of $\mathbb{I}$ on arbitrary scales as a consequence of the fact that $\mathbb{I}$ is the initial closed midpoint algebra.

[54] There are those who say that the property should be known as "Eudoxian." Euclid wrote about it in the *Elements* and Proclus said that the idea was due to one Eudoxus but a case may be made for Archimedes. Euclid's Definition 5 of Book V:

> Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

But Archimedes in his *Quadrature of the Parabola* does a better job of isolating the salient point:

> The excess by which the greater of (two) unequal areas exceeds the less, can by being added to itself be made to exceed any given finite area.

Which is how in the absence of the word *positive* one states that any positive quantity when repeatedly doubled becomes arbitrarily large. (Surely such is equivalent to the assertion that any quantity when repeatedly halved becomes arbitrarily small.) Archimedes, note, did not actually claim originality in this; immediately after the line quoted above he writes:

> The early geometers have also used this lemma.

$u < v$, let n be minimal such that $v \multimap u < (\top|)^{n+1}\bot$. The argument requires induction on n. If $n = 0$, that is if $v \multimap u < \odot$ then the two inequalities $u = \top \multimap u \leq v \multimap u$ and $\dot{v} = v \multimap \bot \leq v \multimap u$ (consequences of the contravariance of $\multimap$ in the first variable and covariance in the second) yield $u < \odot < v$ and we are done. If $n > 0$ we consider the three cases: $u < v \leq \odot$, $u < \odot < v$ and $\odot \leq u < v$. In the 1st case, we have $\hat{u} = \bot = \hat{v}$, hence (using the scale identity) $v \multimap u = (\overset{\vee}{v} \multimap \overset{\vee}{u}) \mid (\hat{v} \multimap \hat{u}) = (\overset{\vee}{v} \multimap \overset{\vee}{u})|\top$ and we obtain $\overset{\vee}{v} \multimap \overset{\vee}{u} < (\top|)^{n}\bot$. Hence by the inductive assumption there is a constant $r \in \mathbb{I}$ such that $\overset{\vee}{u} < r < \overset{\vee}{v}$, thus for $q = r|\bot$ we have $u = \overset{\vee}{u}|\hat{u} = \overset{\vee}{u}|\bot < q < \overset{\vee}{v}|\bot = \overset{\vee}{v}|\hat{v} = v$. In the 2nd case we take, of course, $q = \odot$. In the 3rd case we use the 1st case to obtain $\dot{v} < q < \dot{u}$ and finish with $u < \dot{q} < v$.

There are two remarkable facts. The first is that there are many simple scales, so many that every non-trivial scale has a simple quotient: use Zorn's lemma on the set of $\top$-faces that do not contain $\bot$.

The second is that there are very few simple scales. Because there is a constant between any two elements we know that elements are distinguished by which constants appear below (or for that matter, above) them, hence there can not be more elements in a simple scale than there are sets of constants: therefore a simple scale has at most $2^{\aleph_0}$ elements.[55]

Since there is no flexibility on what homomorphisms do to constants:

10.3. PROPOSITION. *Given a pair of simple scales there is at most one map from the first to the second.*

The full subcategory of simple scales is, thus, a pre-ordered set. It has a maximal element and the name of that maximal element is the closed interval, $\mathtt{I}$. Non-constant maps from simple scales are embeddings, hence every simple scale is uniquely isomorphic to a unique subscale of $\mathtt{I}$. (And for an algebraic construction of $\mathtt{I}$ take any simple quotient of a coproduct over the family of all simple scales.)

Combining the two remarkable facts we obtain

10.4. THEOREM. *Every non-trivial scale—indeed, any non-trivial model of any Lipschitz extension of the theory of scales—has a map to* $\mathtt{I}$.

One quick corollary: add any set of constants to the theory of scales and any consistent set of equations thereon. Necessarily there is an interpretation for all the constants in $\mathtt{I}$ satisfying all the equations. (Recall that if all the new operations in an extension of the theory of scales are constants then it is automatically a Lipschitz extension.) As an example, adjoin just one constant, a, and a maximal consistent set of axioms of the form $q \multimap a = \top$ and $a \multimap q = \top$ where the qs are restricted to constants, (that is, names of elements of $\mathbb{I}$). Such a maximal consistent extension is, of course, called a "Dedekind cut" and this quick corollary of the standard models theorem (to wit, that any such set of conditions can be realized in $\mathtt{I}$) is, of course, called "Dedekind completeness."

[55] Compare with the theory of groups: no quotient of the rational numbers—viewed as a group under addition—is simple. On the other hand there are simple groups of every infinite cardinality; indeed, every group can be embedded in a simple group.

Because every consistent Lipschitz theory of scales can be enlarged to a consistent Lipschitz theory of $\mathtt{I}$-scales (Section 9) we obtain, as an immediate consequence of the existence of simple quotients of non-trivial $\mathtt{I}$-scales:

10.5. THEOREM. ON THE EXISTENCE OF STANDARD MODELS
Every consistent Lipschitz extension of the theory of scales has an interpretation on $\mathtt{I}$.

Note that every non-trivial linear scale has a unique map to $\mathtt{I}$ (the kernel of the map is the $\top$-face of all elements larger than all $(\top|)^n\bot$). We may rephrase this: consider the category of non-trivial scales that satisfy the coalgebra condition, to wit, $\hat{x} = \bot$ or $\overset{\vee}{x} = \top$ for all x. This category has a final object. (This fact is, of course, much much weaker than the usual statement of the coalgebraic characterization of $\mathtt{I}$, but it is comforting to see it arise in such an algebraic manner.)

Since each simple scale has a unique map to $\mathtt{I}$:

10.6. LEMMA. *The maps from a scale to* $\mathtt{I}$ *are known by their kernels.*

Later we will use the fact that the **maximal ⊤-face spectrum** of A, denoted **Max(A)** is canonically equivalent with the set of maps $(A, \mathtt{I})$.

One immediate application:

10.7. THEOREM. *The standard interval,* $\mathtt{I}$*, is an injective object in the category of scales.*

Because, given a subscale S' of S and an $\mathtt{I}$-valued map f' from S' we seek an extension to all of S. Let $\mathcal{F} \subseteq S$ be the $\top$-face generated by $\ker(f')$, (that is, the result of adding all elements in S larger than an element in $\ker(f')$) and note that it remains a proper $\top$-face, hence $S/\mathcal{F}$ is non-trivial and we may choose a map $S/\mathcal{F} \to \mathtt{I}$. Define f to be $S \to S/\mathcal{F} \to \mathtt{I}$. The kernel of $S' \to S \to \mathtt{I}$ must, of course, contain $\ker(f')$. But $\ker(f')$ is maximal, hence f' has the same kernel as $S' \to S \to \mathtt{I}$ and as we just noted, maps to $\mathtt{I}$ are known by their kernels: thus f is an extension of f'.[56]

Bear in mind that all these special properties of $\mathtt{I}$ are maintained for any Lipschitz extension of the theory of scales.

Following the language of ring theory we define the **Jacobson radical** of a scale to be the intersection of all its maximal $\top$-faces and we say that a scale is **semi-simple** if its Jacobson radical is trivial. (The name used in the theory of convex sets for maximal proper faces is "facet," hence we could say that the Jacobson radical is the intersection of all the "top-facets." Doing so, of course, means that one must not be bothered by etymology.) A scale is semi-simple iff its representations into simple quotients are collectively faithful. (Hence, a better term for both rings and scales would have been "residually simple.")

10.8. THEOREM. *A scale is semi-simple iff* $\sup_n (\top|)^n\bot = \top$

In any simple scale the cofinality of $(\top|)^n\bot$ implies the weaker condition that its supremum is $\top$ and such remains the case in any cartesian product of simple scales. To see that

[56]The injectivity of $\mathtt{I}$ is quite enough to yield the existence of a map to $\mathtt{I}$ from every non-trivial scale since we know that any non-trivial scale contains a copy of at least one scale with a map to $\mathtt{I}$, to wit, $\mathbb{I}$.

the condition implies semi-simplicity we need to show that it implies for every $x < \top$ that there is a simple quotient in which x remains less than $\top$. The condition tells us that there is n such that the inequality $(\top|)^n\bot < x$ fails, that is. $((\top|)^n\bot) \multimap x < \top$. By the linear representation theorem we may find a linear quotient in which that failure is maintained, that is, $((\top|)^n\bot) \multimap x$ remains below $\top$, hence in which $x \le (\top|)^n\bot$. Now take its (unique) simple quotient.

10.9. Lemma. *The Jacobson radical of a scale is the set,* $\mathcal{R}$*, of all* x *such that* $(\top|)^n\bot \le x$ *for all* n.

It is clear $\mathcal{R}$ is in the kernel of every simple quotient. For the converse we need to show that $((\bot|)^m\top) \multimap x \equiv \top \pmod{\mathcal{R}}$ for if all m, then $x \in \mathcal{R}$. But $((\bot|)^m\top) \multimap x \equiv \top$ $\pmod{\mathcal{R}}$ says, of course, that $((\bot|)^m\top) \multimap x \in \mathcal{R}$ hence $(\top|)^n\bot \le ((\top|)^m\bot) \multimap x$ all n. In particular $(\top|)^n\bot \le ((\top|)^n\bot) \multimap x$ all n. Use the adjointness lemma to obtain $\widehat{(\top|)^n\bot} \le x$ all n, hence that $(\top|)^{n-1}\bot \le x$ all n, which, of course, says $x \in \mathcal{R}$.

Our definition of the Jacobson radical as the intersection of all the maximal $\top$-faces relied on the axiom of choice. But note that this construction of the Jacobson radical is choice-free. (So it would have been better—with both rings and scales—to use the choice-free construction as the definition.)

11. A Few Applications

For the most algebraic construction of $\mathtt{I}$, take the co-product of all one-generator simple scales and reduce by the Jacobson radical. Put another way: start with a freely generated scale and for each generator reduce by a maximal consistent set of relations that involve only that generator (and, of course, the primitive constants of the theory of scales). Do this in such a way that every possible such set of relations appears for at least one generator. Now take any simple quotient. It is necessarily a copy of $\mathtt{I}$. But we do not need the axiom of choice: there is only one simple quotient and it is the result of reducing by the Jacobson radical. (We are in a case where semi-simplicity implies simplicity.)

The second footnote in this work suggested a way of handling limits of sequences in $\mathtt{I}$. Let's redo it, this time without using the axiom of choice. Again let $\mathtt{I}^{\mathbb{N}} = \prod_{\mathbb{N}} \mathtt{I}$ denote the scale of all sequences in $\mathtt{I}$. The first step is to identify sequences that agree almost everywhere to obtain the quotient scale $\mathtt{I}^{\mathbb{N}}/\mathcal{E}$ (where $\mathcal{E}$ is the $\top$-face of all sequences that are eventually constantly equal to $\top$). The next step (a step we could not take before) is to reduce by the Jacobson radical. $(\mathtt{I}^{\mathbb{N}}/\mathcal{E})/\mathcal{R}$. As already observed, there is never more than one map from $\mathtt{I}$ to a semi-simple scale hence the map $\mathtt{I} \xrightarrow{\Delta} \mathtt{I}^{\mathbb{N}} \to (\mathtt{I}^{\mathbb{N}}/\mathcal{E}) \to (\mathtt{I}^{\mathbb{N}}/\mathcal{E})/\mathcal{R}$ is the unique map from $\mathtt{I}$ to $(\mathtt{I}^{\mathbb{N}}/\mathcal{E})/\mathcal{R}$. Let

$$\begin{array}{ccc} C & \to & \mathtt{I}^{\mathbb{N}} \\ \downarrow & & \downarrow \\ \mathtt{I} & \to & (\mathtt{I}^{\mathbb{N}}/\mathcal{E})/\mathcal{R} \end{array}$$

be a pullback where we view $C \to \mathtt{I}^{\mathbb{N}}$ as an inclusion. C is the subscale of all convergent sequences. The (vertical) map $C \to \mathtt{I}$ is the unique map from C to $\mathtt{I}$ that respects almost-everywhere equivalence. Its standard name is Lim.

We said in the opening section that $C \subseteq \mathtt{I}^{\mathbb{N}}$ could be defined as the joint equalizer of all the closed-midpoint maps from $\mathtt{I}^{\mathbb{N}}/\mathcal{E}$ to $\mathtt{I}$. Since pullbacks of equalizers are equalizers it suffices, obviously, to show that the (unique) map $\mathtt{I} \to (\mathtt{I}^{\mathbb{N}}/\mathcal{E})\mathcal{R}$ is such a joint equalizer. Define the **simple part** of any semi-simple scale to be the joint equalizer of all maps to $\mathtt{I}$; since, by definition, those maps are jointly faithful any map from the simple part to $\mathtt{I}$ is necessarily an embedding and the simple part is, indeed, simple; conversely any simple subscale has a unique map to $\mathtt{I}$, hence is in the simple part. All of which says that any map from $\mathtt{I}$ to a semi-simple scale, e.g. the diagonal map from $\mathtt{I}$ to $\mathtt{I}^{\mathbb{N}}$ followed by the quotient map to $(\mathtt{I}^{\mathbb{N}}/\mathcal{E})\mathcal{R}$, is automatically its simple part.[57]

Even before we took limits of sequences we defined $\mathcal{C}(G)$ as the set of continuous maps and promised that "In time we will obtain a totally algebraic definition." The scale of uniformly continuous $\mathtt{I}$-valued maps on a uniform space, X, is easier to construct. Consider, first, $\mathtt{I}^{X\times X}$, the scale of all functions from $X \times X$ to $\mathtt{I}$. Let $\mathcal{E}$ be the $\top$-face of those functions that are equal to $\top$ on some element of the given uniformity and let $\mathcal{R}$ be the Jacobson radical of $\mathtt{I}^{X\times X}/\mathcal{E}$. The pair of projection maps from $X \times X$ to X induces a pair of maps from $\mathtt{I}^X$ to $\mathtt{I}^{X\times X}$ which, in turn, yields—by composition—a pair of maps from $\mathtt{I}^X$ to $(\mathtt{I}^{X\times X}/\mathcal{E})/\mathcal{R}$. *The scale of uniformly continuous $\mathtt{I}$-valued maps on X is the equalizer of this final pair of maps.*

For ordinary continuity let X be a topological space and $\mathtt{I}^X = \prod_X \mathtt{I}$ denote the set of all functions (continuous or not) from X to $\mathtt{I}$. For $x \in X$ identify functions that agree on a neighborhood of x to obtain $\mathtt{I}^{\mathbb{N}}/\mathcal{E}_x$ (where $\mathcal{E}_x$ is the $\top$-face of all sequences that are equal to $\top$ on some neighborhood of x). Then reduce by the Jacobson radical $(\mathtt{I}^{\mathbb{N}}/\mathcal{E}_x)/\mathcal{R}$. Let

$$\begin{array}{ccc} C_x & \to & \mathtt{I}^X \\ \downarrow & & \downarrow \\ \mathtt{I} & \to & (\mathtt{I}^X/\mathcal{E}_x)/\mathcal{R} \end{array}$$

be a pullback where $C_x \to \mathtt{I}^X$ is an inclusion. C_x is the subscale of all functions from X

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[57] In the opening section we made the joint-equalizer assertion not for scale maps but for closed-midpoint maps. For a proof, let $s_1, s_2, \dots,$ be a sequence in $\mathtt{I}$. Note first that if $q \in \mathbb{I}$ is an upper bound for almost all s_n, then for any closed-midpoint map, $f : \mathtt{I}^{\mathbb{N}}/\mathcal{E} \to \mathtt{I}$ we have $f(s) \le q$, hence $f(s) \le \limsup s$ and similarly $f(s) \ge \liminf s$ In particular, for any convergent s we have $f(s) = \lim s$. Next, if a is an accumulation point of s then we may take infinite $\mathbb{N}' \subseteq \mathbb{N}$ such that s restricted to $\mathbb{N}'$ converges to a. The inclusion map $\mathbb{N}' \to \mathbb{N}$ induces a scale-map $\mathtt{I}^{\mathbb{N}}/\mathcal{E} \to \mathtt{I}^{\mathbb{N}'}/\mathcal{E}$ that carries s to a convergent sequence. The axiom of choice gives us an $\mathtt{I}$-valued scale map, hence any accumulation point of s appears as a value of an $\mathtt{I}$-valued scale-map—*a fortiori*, a closed-midpoint map—from $\mathtt{I}^{\mathbb{N}}/\mathcal{E}$ to $\mathtt{I}$. Since $\mathtt{I}$-valued closed-midpoint maps are closed under convex combinations we may conclude that the closed interval $[\liminf s, \limsup s]$ is the set of all such values. (But the only values of the scale-maps are the accumulation points: if b is not an accumulation point we may chose $\ell < b < u$ such that $s_n \notin (\ell, u)$ for almost all n; then $(s_n \multimap \ell) \vee (u \multimap s_n) = \top$ for almost all n; hence $(f(s) \multimap \ell) \vee (u \multimap f(s)) = \top$ and we may conclude $f(s) \notin (\ell, u)$.)

to $\mathtt{I}$ that are continuous at x, or, put another way:

11.1. PROPOSITION. *The functions in $\mathtt{I}^X$ that are continuous at x is the pullback of the simple part of $(\mathtt{I}^X/\mathcal{E}_x)/\mathcal{R}$.*

Finally:

$$\mathcal{C}(X) = \bigcap_{x \in X} C_x \ \subseteq\ \mathtt{I}^X$$

12. Non–Semi-Simple Scales and the Richter Scale

The **Richter scale**, $\mathtt{R}$, is defined as the interval from $\langle -1, 0\rangle$ through $\langle 1, 0\rangle$ in the lexicographically ordered dy-module $\mathbb{D} \oplus \mathbb{D}$. [58] The set-valued "Jacobson-radical functor" is represented by the Richter scale, that is, the elements of the Jacobson radical of a scale A are in natural one-to-one correspondence with the scale maps from $\mathtt{R}$ to A. Mac Lane's "universal element" (the most important concept in *Algebra*!) may be taken to be $\langle 1, -1\rangle$: for every element, x, in the Jacobson radical of A there is a unique map that carries $\langle 1, -1\rangle$ to x. [59]

The Richter scale is not, of course, simple. But it just misses. It has just one quotient neither entire nor trivial, to wit, its semi-simple reflection, $\mathbb{I}$. Hence the Richter scale appears as a subscale of every non–semi-simple scale: the necessary and sufficient condition for semi-simplicity is that a scale not contain a copy of the Richter scale.

A non-simple scale, A, is SDI iff there is an embedding $\mathtt{R} \to A$ such that $\mathtt{R}_*$ is cofinal in A_* (where the "lower star" means "remove the top").[60]

It is worth having at our disposal examples of arbitrarily large SDIs. Let B be any ordered set with top and bottom. Consider the set of functions from B to $\mathbb{D}$ with "finite support" (that is, the set of functions that are zero almost everywhere). Lexicographically order these functions and define $\top$ as the characteristic map of $-\infty$ (the bottom of B), define $\bot$ as its negation and define A to be the interval $[\bot, \top]$. We may regard $\mathtt{R}$ as the subscale of A of all functions whose support is confined to $\pm\infty$ (where $+\infty$ is the top of B). Every element in A not in this copy of $\mathtt{R}$ is less than any element in $\mathtt{R}$'s Jacobson

[58]If we view $\mathbb{D} \oplus \mathbb{D}$ as the ring $\mathbb{D}[\hbar]/(\hbar^2)$, ordered so that $\hbar$ is "infinitesimal," then the Richter scale is just the standard interval in the "ring of dyadic dual numbers."Its Jacobson radical is the set of all of its pairs of the form $\langle 1, q\rangle$ (note that q must be non-positive).

[59]For a proof, start with the following:

$$\begin{array}{rcl} f\langle 1, -m2^{-n}\rangle &=& (\top|)^n((x \overset{\wedge}{|}\,)^m \top) \\ f\langle 0, -m2^{-n}\rangle &=& \bot|f\langle 1, -m2^{1-n}\rangle \\ f\langle 0, +m2^{-n}\rangle &=& (f\langle 0, -m2^{-n}\rangle)^{\bullet} \\ f\langle q, \pm m2^{-n}\rangle &=& \odot\triangleleft\,(q|f\langle 0, \pm m2^{-n}\rangle) \end{array}$$

These formulas can be used, at least, for the uniqueness of f, but the fact they describe a scale map requires a bit of work. When we have in hand the representation theorem of Section 20 for the free scale on one generator an easier proof will be available.

[60]We will see later that all such $\mathtt{R} \to A$ are "essential extensions."

radical and any element in that radical less that $\top$ serves as an element whose multiples are cofinal among the elements below $\top$ in all of A. The existence of such an element, recall, is equivalent with A being an SDI .

13. A Construction of the Reals

Among the many ways of constructing the reals perhaps the best is as the set of "germs of midpoint- and $\odot$-preserving maps" (that is, a real is named by such a map defined on some open subinterval containing $\odot$; two such partial maps name the same real if they agree on the intersection of their domains).

The entire ordered-field structure is inherent: 0 is named by the constant map; 1 by the identity map; negation by dotting; $r + s$ is characterized by $\odot|((r + s)x) = (rx)|(sx)$ for all x near $\odot$; multiplication is, of course, defined as composition; $r \leq s$ iff $rx \leq sx$ for all x near $\odot$. The standard interval in this ring has a (unique) scale-isomorphism to $\mathtt{I}$, to wit, the map that sends r to $r\top$.

To fill in the details, note first that a midpoint- and $\odot$-preserving $\mathtt{I}$-valued map on an interval is necessarily monotonic:

13.1. LEMMA. *Any function from* $\mathtt{I}$ *to* $\mathtt{I}$ *that preserves midpoints is monotonic.*

It suffices to show that if f preserves midpoints then it preserves betweenness. Suppose that $a < b < c$ and that $f(b)$ is *not* between $f(a)$ and $f(c)$. We will regularize the example by replacing f with $\dot{f}$, if necessary, to ensure that $f(a) < f(c)$. Either $f(b) < f(a) < f(c)$ or $f(a) < f(c) < f(b)$. Symmetry implies that without loss of generality we may assume the latter. It suffices to show that there is another point b' between a and c which is not only a counterexample, as is b, but doubles, at least, the extent to which it is a counterexample, that is, we will obtain the inequality $f(b') - f(c) \geq 2(f(b) - f(c))$. This suffices because if we iterate the construction then this distance will eventually be greater than the distance from c to $\top$. The construction of b' is by cases:

$$b' = \begin{cases} c \lhd b & \text{if } b > a|c \\ a \lhd b & \text{if } b < a|c \end{cases}$$

In the first case we have that $f(b) = f(c)|f(b')$, hence $f(b') - f(c) = 2(f(b) - f(c))$. In the second case we have $f(b) = f(a)|f(b')$, hence $f(b') - f(c) = (f(b') - f(a)) + (f(a) - f(c)) = 2(f(b) - f(a)) + (f(a) - f(c)) = 2f(b) - (f(a) + f(c)) \geq 2f(b) - 2f(c)$. [61]

[61] The monotonicity of midpoint-preserving maps requires linear ordering. Consider the non-constant midpoint-preserving map, $\mathtt{I} \times \mathtt{I} \to \mathtt{I}$ that sends $\langle u, v\rangle$ to $u|\dot{v}$. It sends both $\langle\top, \top\rangle$ and $\langle\bot, \bot\rangle$ to $\odot$. and any monotonic map that collapses top and bottom must, of course, be constant. But simplicity—not just linear ordering—is required: the non-constant map on the Richter scale that sends $\langle x, v\rangle$ to $\langle 0, v\rangle$ preserves midpoints and, again, sends both ends to $\odot$.

13.2. Corollary. *Any* $\mathtt{I}$*-valued midpoint-preserving map is determined by its values on any two of its points.*

It is easy enough to define the midpoint operation on reals: given partial midpoint- and $\odot$-preserving maps r and s define $(r|s)x = (rx)|(sx)$. (The medial law is just what is needed to show that $r|s$ preserves midpoints.) The idempotent, commutative and medial laws for mediation are automatic. Using lemma 8.4 (that the image of the central contraction, $\odot|$, is the "middle half", to wit, the sub-interval $[\bot|\odot, \odot|\top]$) we name the real 2 with the map $\odot\!\lhd$ defined on the middle half. (To check that it preserves midpoints it suffices to check that $\odot|(2(x|y)) = \odot|((2x)|(2y).)$ We construct $r + s$ as $2(r|s)$ and check that it satisfies $\odot|((r+s)x) = (rx)|(sx)$ for all x near $\odot$. We easily verify—using self-distributivity and the cancellation law—the medial law for addition: $\odot(\odot|((r{+}s){+}(t{+}u))) = \odot(|((r{+}s)|(t{+}u))) = (\odot|((r{+}s))|(\odot|(t{+}u)) = (r|s)|(t|u)$ and similarly $\odot(\odot|((r+t)+(s+u))) = (r|t)|(s|u)$. By fiat 0 is a unit for addition. Associativity is then a consequence of mediation: $(r{+}s){+}u = (r{+}s){+}(0{+}u) = (r{+}0){+}(s{+}u) = r{+}(s{+}u)$. For commutativity of multiplication it suffices to verify it for reals named by contraction at elements of $\mathtt{I}$ (because by repeated central contractions we can reduce any two arbitrary reals to such). For the commutativity of the multiplicative structure on $\mathtt{I}$ it suffices to verify it on an order-dense subset, to wit, $\mathbb{I}$.

Finally, for distributivity: $r(s+t) = r(2(s|t)) = (r2)(s|t) = (2r)(s|t) = 2(r(s|t)) = 2((rs)|(rt)) = (rs) + (rt)$.

14. The Enveloping Dy-Module

Given a scale, A, we construct its **enveloping dy-module**, M, as the direct limit of:

$$A \xrightarrow{\odot|} A \xrightarrow{\odot|} A \xrightarrow{\odot|} \cdots$$

More explicitly, its elements are named by pairs, $\langle x, m\rangle$, where x is an element of A and m is a natural number. The pair $\langle y, n\rangle$ names the same element iff $(\odot|)^n x = (\odot|)^m y$. Addition is defined by $\langle x, m\rangle + \langle y, n\rangle \;=\; \langle ((\odot|)^n x) \bigm| ((\odot|)^m y), m+n+1\rangle$. It is routine to check that the definition is independent of choice of name. Commutativity is immediate. The zero-element is named by $\langle \odot, 0\rangle$ and it is routine to see that it is a unit for addition. The mediation law can be verified by straightforward application of the definitions. Associativity is then a consequence: $(a{+}b){+}c = (a{+}b){+}(0{+}c) = (a{+}0){+}(b{+}c) = a{+}(b{+}c)$. The negation of $\langle x, m\rangle$ is named by $\langle \dot{x}, m\rangle$. Scalar multiplication by 1/2 sends $\langle x, m\rangle$ to $\langle \odot|x, m\rangle$.

Embed A into M by sending x to $\langle x, 0\rangle$. [62]

[62] We have used so far only that A is a closed symmetric midpoint algebra, which, recall, was defined to be a model of the three equations for midpointing, the one equation for dotting and the (non-equational) Horn condition of cancellation.

With a little more work one may drop dotting and obtain a representation for plain closed midpoint algebras as follows. Given an object A with a binary operation satisfying the idempotent, commutative

We order M by $\langle x, m\rangle \leq \langle y, n\rangle$ iff $(\odot|)^n x \leq (\odot|)^m y$. We obtain a midpoint-isomorphism from A to the set of all elements in M from $\langle\bot, 0\rangle$ through $\langle\top, 0\rangle$. That is, $\langle\bot, 0\rangle \leq \langle x, n\rangle \leq \langle\top, 0\rangle$ implies there is $y \in A$ such that $\langle x, n\rangle = \langle y, 0\rangle$ and that is because the two M-inequalities translate to the two A-inequalities: $(\odot|)^n\bot \leq x \leq (\odot|)^n\top$. When we showed that the image of the central contraction is the sub-interval from $\odot|\bot$ through $\odot|\top$, hence that the image of the nth iterate of the central contraction is the sub-interval $[(\odot|)^n\bot, (\odot|)^n\top]$, we showed—precisely—that it is possible to solve for $\langle x, n\rangle = \langle y, 0\rangle$.

We may thus infer:

14.1. Lemma. *Every scale has a faithful representation as a closed interval in a lattice-ordered dy-module.*[63]

15. The Semi-Simplicity of Free Scales

15.1. Theorem. *An equation in the theory of scales holds for all scales iff it holds for the initial scale, $\mathbb{I}$.*

We need to show that if an equation in the operators for scales fails in any scale it fails in $\mathbb{I}$. It suffices, note, to find a failure in $\mathtt{I}$ since the operators are continuous—if a pair of continuous functions disagree anywhere on $\mathtt{I}^n$ they must disagree somewhere on $\mathbb{I}^n$. For reasons to become clear later, we will settle here for a failure in between: we will find a failure on the standard rational interval, $\mathtt{I} \cap \mathbb{Q}$. Given an equation in the operators for scales we already know that if there is a counterexample then there is a counterexample

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and medial laws and the Horn condition of cancellation, define a congruence on the cartesian square $A\times A$ by $\langle u, v\rangle \equiv \langle w, x\rangle$ iff $u|x = v|w$. Reflexivity and symmetry are immediate. For transitivity suppose, further, that $\langle w, x\rangle \equiv \langle y, z\rangle$. Then from $u|x = v|w$ and $w|z = x|y$ we may infer $(u|x)\big|(w|z) = (v|w)\big|(x|y)$ hence $(w|x)\big|(u|z) = (w|x)\big|(v|y)$ which by cancellation wields $\langle u, v\rangle \equiv \langle y, z\rangle$. That it is a congruence is immediate: $\langle u, v\rangle \equiv \langle w, x\rangle$ easily implies $\langle y|u, z|v\rangle \equiv \langle y|w, z|x\rangle$. Let $A \times A \to S$ be the quotient structure. The three equations automatically hold in S but the cancellation condition must be verified (if $\langle y|u, z|v\rangle \equiv \langle y|w, z|x\rangle$ then from $(y|u)\big|(z|x) = (z|v)\big|(y|w)$ we may infer $(y|z)\big|(u|x) = (y|z)\big|(v|w)$ and use cancellation in A to yield $\langle u, v\rangle \equiv \langle w, x\rangle$.) Define dotting on S by $\dot{\langle u, v\rangle} = \langle v, u\rangle$ and verify the constant law. Map A into S by choosing an element c and sending $x \in A$ to $\langle x, c\rangle \in S$. The cancellation law must be used one more time to prove that this is a faithful representation.

In a Section-3 footnote when verifying that $\mathbb{I}$ is the initial scale it was pointed out that there are cyclic closed symmetric midpoint algebras. We may eliminate them by imposing a torsion-freeness condition, to wit, by adding the further Horn conditions $[(x|)^n y = y] \Rightarrow [x = y]$, one such condition for each positive integer, n. Then one may prove that the enveloping dy-module will be torsion-free. If one starts with a finitely generated torsion-free symmetric or plain midpoint algebra one ends in a finite-rank free dy-module (using that $\mathbb{D}$ is a principal ideal ring). And one may then embed that module into $\mathbb{R}$, all of which yields a completeness result, to wit, every universally quantified first-order assertion about the (symmetric) plain midpoint algebra $\mathtt{I}$ is true for all torsion-free (symmetric) midpoint algebras. Continuity considerations suffice for the same result with $\mathtt{I}$ replaced by $\mathbb{I}$.

[63]One immediate consequence is the generalization to all contractions of the lemma just used about central contractions. That is, the image of the contraction $a|$ on a scale is the subscale from $a|\bot$ through $a|\top$. (Because the inequalities allow one to prove that $a \triangleleft x = 2x - a$.) An equational proof that $a|\bot \leq x \leq a|\top$ implies $x = a|(a \triangleleft x)$ must therefore exist but it would appear to be quite incomprehensible.

in a linear scale and consequently in a closed interval in an ordered dy-module. There are only finitely many elements in the counterexample, hence the ordered dy-module may be taken to be finitely generated. The ring $\mathbb{D}$ is a principal ideal domain, hence every finitely generated dy-module is a product of cyclic modules, to wit, copies of $\mathbb{D}$ or finite cyclic groups of odd order. But the existence of a total ordering rules out the finite cyclic groups. We are thus in an interval in a totally ordered finite-rank free dy-module, hence, *a fortiori*, a totally ordered finite-dimensional $\mathbb{Q}$-vector space. We need to move the counterexample into a totally ordered one-dimensional $\mathbb{Q}$-vector space which, of course, will be taken to be $\mathbb{Q}$ with its standard ordering.

We first translate the given counterexample into a set of $\mathbb{Q}$-linear equalities and inequalities. Besides the variables $x_1, x_2, \ldots, x_n$ that appear in the counterexample we treat $\top$ and $\bot$ as variables. The equation that fails is replaced with a strict inequality, namely, the strict inequality that results when the given counterexample is instantiated. For each i we add the two inequalities $\bot \leq x_i \leq \top$. And, of course, we add the strict inequality $\bot < \top$. We eliminate the scale operations by iterating the following substitutions (where A and B are terms free of scale operations, that is, are linear combinations of the variables): replace $A|B$ with $(A+B)/2$; replace $\dot{A}$ with $\bot + \top - A$; replace $\hat{A}$ with either $\bot$ or $2A - \top$, whichever is correct for the given counterexample and if $\hat{A} = \bot$ add to the set of conditions to be satisfied the condition $2A \leq \bot + \top$ else $2A > \bot + \top$.

Next, replace each weak inequality either with an equality or strict inequality, depending, once again, on which obtains for the given counterexample. We now eliminate the equations by using the standard substitution technique to eliminate for each equation, one variable (and one equation). We thus are given a finite set of strict linear inequalities and we know that there is a simultaneous solution in a totally ordered finite-dimensional $\mathbb{Q}$-vector space. We wish to find a solution in $\mathbb{Q}$.

We know two proofs, one geometric and the other syntactical.[64]

The geometric argument takes place in Euclidean space. We first standardize the strict linear inequalities to a set $\mathcal{A}$ of "positivities," that is a set of linear combinations of the variables to be modeled as positive elements. We are given a totally ordered finite-dimensional $\mathbb{R}$-vector space with an $(n+2)$-tuple of points one for each variable $\bot, \top, x_1, x_2, \ldots, x_n$. Consider the convex polytope, $\mathtt{P}$, whose vertices are the linear combinations of those points, one for each $A \in \mathcal{A}$. Our one use of the total ordering on the vector space is the knowledge that $\mathtt{P}$ *does not contain the origin.* Given that fact one simply takes the line $\mathtt{L}$ through the origin and the nearest point in $\mathtt{P}$ and applies the orthogonal projection onto $\mathtt{L}$. The image of $\mathtt{P}$ on $\mathtt{L}$ lies on only one side of the origin which, of course, we declare its positive side. The image of the variables on that line give us an $\mathbb{R}$-instantiation as needed.[65]

The syntactical proof uses an induction on the number of variables $\bot, \top, x_1, x_2, \ldots, x_n$.

[64] I learned the latter from Dana Scott, who claimed he was only following the lead of Tarski.

[65] For later purposes, we will need to know that there is a $\mathbb{Q}$-instantiation. Since the problem has been reduced to modeling strict inequalities we may use the continuity of the operations to insure rationality, even dyadic rationality.

Let $\mathcal{C}$ be the set of strict inequalities that do not involve the variable x_n. Recast each remaining inequality in the form either

$$a_{\perp}\perp + a_{\top}\top + a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} < x_n$$

or

$$x_n < a_{\perp}\perp + a_{\top}\top + a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1}$$

where the a_is are rational. Let $\mathcal{L}$ be the set of linear combinations that are to be modeled as strictly less than x_n, and $\mathcal{R}$ the set to be modeled as strictly larger that x_n. If $\mathcal{L}$ is empty use the inductive hypothesis to find an instantiation that models all the inequalities in $\mathcal{C}$ and then choose an instantiation of x_n smaller than all the modeled values of the combinations in the set $\mathcal{R}$. Dually if $\mathcal{R}$ is empty. If neither is empty, model all the inequalities in $\mathcal{C}$ and all inequalities of the form $L < R$ where $L \in \mathcal{L}$ and $R \in \mathcal{R}$. Then model x_n as the midpoint of largest of the modeled values of the forms in $\mathcal{L}$ and the smallest in $\mathcal{R}$.

A consequence is that the maps from a free scale into $\mathtt{I}$ are collectively faithful because if two terms can not be proved equal then necessarily there is a counterexample in the free scale, hence in $\mathbb{I}$, *a fortiori* in $\mathtt{I}$. That is, there are elements of $\mathtt{I}$ which—when they are used to instantiate the variables—produce different values for the two terms. Such, of course, describes a scale homomorphism from the free scale that separates the two terms. Hence:

15.2. THEOREM. *Every free scale appears as a subscale of a cartesian power of* $\mathtt{I}$.

An immediate corollary is the semi-simplicity of free scales:

15.3. THEOREM. *Every free scale is embedded into the product of its simple quotients.*

Note that the constructions used for counterexamples of equations in $\mathtt{I}$ can as easily be used for counterexamples of universal Horn sentences: the constructions not only preserve strict inequalities but (in the process of elimination of variables) any number of equalities. Thus given a sentence of the form

$$(s_1 = t_1)\,\&\cdots\&\,(s_n = t_n) \quad \Rightarrow \quad (u = v)$$

with a counterexample anywhere the constructions produce counterexamples in $\mathtt{I}$, indeed, in the rational points in $\mathtt{I}$. (There are Horn sentences true for some non-trivial scale that do not hold for $\mathtt{I}$. A universal Horn sentence is consistent iff it holds for the initial scale, $\mathbb{I}$. An example of such that does not hold for $\mathtt{I}$ is $[x|(x|\perp) = \top|\perp] \Rightarrow [y = z]$.) We may add one non-equational condition, $\overset{\vee}{x} = \top$ *or* $\hat{x} = \perp$ (which for good historical reasons will be called the **coalgebra condition**) that yields a completeness theorem for the entire universally quantified first-order theory: such a sentence is a consequence of the defining equations for scales plus the coalgebra condition iff it is true for $\mathtt{I}$.

16. Diversion: Harmonic Scales and Differentiation

The theory of **harmonic scales** is given by a binary operation whose values are denoted with catenation, xy, satisfying the equations:

$$\begin{array}{rcl} \top x &=& x \;=\; x\top \\ \odot x &=& \odot \;=\; x\odot \\ x\,(y|z) &=& (xy)|(xz) \\ (x|y)\,z &=& (xz)|(yz) \\ (u \multimap\!\circ\, v)|(x \multimap\!\circ\, y) &\le& \top|(ux \multimap\!\circ\, vy) \end{array}$$

We'll refer to the top two equations as the "unit condition," the next four as the "bilinear condition." The last equation is, of course, the Lipschitz condition.[66]

Standard multiplication is the unique interpretation of these equations on $\mathtt{I}$ (and there is at most one interpretation on any simple scale).

A few lemmas we'll need:

16.1. Lemma.

$$\begin{array}{rcl} \dot{u}\,v &=& (uv)^{\bullet} \\ \bot\, v &=& \dot{v} \\ \odot|(uv) &=& (\odot|u)v \end{array}$$

For the 1st equation use cancellation on $(uv)|(\dot{u}\,v) = (u|\,\dot{u})|v = \odot|v = \odot = (uv)|(uv)^{\bullet}$. For the 2nd: $\bot\, v = \dot{\top}\, v = (\top v)^{\bullet} = \dot{v}$. For the 3rd: $(\odot|u)v = (\odot v)|(uv) = \odot|(uv)$.

The harmonic structure greatly extends our descriptive power. Among many other things it allows us to identify not just any algebraic number in $\mathtt{I}$ but many transcendentals. As just one example, we will see below that if we add a unary operator f satisfying the two conditions:

$$\begin{array}{c} f\odot \;=\; \top|(\bot|\odot) \\ \{[(fu)^{\bullet}|(fv)] \;\big|\; [(fv)(u|\dot{v})]\}^2 \;\le\; (u|\dot{v})^4 \end{array}$$

Then for any non-trivial model, S, and any $q \in \mathbb{I} \subseteq S$:

$$q \gtrless f\top \quad \text{iff} \quad q \gtrless \mathrm{e}/4$$

(where the second $\gtrless$ is as defined in the standard interval).

The harmonic structure allows us to identify values of derivatives. Suppose f is a unary operator on $\mathtt{I}$ and $a, b \in \mathtt{I}$ are constants:

[66] The standard interval in any ordered unital ring with $\frac{1}{2}$ satisfies the first six equations. Consider again the ring we used before to show the necessity of a Lipschitz condition: the polynomial ring $\mathbb{R}[\varepsilon]$ ordered by stipulating $P(\varepsilon) \ge 0$ iff $P(1/n) \ge 0$ for all sufficiently large n. If we use, instead, the nonstandard multiplication, $(\sum a_n\varepsilon^n) \circ (\sum b_n\varepsilon^n) = \sum(a_0b_n + a_nb_0 - 2^n a_nb_n)\varepsilon^n$, then on the standard interval multiplication is not Lipschitz.

16.2. PROPOSITION. *For any natural number, n:*

$$[(u|\dot{a})^2]^{\bullet} \ \leq\ (\odot|)^n[((fu)^{\bullet}|fa) \mid (b(u|\dot{a}))] \ \leq\ (u|\dot{a})^2 \ \ \textit{implies}\ \ f'a = b$$

.

Switching to usual notation, this is saying that if

$$\frac{fu - fa - b(u-a)}{(u-a)^2}$$

is bounded then $f'a = b$. We may rewrite: if there is a constant K such that

$$-K|u-a| \ \leq\ \frac{fu - fa}{u-a} - b \ \leq\ K|u-a|$$

then

$$\lim_{u\to a} -K|u-a| \ \leq\ \lim_{u\to a} \frac{fu-fa}{u-a} - b \ \leq\ \lim_{u\to a} K|u-a|$$

hence

$$0 \ \leq\ f'a - b \ \leq\ 0$$

Therefore, if f and g are unary operators on $\mathtt{I}$ satisfying the conditions

$$[(u|\dot{v})^2]^{\bullet} \ \leq\ (\odot|)^n[((fu)^{\bullet}|fv) \mid ((gv)(u|\dot{v}))] \ \leq\ (u|\dot{v})^2$$

we have $f' = g$ and—since g is bounded—that f is Lipschitz. We will need that last fact, not just on simple scales, but for all scales. Even better, g is also Lipschitz:

16.3. LEMMA. *If f and g are unary operations on a scale and $n \in \mathbb{N}$ such that*

$$[(u|\dot{v})^2]^{\bullet} \ \leq\ (\odot|)^n[((fu)^{\bullet}|fv) \mid ((gv)(u|\dot{v}))] \ \leq\ (u|\dot{v})^2$$

then f and g are Lipschitz.

Because $\odot|$ is a homomorphism with respect to dotting and midpointing we may replace f with $(\odot|)^n f$ and g with $(\odot|)^n g$ and reduce to the case

$$[(u|\dot{v})^2]^{\bullet} \ \leq\ ((fu)^{\bullet}|fv) \mid ((gv)(u|\dot{v})) \ \leq\ (u|\dot{v})^2$$

We may then apply $(\odot|)$ twice more and replace f with $(\odot|)^2 f$ and g with $(\odot|)^2 g$ thus reducing to the case

$$(\odot|)^2[(u|\dot{v})^2]^{\bullet} \ \leq\ ((fu)^{\bullet}|fv) \mid ((gv)(u|\dot{v})) \ \leq\ (\odot|)^2(u|\dot{v})^2$$

As often when working with harmonic scales we resort to the standard notation appropriate to the enveloping dy-module (and, in this case, also negate the terms):

$$-(u-v)^2/4 \ \leq\ fu - fv - (gv)(u-v) \ \leq\ (u-v)^2/4$$

Because g was replaced with $(\odot|)^2 g$ we have the additional information $-1/4 \le g \le 1/4$, hence

$$-|u-v|/4-(u-v)^2/4 \;\le\; fu-fv \;\le\; (u-v)^2/4+|u-v|/4$$

If we now use the inequality $(u-v)^2/4 \le |u-v|/2$ we obtain

$$-3|u-v|/4 \;\le\; fu-fv \;\le\; 3|u-v|/4$$

which, of course, says that f is Lipschitz.

The proof for g is easier. After replacing f and g with $(\odot|)^n f$ and $(\odot|)^n g$ and moving to standard notation we have the two inequalities:

$$-(u-v)^2 \;\le\; fu-fv-(gv)(u-v) \;\le\; (u-v)^2$$

By switching u and v we also have (obviously):

$$-(v-u)^2 \;\le\; fv-fu-(gu)(v-u) \;\le\; (v-u)^2$$

When we add the two rows we obtain:

$$-2(u-v)^2 \;\le\; (gu-gv)(u-v) \;\le\; 2(u-v)^2$$

hence

$$|gu-gv| \;\le\; 2|u-v|$$

Going back to the two conditions that allow us to identify e/4, the 2[nd] condition says that in any model on $\mathtt{I}$ we have $f=f'$ hence that $fu = A\mathrm{e}^u$ for some constant A and the 1[st] condition forces $A=1/4$. Hence $f\top = \mathrm{e}/4$ in any model in $\mathtt{I}$. Since f is necessarily Lipschitz we know that in any simple quotient we have $q \gtreqless f\top$ iff $q \gtreqless \mathrm{e}/4$ for all $q \in \mathbb{I}$, hence such is the case in any linear quotient. The linear representation theorem says, therefore, that in any model, $q \gtreqless f\top$ iff $q \gtreqless \mathrm{e}/4$ for all $q \in \mathbb{I}$. We can infer more: let F be the initial algebra for the theory of harmonic scales with a unary operator satisfying the two conditions we have used to identify e/4; then its semi-simple reflection, $F/\mathcal{R}$, is simple (to wit, the smallest harmonic subscale of $\mathtt{I}$ closed under the action of f). It suffices to show that $F/\mathcal{R}$ is its own simple part: but for any element, a, we have either $q \le a$ or $a \le q$ for all $q \in \mathbb{I}$, forcing a to be in the equalizer of all maps to $\mathtt{I}$, that is, forcing a to be in the simple part.

A function may have a derivative at a without $[-fu+fa+(f'a)(u-a)]/(u-a)^2$ being bounded (e.g. the derivative of $u^{4/3}$ at 0) but not when the derivative is Lipschitz; we have the converse of the last lemma:

16.4. LEMMA. *If $f:\mathtt{I}\to\mathtt{I}$ is differential and if, further, f' is Lipschitz then there is $n\in\mathbb{N}$ such that*

$$[(u|\dot{v})^2]^{\bullet} \;\le\; (\odot|)^n[((fu)^{\bullet}|fv) \mid ((f'v)(u|\dot{v}))] \;\le\; (u|\dot{v})^2$$

For a proof let

$$L = \frac{1}{2} \inf_{u \neq v} \frac{f'u - f'v}{u - v} \qquad\qquad R = \frac{1}{2} \sup_{u \neq v} \frac{f'u - f'v}{u - v}$$

We wish to prove the two inequalities:

$$L(u-v)^2 \;\leq\; fu - fv - (f'v)(u-v) \;\leq\; R(u-v)^2$$

For each $a \in \mathtt{I}$ let

$$h_a(u) = fu - fa - (f'a)(u-a) - L(u-a)^2$$

The 1st inequality is equivalent to $0 \leq h_a$. Clearly $h_a a = h'_a a = 0$. It thus suffices to show that a is an absolute minimal point for h_a and for that it suffices to show that $h'_a u \leq 0$ for $u < a$ and $h'_a u \geq 0$ for $u > a$. Since $2L \leq \frac{f'u - f'a}{u-a}$ we have for $u < a$ that

$$h'_a u = f'u - f'a + 2L(a-u) \leq f'u - f'a + \frac{f'u - f'a}{(u-a)}(a-u) = f'u - f'a - (f'u - f'a) = 0$$

The argument for $u > a$ is similar. The 2nd inequality may be reduced to the 1st by replacing f and g with their negations.

The last two lemmas yield:

16.5. Theorem. *$f : \mathtt{I} \to \mathtt{I}$ is differential with a Lipschitz derivative f' iff there is $n \in \mathbb{N}$ such that:*

$$[(u|\dot{v})^2]^{\bullet} \;\leq\; (\odot|)^n[((fu)^{\bullet}|fv) \;\big|\; ((f'v)(u|\dot{v}))] \;\leq\; (u|\dot{v})^2$$

Consequently the harmonic structure allows an exploration of differential equations.

In case one wishes to identify π rather than e, add to the theory of harmonic scales two unary operators, s, c, subject to the conditions:

$$\begin{aligned} c\odot \;&=\; \top \\ \{[(su)^{\bullet}|(sv)] \;\big|\; [(cv)(u|\dot{v})]\}^2 \;&\leq\; (u|\dot{v})^4 \\ \{[(cu)^{\bullet}|(cv)] \;\big|\; [(sv)^{\bullet}(u|\dot{v})]\}^2 \;&\leq\; (u|\dot{v})^4 \end{aligned}$$

Then for any non-trivial model, S, and any $q \in \mathbb{I} \subseteq S$:

$$sq \gtreqless cq \quad \text{iff} \quad q \gtreqless \pi/4$$

The 2nd condition says $s' = c$ in any model in $\mathtt{I}$ and the 3rd condition $c' = -s$. Necessarily, then, s and c are each of the form $A\cos(u+\theta)$ where A is bounded by 1 (since s and c are). The 1st condition says, among other things, that c attains the value 1, hence that $A = 1$. The 1st condition then goes on to say that 0 is a maximal point for c forcing $cu = \cos u$

and, then, $su = \sin u$. Again we may proceed to show that the semi-simple reflection of the free model for these operators is simple.[67]

The increase in expressive power comes, as usual, with a cost: whereas the first-order theory of harmonic $\mathtt{I}$-scales is decidable (as a consequence of Tarski's proof of the completeness of the first-order theory of real closed fields), the addition of further Lipschitz equational structure can make it possible to capture all of first-order number theory.[68]

17. Subintervals

Given scale elements $b < t$ the interval of all elements x such that $b \le x \le t$ is, as usual, denoted $[b,t]$. It is, of course, closed under midpointing, but not, in general, under dotting or zooming. It does have an induced scale-structure. Define a new dilatation operation that sends $x, y \in [b,t]$ to $b \vee (x \triangleleft y) \wedge t$ (in a distributive lattice $b \vee z \wedge t$ is unambiguous when $b \le t$). For elements in $[b,t]$ this enjoys the characterizing properties of dilatations, to wit, it is covariant in y and undoes contraction at x. We then define the zoom operations on $[b,t]$ as the dilatations at b and t. Dotting is obtained by dilitating "into" its center: $\dot{x} = x \triangleleft (b|t)$.

The verification of the scale axioms is most easily dispatched by using the semi-simplicity of free scales. If there were a counterexample anywhere there would be one in $\mathtt{I}$. It is easy to see that the induced structure on any non-trivial subinterval of $\mathtt{I}$ makes it isomorphic to $\mathtt{I}$. [69]

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[67] It may be the case that the free model is semi-simple (hence simple). To show otherwise we need a term provably greater that $(\top|)^n \bot$ all $n \in \mathbb{I}$ but not provably equal to $\top$. The same goes for the initial model for $e^u/4$.

[68] One way is as follows: the set of positive elements of $\mathtt{I}$ under ordinary multiplication and a non-standard "addition" characterized by $(x \tilde{+} y)(x|y) = \odot|(xy)$ is isomorphic—via reciprocation—to the real half-line $[1, +\infty)$ (with its usual multiplication and addition). We can identify the reciprocals of positive integers by using the differential equation $x^4 h''x - 10x^3 h'x + (30x^2 + 1)hx = 0$: the further equations $h(\pm\pi^{-1}) = 0$ and $h'(\pm\pi^{-1}) = 2^{-6}\pi^{-4}$ identify hx as $(x/2)^6 \sin x$; hence $x \in \mathtt{I}$ is the reciprocal of a positive integer iff $x > 0$ and $h(\pi^{-1}x) = 0$. Given π^{-1} it thus suffices to add three unary operations h, h', h'' and three conditions:

$$\begin{aligned} \{[(hu)^{\bullet}|(hv)] \,\big|\, [(h'v)(u|\dot{v})]\}^2 &\le (u|\dot{v})^4 \\ \{[(h'u)^{\bullet}|(h'v)] \,\big|\, [(h''v)(u|\dot{v})]\}^2 &\le (u|\dot{v})^4 \\ (\tfrac{-1}{32}x^4 h''x)\big|(\tfrac{5}{16}x^3 h'x) &= (\tfrac{15}{16}x^2|\tfrac{1}{32})hx \end{aligned}$$

where (following Section 3):

$$\begin{aligned} \tfrac{-1}{32} &= \bot|(\top|(\top|(\top|(\top|\odot)))) \\ \tfrac{5}{16} &= \top|(\bot(|\top|(\bot|\odot))) \\ \tfrac{15}{16} &= \top|(\top|(\top|(\top|\odot))) \\ \tfrac{1}{32} &= \top|(\bot|(\bot|(\bot|(\bot|\odot)))) \end{aligned}$$

[69] But the subintervals of $\mathbb{I}$ are not all isomorphic to each other. We constructed isomorphisms between

Consider the example used in the opening section's footnote-definition of derivatives: we considered the set, F, of functions from the standard interval to itself, such that $|f(x)| \leq |x|$ for all x. If we view F as a subset of all functions from the standard interval to itself it is an example of a **twisted interval**. It may be described as the set of functions whose values "lie between the identity function and its negation." Given any scale and elements b, t therein we can formalize the notion by defining the twisted interval $[\![\boldsymbol{b}, \boldsymbol{t}]\!]$ as the set of all elements, x, such that in every linear representation it is the case that x is between b and t. This results, easily enough, in the ordinary interval $[b \wedge t, b \vee t]$. But the scale structure we want on $[\![b, t]\!]$ is different: the bottom is to be b not $b \wedge t$ and the top is to be t not $b \vee t$. We simply repeat the construction as for ordinary intervals but with that one change—the new dilatation operator is still obtained by contracting the output of the ambient operation to the ordinary interval $[b \wedge t, t \vee b]$ it being understood that the top and bottom are not the standard endpoints but rather b and t. We know such yields a scale because we know that on every linear quotient it does so (albeit that on some of those linear quotients the order is not the induced order but its opposite).

18. Extreme Points

There is a curious similarity between idempotents in rings and extreme points in scales. First:

18.1. Proposition. *The following are equivalent:*

$$x \textit{ is an extreme point}$$
$$\hat{x} = x$$
$$\overset{\vee}{x} = x$$
$$x \vee \dot{x} = \top$$
$$x \wedge \dot{x} = \bot$$
$$\exists_v[(x \vee v) \wedge (\dot{x} \vee \dot{v}) = \top]$$

Because the law of compensation easily shows, first, that fixed points for either $\top$- or $\bot$-zooming are automatically fixed points for the other and, second, that extreme points are fixed points. Conversely, to show that fixed points are extreme suppose that $x|y$ is a fixed point; the only cleverness needed is $x = x \overset{\vee}{|} \bot \leq x \overset{\vee}{|} y = x|y = x \overset{\wedge}{|} y \leq x \overset{\wedge}{|} \top = x$.

Before dispatching the remaining conditions note that the interval coalgebra condition, $\overset{\vee}{x} = \top$ *or* $\hat{x} = \bot$, known to be equivalent in scales with linearity, implies that there are just

subintervals of the same length at the end of Section 4 (actually, we constructed an isomorphism between any interval and the interval of the same length of the form $[b, \top]$). It is easy to see that subintervals are isomorphic if the ratio of their lengths is a power of 2. The odd part of the numerator of the dyadic rational that measures the length is a complete isomorphism invariant: clearly if a pair of subintervals have the same odd part of their length we may construct an isomorphism; for the converse note that universal Horn sentences of the form $[(x \overset{\vee}{|}\ )^n(\bot) = \odot] \Rightarrow [y = z]$ hold for the induced scale-structure of a subinterval of $\mathbb{I}$ iff n does not divide the odd part of its length.

the two fixed points, $\top$ and $\bot$. On a linear scale either equation $x \vee \dot{x} = \top$ or $x \wedge \dot{x} = \bot$ is thus clearly equivalent with x being extreme and therefore a fixed point, hence such is the case in any scale. And, similarly, on a linear scale the existence of a complement is also clearly equivalent with being a fixed point, which dispatches the last condition above.

We will use $\boldsymbol{\mathcal{B}(S)}$ to denote the set of fixed points/extreme points of a scale, S. The set of extreme points of any scale is closed under dotting and the lattice operations and we will regard $\mathcal{B}$ as a (covariant) functor from scales to Boolean algebras.

Recalling that $\boldsymbol{\mathcal{C}(X)}$ denotes the scale of $\mathtt{I}$-valued continuous functions on a Hausdorff space, X, we see that $\mathcal{B}(\mathcal{C}(X)$ is isomorphic to the Boolean algebra of clopens in X. Following ring-theoretic language, we will say that a scale, S, is **connected** if $\mathcal{B}(S)$ has just two elements, $\top$ and $\bot$.

If A and B are scales then in $A{\times}B$ the elements $\langle\top,\bot\rangle$ and $\langle\bot,\top\rangle$ are a complementary pair of extreme points. Every such pair of extreme points so arises: first, note that if e is an extreme point in a scale S then the principal $\top$-face $(\!(e)\!)$ generated by e is the interval $[e,\top]$ (indeed, any subset that is both an interval and a face must have extreme points as endpoints). The quotient structure $S/(\!(e)\!)$ is isomorphic to the induced scale on the interval $[\bot,e]$ via the map that sends x to $e \wedge x$. The map $S \to [\bot,e]\times[\bot,\dot{e}]$ that sends x to $\langle e \wedge x, \dot{e} \wedge x\rangle$ is an isomorphism; its inverse sends $\langle x,y\rangle$ to $x \vee y$. This is, of course, just the analog of Peirce decomposition for central idempotents.[70] (The fact that $e \wedge x$ describes a homomorphism is most easily dispatched using the linear representation theorem. For example, its preservation of midpointing is the Horn sentence $[\hat{e} = e] \Rightarrow [e\wedge(x|y) = (e\wedge x) \bigm| (e\wedge y)]$, a triviality in any linear scale since its only extreme points are $\bot$ and $\top$.)

The atoms of $\mathcal{B}(S)$ thus correspond to the connected components of S. One consequence is the uniqueness of product decompositions. In a finite product of connected scales $\mathcal{B}$ is finite; its atoms yield the only decomposition into indecomposable products it has. All of this is just as it is for central idempotents in the theory of rings.[71]

If X is totally disconnected then every $\top$-face in $\mathcal{C}(X)$ is generated by the extreme points it contains; the lattice of $\top$-faces is canonically isomorphic to the lattice of filters in $\mathcal{B}(\mathcal{C}(X))$.

In a product of connected scales $\prod_J S_j$ the extreme points are the characteristic functions of subsets of J. An ultrafilter of $\mathcal{B} = 2^J$ generates a maximal $\top$-face of the product. The quotient structure is usually called an **ultraproduct**. It has the wonderful feature that any first-order sentence is modeled by the ultraproduct iff it is modeled by enough S_js, that is, iff the set of j such that the sentence is modeled by S_j is one of the sets in the ultrafilter.[72]

70 I am a terrible speller myself, but the great first American mathematician deserves to have his name spelled correctly. And pronounced correctly—he and his family rhyme it with *terse*.

71 But there are a few isomorphisms that are not reminiscent of Peirce decomposition. $[\bot,e]$ is isomorphic to $[\dot{e},\top]$ via the map that sends $x \in [\bot,e]$ to $\dot{e} \vee x \in [\dot{e},\top]$. The inverse isomorphism sends $y \in [\dot{e},\top]$ to $e \wedge y \in [\bot,e]$. The product decomposition arising from an extreme point e can be re-described as the isomorphism to $[\bot,e]{\times}[e,\top]$ that sends x to $\langle e{\wedge}x, e{\vee}x\rangle$. Its inverse sends $\langle x,y\rangle \in [\bot,e]{\times}[e,\top]$ to $x \vee \dot{e} \wedge y$.

72 One may show that any linear scale may be embedded in a subinterval (with its induced scale-

In a twisted interval $[\![b,t]\!]$ let $b \wedge t$ and $b \vee t$ denote the elements as defined in the ambient scale (since b is bottom according to the intrinsic ordering on the twisted interval the use of the intrinsic—instead of the induced—lattice operations would be unproductive). $b \wedge t$ and $b \vee t$ are a complementary pair of extreme points in the twisted interval and the pair yields an isomorphism $[\![b,t]\!] \to [b \wedge t, b]^{\circ} \times [b, b \vee t]$ where $^{\circ}$ denotes the **opposite scale**: the one obtained by swapping $\top$- with $\bot$-zooming and top with bottom. But any scale is isomorphic to its opposite via the dotting operation hence $[\![b,t]\!]$ is isomorphic to $[b \wedge t, b] \times [b \wedge t, t]$ via the map that sends $x \in [\![b,t]\!]$ to $\langle((x \triangleleft (t|b)) \wedge b), x \wedge t\rangle$ (using $(x \wedge b) \triangleleft ((b \wedge t)|b) = (x \triangleleft (t|b)) \wedge b$). And that yields the isomorphism from $[\![b,t]\!]$ to $[b \wedge t, b \vee t]$ that sends x to $((x \triangleleft (t|b)) \wedge b) \vee (x \wedge t)$. All of which totally obscures the geometry of the opening section's construction of derivatives.

19. Chromatic Scales

A **chromatic scale**[73] is a scale with a (non-Lipschitz, indeed, discontinuous) unary **support operation**, whose values are denoted $\overline{\boldsymbol{x}}$, satisfying the equations:

$$
\begin{aligned}
\overline{\bot} &= \bot \\
\overset{\wedge}{\overline{x}} &= \overline{x} \\
x \wedge \overline{x} &= x \\
\overline{x \wedge y} &= \overline{x} \wedge \overline{y}
\end{aligned}
$$

Note that the first three equations have a unique interpretation on any connected scale and the 4[th] equation holds iff the connected scale is linear.[74]

These equations say, in concert:

19.1. Lemma. *$\overline{x}$ is the smallest extreme point above x.*

(The 3[rd] equation says, of course, that $\overline{x} \geq x$; next, if e is an extreme point then $e = e \vee \overline{\bot} = e \vee \overline{e \wedge \dot{e}} = e \vee (\overline{e} \wedge \overline{\dot{e}}) = (e \vee \overline{e}) \wedge (e \vee \overline{\dot{e}}) = \overline{e} \wedge (e \vee \overline{\dot{e}}) \geq \overline{e} \wedge (e \vee \dot{e}) = \overline{e} \wedge \top = \overline{e}$; third, if e is an extreme point above x then since the 4[th] equation implies that the support operation is covariant we have $e = \overline{e} \geq \overline{x}$).

Note that it follows that the support operator distributes not just with meet but with join (its covariance yields $\overline{x} \vee \overline{y} \leq \overline{x \vee y}$ and the characterization of $\overline{x \vee y}$ as the smallest extreme point above $x \vee y$ yields $\overline{x \vee y} \leq \overline{x} \vee \overline{y}$). The **co-support**, $\underline{\boldsymbol{x}}$, of x is the largest extreme point below x. It is easily constructible as $\underline{x} = \dot{\overline{\dot{x}}}$.

structure) of an ultrapower of $\mathbb{I}$.

[73]To some extent, chromatic scales are to measurable functions as scales are to continuous. See, in particular, Section 25.

[74]The 2[nd] equation becomes redundant if the 3[rd] equation is replaced with $x \wedge \overline{\dot{x}} = \bot$. See Section 29.

19.2. THEOREM. *A scale is a simple chromatic scale iff it is linear. Every chromatic scale is semi-simple, that is, any chromatic scale is embedded (as such) in a product of linear scales. The defining equations for the support operator are therefore complete: any equation—indeed any universal Horn sentence—true for the support operator on all linear scales is a consequence of the four defining equations.*

We have already noticed that connected scales have a support operator iff they are linear, so among chromatic scales connectivity and linearity are equivalent. But connected easily implies simple: if a congruence is non-trivial then its kernel has an element, x, below $\top$. But then $\underline{x} \neq \top$ hence $\bot = \underline{x} \equiv \underline{\top} = \top$.

If $\mathcal{F} \subseteq \mathcal{B}$ is a filter in the Boolean algebra of extreme points then the $\top$-face it generates in the scale, $\mathcal{F}^{\Uparrow}$, consists of all elements x such that $\underline{x} \in \mathcal{F}$ (because clearly the set of such x is a zoom-invariant filter).

The support operator is discontinuous on the standard interval but among discontinuous operations it appears to play something of a universal role. The Heyting arrow operation, $x \to y$ can be constructed as $\underline{x \multimap y} \vee y$ (the defining equations for the operation hold for this construction on linear scales, hence the representation theorem implies that they hold on all chromatic scales).[75] We observed in Section 5 that Girard's $!\Phi$ is $\underline{\Phi}$ and his $?\Phi$ is $\overline{\Phi}$.

If $\mathcal{F}$ is an ultrafilter, then for extreme points e and e' if $e \vee e' \in \mathcal{F}$ either $e \in \mathcal{F}$ or $e' \in \mathcal{F}$ consequently if $x \vee x' \equiv \top \pmod{\mathcal{F}^{\Uparrow}}$ then either $x \equiv \top$ or $x' \equiv \top$ forcing the quotient scale to be linear. The scale map to the quotient structure clearly preserves the co-support—and hence the support—operation.

Given any $x \neq \top$ we can find an ultrafilter of extreme points excluding $\underline{x}$ hence x remains below $\top$ in the corresponding quotient structure.[76]

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[75] The support operator is definable, in turn, from the Heyting operation, indeed, just from the Heyting negation: $\overline{x} = (x \to \bot)^{\bullet}$. (Hence the less colorful alternate name, "Heyting scales.")

[76] The analogous material for rings and idempotents is the following equational theory (which I have assumed for at least 40 years must already be known):

Define a support operation on a ring to be a unary operation satisfying:

$$\begin{array}{rcl} \overline{0} & = & 0 \\ (\ \overline{x}^{\,2} & = & \overline{x}\ \ ) \\ x\overline{x} & = & x \\ \overline{xy} & = & \overline{x}\,\overline{y} \end{array}$$

(The 2nd equation is, in fact, redundant. It is present only to emphasize the analogy with chromatic scales. See Section 29.)

For one source of examples take any strongly regular von Neumann ring and take $\overline{x} = xx^*$. For a better source note that the first three equations say, in concert, that a connected ring has a unique support operator and it satisfies the 4th equation iff the ring is a domain (that is, a ring, commutative or not, without zero divisors). These equations are complete for such examples: any equation—indeed any universal Horn sentence—true for all domains is a consequence of these equations because any algebra is embedded (as such) into a product of domains. To prove it, first note that if $x^2 = 0$ then $x = x\overline{x} = x\overline{x}^2 x = x\overline{x^2} = x\overline{0} = x0 = 0$. Any idempotent is central ($(1-e)xe$ is 0 since its square is 0 hence $xe = exe$ and similarly $ex = exe$). For any idempotent e we have $e = \overline{e}$ (because $e = e + (1-e)\overline{0} =$

Every Boolean algebra arises in the form $\mathcal{B}(S)$. First note that the functor $\mathcal{B}$ has a left adjoint: given any scale S and Boolean algebra B let $S[B]$ be the scale generated by S and the elements of B subject to relations that, first, make those elements fixed points and, second, obey all the lattice relations that obtain in B; then the maps from $\mathbb{I}[B]$ to any scale T are in natural correspondence with the maps from B to $\mathcal{B}(T)$.

It is the case that the adjunction map from B to $\mathcal{B}(\mathbb{I}[B])$ is an isomorphism. For a quick and dirty proof note that we may construct $\mathbb{I}[B]$ as the $\mathbb{I}$-valued continuous functions from the space of ultrafilters on B. More constructively we gain a handle on $S[B]$ as follows: we will say that a term is "pre-canonical" if it is of the form $(s_1 \wedge e_1) \vee (s_2 \wedge e_2) \vee \cdots (s_n \wedge e_n)$ where $e_1, e_2, \ldots, e_n$ is a partition of unity (pairwise disjoint and $e_1 \vee e_2 \vee \cdots \vee e_n = \top$); and "canonical" if, further, none of the e_is equals $\bot$ and the s_is are distinct. We need only show that every element of $S[B]$ is described by a canonical form, unique up to the ordering of the partition of unity. For the existence of the form it suffices to show that elements named by canonical terms are closed under the scale operations. Zooming is easy since it distributes with the lattice operations and preserves the extreme points: $((s_1 \wedge e_1) \vee (s_2 \wedge e_2) \vee \cdots (s_n \wedge e_n))^\wedge = (\hat{s}_1 \wedge e_1) \vee (\hat{s}_2 \wedge e_2) \vee \cdots (\hat{s}_n \wedge e_n)$. The latter term may be only pre-canonical but it is clear that every pre-canonical term is equal to a canonical term. Dotting requires a little work: to see that $((s_1 \wedge e_1) \vee (s_2 \wedge e_2) \vee \cdots (s_n \wedge e_n))^{\bullet} = (\dot{s}_1 \wedge e_1) \vee (\dot{s}_2 \wedge e_2) \vee \cdots (\dot{s}_n \wedge e_n)$ it suffices to check the equation on linear scales. But we know that in a linear scale all but one of the e_is will be $\bot$ and the one that is not will be $\top$ and that is quite enough. For midpointing suppose that $(s'_1 \wedge e'_1) \vee (s'_2 \wedge e'_2) \vee \cdots (s'_m \wedge e'_m)$ is another canonical term. Then:

$$\bigvee_{i=1}^{n}(s_i \wedge e_i) \mid \bigvee_{j=1}^{m}(s'_j \wedge e'_j) = \bigvee_{i=1}^{n}\bigvee_{j=1}^{m}((s_i|s'_j) \wedge e''_{i,j})$$

where $e''_{i,j} = e_i \wedge e'_j$ (the "joint refinement" of the two given partitions of unity). Again, the easiest proof is simply to consider the equality in the case of a linear scale. For the uniqueness of the canonical term note first that in a pre-canonical term for $\odot$ it must be the case that $s_i = \odot$ whenever $e_i \neq \bot$ (just specialize—once again—to any linear quotient). Hence if $(s_1 \wedge e_1) \vee (s_2 \wedge e_2) \vee \cdots (s_n \wedge e_n) = (s'_1 \wedge e'_1) \vee (s'_2 \wedge e'_2) \vee \cdots (s'_m \wedge e'_m)$ then we know that $\dot{s}_i|s'_j = \odot$ whenever $e''_{i,j} \neq \bot$ which, in turn, says that $s_i = s'_j$ whenever $e_i \wedge e'_j \neq \bot$. We may thus infer that for every $1 \leq i \leq n$ there is $1 \leq j \leq m$ such that $s_i = s'_j$ and vice versa. Because the s_is in a canonical term are distinct this forces $n = m$

$e + (1-e)\overline{(1-e)e} = e + (1-e)\overline{(1-e)}\ \overline{e} = e + (1-e)\overline{e} = e + \overline{e} - e\overline{e} = e + \overline{e} - e = \overline{e}$). The equations characterize $\overline{x}$ as the smallest element in the Boolean algebra of idempotents that acts like the identity element when multiplied by x. If $\mathfrak{A}$ is an ideal in that Boolean algebra then the ideal it generates in the ring consists of all elements x such that $\overline{x} \in \mathfrak{A}$ (if $e \in \mathfrak{A}$ then $\overline{ex} = e\overline{x} \in \mathfrak{A}$ and if, further, $e' \in \mathfrak{A}$ then $(ex+e'x') = (e \vee e')(ex+e'x')$ where $e \vee e' = e+e'-ee' \in \mathfrak{A}$ which gives $e \vee e' \geq \overline{ex+e'x} \in \mathfrak{A}$). When $\mathfrak{A}$ is a maximal Boolean ideal then the ring ideal it generates is prime (since xy is in it iff $\overline{x}\,\overline{y}$ is and a maximal Boolean ideal is a prime ideal in the Boolean algebra) hence the corresponding quotient is a domain. The ring-homomorphism down to the quotient is easily checked to be a homomorphism with respect to the support operation. Finally, given any $x \neq 0$ we can find a maximal Boolean ideal containing $1 - \overline{x}$ hence x remains non-zero in the corresponding quotient.

and the existence of a permutation π such that $s_i = s'_{\pi(i)}$. Since $e_i \wedge e'_j = \bot$ whenever $j \neq \pi(i)$ we may infer that $e_i = e'_{\pi(i)}$. [77]

Note that that $\overset{\wedge}{|}$ and $\overset{\vee}{|}$ are lattice operations in $\mathcal{B}(S)$ and $x \multimap y$ is the same as the standard material implication. (Scales of the form $\mathtt{I}[B]$ thus succeed in mixing Łukasiewicz, classical, and Girard's linear logic. The support and co-support operators serve to translate from Łukasiewicz to classical: $\overline{\hat{x}}$ is the assertion, for example, that x is more probable than not.)

20. The Representation Theorem for Free Scales

20.1. Theorem. *The free scale on n generators is isomorphic to the scale of all continuous piecewise dy-affine functions from the standard n-cube, $\mathtt{I}^n$, to $\mathtt{I}$.*

We need some definitions.

Given a scale, S, let $\boldsymbol{S[x_1, \ldots, x_n]}$ denote the scale that results from freely adjoining n new elements, traditionally called "variables," $x_1, \ldots, x_n$, to S. (The elements of $S[x_1, \ldots, x_n]$ are named by scale terms built from the elements of S and the symbols $x_1, \ldots, x_n$ with two terms naming the same element iff the equational laws of scales say they must.)

The free scale on n generators is thus $\mathbb{I}[x_1, x_2, \ldots, x_n]$. Each of its simple quotients is the image of a unique map to $\mathtt{I}$, hence is determined by where it sends the generators $x_1, x_2, \ldots, x_n$. And, of course, each x_i may be sent to an arbitrary point in $\mathtt{I}$. Thus $\mathrm{Max}(\mathbb{I}[x_1, x_2, \ldots, x_n])$ may be identified with $\mathtt{I}^n$. Because we now have the semi-simplicity of $\mathbb{I}[x_1, x_2, \ldots, x_n]$ we have a faithful representation $\mathbb{I}[x_1, x_2, \ldots, x_n] \to \mathcal{C}(\mathtt{I}^n)$. In this section we will henceforth treat $\mathbb{I}[x_1, x_2 \ldots, x_n]$ as a subscale of $\mathcal{C}(\mathtt{I}^n)$

We will show that any function in $\mathbb{I}[x_1]$ is what is traditionally called "piecewise linear," that is, a function $f : \mathtt{I} \to \mathtt{I}$ for which there is a sequence $\bot = c_0 < c_1 < \cdots < c_k = \top$ such that f is an affine function whenever it is restricted to the closed subinterval from c_i through c_{i+1}. We need to generalize the notion to higher dimensions.

We are confronted with a terminological problem. Tradition has it that piecewise affine functions be called "piecewise linear" but we will need both notions. (Free lattice-ordered abelian groups will be represented as the functions on $\mathbb{R}^n$ that—informally stated—allow a dissection of $\mathbb{R}^n$ into a finite number of polytopal collections of rays on each of which the function is linear.) Free scales lead not to piecewise linear but piecewise affine functions. Informally: a piecewise affine function is one whose domain may be covered with a finite family of closed polytopes on each of which the function is affine.

So let us start at the beginning. An **affine function** from a convex subset of $\mathbb{R}^n$ to $\mathbb{R}$ can be defined as a continuous function that preserves midpoints.[78] (When the

[77]Having found this construction for scales, I assume that it is ancient knowledge that the analog construction for Boolean algebras of central idempotents works as well. The case $\mathbb{Z}_2[B]$ returns us, of course, to Boole's original *Laws of Thought*.

[78]One need not require continuity if the target is bounded: we pointed out in Section 13 that if a

domain is all of $\mathbb{R}^n$ such is equivalent to the preservation of **affine combinations**, that is, combinations of the form $ax + by$ where $a + b = 1$. Continuity is then automatic.) It is routine, of course, that a function $f : \mathbb{R}^n \to \mathbb{R}$ is affine iff there are constants $a_0, a_1, \ldots, a_n$ such that $f\langle x_1, x_2, \ldots, x_n\rangle = a_0 + a_1x_1 + \cdots + a_nx_n$. If f preserves the origin, equivalently if $a_0 = 0$, it is said to be **linear**.

We will say that an affine function is **dy-affine** if it carries dyadic rationals to dyadic rationals. It is routine that such is equivalent to the a_is all being in $\mathbb{D}$.

For our purposes the simplest—and technically most useful—definition is that $f : \mathtt{I}^n \to \mathtt{I}$ is **continuous piecewise affine** if it is continuous and if there exists an **affine certification**, to wit, a finite family, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, of global affine functions, such that $f(x) \in \{A_1(x), A_2(x), \ldots, A_n(x)\}$ all $x \in \mathtt{I}^n$. Given f and an affine certification, $\mathcal{A}$, we construct its **canonical polytopal dissection**, $\mathcal{P}$, as follows: starting with the interior of $\mathtt{I}^n$ remove all hyperplanes that arise as equalizers of pairs of members of $\mathcal{A}$. The dense open set that remains falls apart as the disjoint union of a finite family of open convex polytopes. We take $\mathcal{P}$ to be the family of closures of these open polytopes.

Given any $P \in \mathcal{P}$ we know that the functions in $\mathcal{A}$ nowhere agree on $\mathring{P}$, the interior of P. We will use the following observation several times: *If a function g is continuous piecewise affine on a connected set and the functions in its certification nowhere agree on that set then g is not just piecewise affine but affine* This observation does not use anything about affine functions other than their continuity: the equalizers of g and the functions in its certification form a finite family of closed subsets that partition the domain, hence each is a component. Thus f agrees with one element of $\mathcal{A}$ on $\mathring{P}$ and—by continuity—on all of P. We denote that affine function as A_P.

Note that there is unique minimal affine certification for f (given any certification retain only those elements of the form A_P). The resulting canonical polytopal dissection will, in general, be simpler for smaller certifications.[79]

By a **CPDA function** we mean a continuous piecewise affine function whose certification consists only of dy-affine functions.

The fact that CPDA functions are closed under the scale operations is easily established: suppose that g is another CPDA function and that $\mathcal{B}$ is its affine certification; then the finite family $\{\ A|B \ :\ A \in \mathcal{A}, B \in \mathcal{B}\ \}$ certifies $f|g$; easier is that $\{-A \ :\ A \in \mathcal{A}\}$ certifies $\dot{f}$ and $\{-1\} \cup \{\ A \ :\ 2A{-}1 \in \mathcal{A}\ \}$ certifies $\hat{f}$. Hence $\mathbb{I}[x_1, x_2, \ldots, x_n]$ viewed as a subset of $\mathcal{C}(\mathtt{I}^n)$, consists only of CPDA functions. (Clearly the generators and constants name dy-affine functions.) What we must work for is the converse: that every CPDA function from

midpoint-preserving function lies in a closed interval then it is monotonic, in particular, continuous. Hence if f is a midpoint-preserving map from a convex subset, C, of $\mathbb{R}^n$ to $\mathtt{I}$, then f restricted to any slice through C is continuous and perforce preserves affine combinations. But the preservation of affine combinations is, by definition, something that takes place on slices.

[79]Note, though, that canonical polytopal dissections are not necessarily minimal. With $n = 2$ consider the piecewise dy-affine function $(x_1 \wedge x_2) \vee 0$. Its minimal certification is, obviously, $\{x_1, x_2, 0\}$ which yields a canonical polytopal dissection with 6 polytopes. But there are dissections with only 4 convex polytopes.

$\mathtt{I}^n$ to $\mathtt{I}$ so appears. It is fairly routine (but a bit tedious) to verify that $-1 \vee A \wedge +1$ [80] is in $\mathbb{I}[x_1, x_2, \ldots, x_n]$ for every dy-affine A. [81]

Let $f : \mathtt{I}^n \to \mathtt{I}$ be an arbitrary CPDA function and $\mathcal{P}$ the canonical polytopal dissection for its minimal certification $\mathcal{A}$. We will construct for each pair $P, Q \in \mathcal{P}$ a function $f_{P,Q} \in \mathbb{I}[x_1, x_2, \ldots, x_n]$ such that:

$$f_{P,Q}(x) \geq f(x) \text{ for } x \in P;$$

$$f_{P,Q}(y) \leq f(y) \text{ for } y \in Q.$$

(Note that it follows that $f_{P,P}(x) = f(x)$ for $x \in P$.)

Then necessarily:

$$f = \bigvee_P (\bigwedge_Q f_{P,Q}).$$

We take $f_{P,P}$ (of course) to be $-1 \vee A_P \wedge +1$.

To construct $f_{P,Q}$ for $P \neq Q$ let $\mathcal{B}_P$ denote the set of the functions of the form $A_R - A_S$ that are non-negative on P and let $B_1, B_2, \ldots, B_k$ be the functions in $\mathcal{B}_P$ that are non-positive on Q. Define

$$f_{P,Q} = -1 \vee A_{P,Q} \wedge +1$$

where

$$A_{P,Q} = A_P + m(B_1 + B_2 + \cdots + B_k)$$

for suitably large integer m.

The two inequalities for the $f_{P,Q}$s are thus reduced to finding $m \geq 0$ such that

$$A_{P,Q}(x) \geq A_P(x) \text{ for } x \in P$$

$$A_{P,Q}(y) \leq A_Q(y) \text{ for } y \in Q$$

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[80] In any distributive lattice $\ell \vee (a \wedge u) = (\ell \vee a) \wedge u$ whenever $\ell \leq u$.

[81] For a proof, say that A is "small" if its values on $\mathtt{I}^n$ lie in $\mathtt{I}$, that is, if $-1 \vee A \wedge +1 = A$. The set of small affine functions is clearly closed under dotting and midpointing. We first show that every small dy-affine function is in $\mathbb{I}[x_1, x_2, \ldots, x_n]$ and we will do that by induction. Given a small dy-affine $f\langle x_1, x_2, \ldots, x_n\rangle = a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n$ we will say that it is of type m if all the a_is can be expressed as dyadic rationals with denominator at most 2^m. There are only $2n+3$ small dy-affine functions of type 0, to wit, $0, \pm 1, \pm x_1, \pm x_2, \ldots, \pm x_n$. Suppose that we have obtained all small dy-affine functions of type m. Given f of type $m{+}1$ let $\{\sigma_i\}_i$ and $\{b_i\}_i$ be such that $a_i = \sigma_i b_i 2^{-(m+1)}$ where $\sigma_i = \pm 1$ and b_i is a natural number. The smallness condition is equivalent to the inequality $b_0 + b_1 + \cdots + b_n \leq 2^{m+1}$. Let $\{c_i\}_i$ and $\{d_i\}_i$ be sequences of natural numbers such that $b_i = c_i + d_i$ each i and $c_0 + c_1 + \cdots + c_n \leq 2^m$, $d_0 + d_1 + \cdots + d_n \leq 2^m$. Define the small dy-affine functions $g\langle x_1, x_2, \ldots, x_n\rangle = \sigma_0 c_0 2^{-m} + \sigma_1 c_1 2^{-m} x_1 + \cdots + \sigma_n c_n 2^{-m} x_n$ and $h\langle x_1, x_2, \ldots, x_n\rangle = \sigma_0 d_0 2^{-m} + \sigma_1 d_1 2^{-m} x_1 + \cdots + \sigma_n d_n 2^{-m} x_n$. Then $f = g|h$. (There are many ways of finding the c_is and d_is, none of which seems to be canonical. Perhaps the easiest to specify is obtained by first defining $e_i = 2^m \wedge \sum_{j=0}^{i} b_j$, then $c_i = e_i - e_{i-1}$ and $d_i = b_i - c_i$.)

Given arbitrary dy-affine A let m be such that $2^{-m}A$ is small. Then $-1 \vee A \wedge +1 = (\odot\!\lhd)^m(2^{-m}A)$.

The 1st inequality holds regardless of $m \geq 0$. For the 2nd suppose, first, that $(B_1 + B_2 + \cdots + B_k)(y) = 0$; then necessarily $B_1(y) = \cdots = B_k(y) = 0$ and since P is the intersection of closed half-spaces

$$P = \bigcap_{B \in \mathcal{B}_P} \{ x : B(x) \geq 0 \}$$

we may conclude that $y \in P$ hence $A_{P,Q}(y) = A_Q(y) = A_P(y)$. If instead $(B_1 + B_2 + \cdots + B_k)(y) < 0$ then for sufficiently large m we obtain $A_{P,Q}(y) = A_P(y) + m(B_1 + B_2 + \cdots + B_k)(y) \leq A_Q(y)$. Because the functions are affine, we need this inequality only for the finite set of extreme points, hence can choose an m that works for all of them. And that completes the construction.

(One immediate application of all this is a construction for the Richter scale that emphasizes its role as the representor for the Jacobson-radical functor. Start with the free scale on one generator, x, and reduce by the $\top$-face, $\mathcal{F}$, that says that the generator is in the Jacobson radical, that is, the $\top$-face generated by all elements of the form $((\top|)^n\bot) \multimap x$. It is easy to verify that $\mathcal{F}$ is the set of all CPDAs that are constantly equal to $\top$ on some non-trivial interval ending at $\top$; the congruence induced by $\mathcal{F}$ thus identifies two CPDAs precisely when they represent the same germ at $\top$. The congruence class of a CPDA f is determined by the value $f(\top)$ and the left-hand derivative of f at $\top$. Note the curious reversal of sign that is needed to establish an isomorphism with our previous construction of the Richter scale: the element $\langle 1, -1 \rangle$ corresponds to a CPDA with a *positive* left-hand derivative at $\top$.)

21. Finitely Presented Scales

By a **closed piecewise affine subset** of Euclidean space we mean a subset of the form $h^{-1}(1)$ where h is a continuous piecewise affine function. If X is closed piecewise dy-affine we will denote the scale of CPDA $\mathtt{I}$-valued functions on X as $\boldsymbol{\mathcal{CPDA}(X)}$. (In the last section we showed that $\mathcal{CPDA}(\mathtt{I}^n)$ is isomorphic to $\mathbb{I}[x_1, x_2, \ldots, x_n]$.)

21.1. THEOREM. *Given a closed piecewise dy-affine subset X of $\mathtt{I}^n$ we obtain a scale homomorphism $\mathcal{CPDA}(\mathtt{I}^n) \to \mathcal{CPDA}(X)$ that is onto. Moreover $\mathcal{CPDA}(X)$ is a finitely presented (f.p.) scale, and all f.p. scales so arise.*

Because the evaluation maps for points in X are thus collectively faithful we will obtain the immediate corollary:

21.2. THEOREM. *All finitely presented scales are semi-simple.*[82]

First note that a finitely presented scale needs just one relation, equivalently any finitely generated $\top$-face is principal: it suffices to note that the $\top$-face generated by

[82]The injectivity of $\mathtt{I}$ implies that this corollary can be strengthened to all "locally f.p. scales," indeed, it suffices for semi-simplicity that each element be contained in an f.p. subscale.

elements $a_1, a_2, \ldots, a_k$ is generated by the single element $a_1 \wedge a_2 \wedge \ldots \wedge a_k$ (one may use $|$ instead of $\wedge$). When we view the free scale on n generators as the scale of CPDA functions on the standard n-cube it is easily seen that a dy-affine function, f, is in the $\top$-face generated by h iff $h^{-1}(1) \subseteq f^{-1}(1)$, hence functions f and g are congruent mod $(\!(h)\!)$ iff they behave the same on the closed dy-affine set $S = h^{-1}(1)$. If we had the lemma that CPDA functions on closed dy-affine subsets of the cube extend to CPDA functions on the entire cube we would be done. (We will, in passing, prove such to be the case since we will show that all such functions are given by an element in the f.p. scale, hence are describable by a term in the free scale and any such term describes a CPDA function on the standard cube, indeed, on the entire Euclidean space.)

So we must redo the previous proof, this time not for the n-cube but for an arbitrary closed dy-affine subset thereof. The only serious complication is that the polytopes of interest are no longer all of dimension n. This complication turns out to be mostly in the eye of the beholder.

Let h be a CPDA function with certification $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$, let $X = h^{-1}(1)$ and let f be an arbitrary CPDA function on X with certification $\mathcal{A} = \{A_1, A_2, \ldots, A_l\}$. We seek a term in $\mathbb{I}[x_1, x_2, \ldots, x_n]$ that describes f on X. Given $s \in X$ let $\mathcal{B}_s$ be the set of functions of the form $D_i - D_j$ and $A_i - A_j$ that are non-negative on s. Note that the negation of some functions in $\mathcal{B}_s$ can also be in $\mathcal{B}_s$ (to wit, all those that are zero on s).

Define

$$P_s = \bigcap_{B \in \mathcal{B}_s} \{\, x \,:\, B(x) \geq 0 \,\}$$

and let $\overset{\circ}{P_s}$ now be—not the interior but—the *inside* of P_s, that is, the points not contained in any of its proper faces.

Note that if two functions in $\mathcal{D}$ agree anywhere on $\overset{\circ}{P_s}$, they agree everywhere on P_s (and if such happens, we know that P_s is of lower dimension than the ambient Euclidean space). For convenience let $\mathcal{D}_s \subseteq \mathcal{D}$ be a minimal certification of h on P_s. Using the lemma from the previous section, we know that h is affine on P_s and since $h(s) = 1$ we know that h is constant on P_s. That is $P_s \subseteq X$. Similarly choose a minimal certification $\mathcal{A}_s \subseteq \mathcal{A}$ of f on P_s. The same lemma says that f agrees with an element of $\mathcal{A}_s$ everywhere on P_s.

If two polytopes of the form P_s overlap their intersection appears as a face of each and is itself of the form P_s. Let $\mathcal{P}$ be the polytopes of form P_s not contained in any other. For each $P \in \mathcal{P}$ chose a function A_P in $\mathcal{A}$ that agrees with f on P. We may now proceed with the construction just as before, starting with the definition of the functions denoted $f_{P,Q}$ (where we understand $\mathcal{B}_P$ to be $\mathcal{B}_s$ for $s \in \overset{\circ}{P_s}$).

We may put this material together to obtain:

21.3. THEOREM. *The full subcategory of f.p. scales is dual to the category of closed piecewise dy-affine sets and* CPDA *maps.*

A quite different notation presents itself, one that emphasizes the pivotal role played by $\mathtt{I}$ (to use a popular phrase—avoided by all those bothered by etymology—it is the duality's "schizophrenic object").

Tradition insists that we change notation. Instead of $\mathcal{C}PDA(X)$ we'll use $\boldsymbol{X}^\star$. The functor that sends X to $X^\star$ may be viewed as an contravariant algebra-valued representable functor, that is, we could also denote $X^\star$ as $(X, \mathtt{I})$ where $\mathtt{I}$ is viewed as a scale algebra in the category of closed piecewise dy-affine sets.

When S is a scale, let $\boldsymbol{S}^\star$ denote the set of $\mathtt{I}$-valued scale-homomorphisms on S. Because $\mathtt{I}$-vauled scale maps are known by their kernels (and because all simple scales are uniquely embeddable in $\mathtt{I}$) $S^\star$ is naturally equivalent to $\mathrm{Max}(S)$

The fact that this pair of functors is an equivalence of categories is equivalent to the "adjunction maps" $X \to X^{\star\star}$ and $S \to S^{\star\star}$ being isomorphisms. To establish the first isomorphism suppose that X is a subset of $\mathtt{I}^n$ of the form $h^{-1}(\top)$ where h is continuous piecewise dy-affine. Then $X^\star$ may be taken as $F_n/\,(\!(h)\!)$ where F_n is the free scale on n generators, $\mathtt{I}\langle x_1, x_2, \ldots, x_n\rangle$. The adjunction map $X \to X^{\star\star}$ sends $x \in X$ to the evaluation map that sends $f \in X^\star$ to $f(x)$. The semi-simplicity of $F_n/\,(\!(h)\!)$ says that $X \to X^{\star\star}$ is monic. To prove that it's onto, let $g \in X^{\star\star}$, that is, $g : X^\star \to \mathtt{I}$. We know that there's $x \in \mathtt{I}^n$ such that $F_n \to X^\star \to \mathtt{I}$ is the evaluation map at x. We need only show that x is in X. Suppose not. It suffices to find $k \in X^\star$, that is, $k : X \to \mathtt{I}$ such that $g(k) \neq k(x)$. It's easy: take $k = h$. The argument for $S \to S^{\star\star}$ is essentially the same.

It is worth noting that the standard notion of homotopy translates rather nicely into this setting. The cylinder over $S^\star$ is $S[x]^\star$ where $S[x]$ is the "polynomial algebra" over S. A pair of maps $f_\perp, f_\top : T \to S$ gives rise to a pair of homotopic maps from $S^\star$ to $T^\star$ iff there is a map $H : T \to S[x]$ such that for $e = \perp, \top$ we have $T \overset{H}{\to} S[x] \overset{v_e}{\to} S \;=\; T \overset{f_e}{\to} S$ where v_e is the map that evaluates a polynomial at e. [83]

We thus capture the dual of the homotopy category of finite simplicial complexes as an algebraically defined quotient category of the category of f.p. scales.

22. Finitely Presented $\mathtt{I}$-Scales and Categories of Closed Piecewise Affine Spaces

Recall that the theory of $\mathtt{I}$-scales in obtained by adding, for each $r \in \mathtt{I}$ a unary operation and the equations:

$$r\odot = \odot$$

$$r(x|y) = rx|ry$$

Whenever $r > 0$:

$$rx \multimap ry \;\leq\; x \multimap y$$

[83]Note that the transitivity of homotopy uses—directly—the coalgebra structure. If $f_\perp$ is homotopic to $f_\odot$ via $H_\perp$ and $f_\odot$ to $f_\top$ via $H_\top$ then $f_\perp$ is homotopic to $f_\top$ via the unique map, $H_\odot$, such that $T \overset{H_\odot}{\to} S[x] \overset{v_{\check{x}}}{\to} S[x] \;=\; T \overset{H_\perp}{\to} S[x]$ and $T \overset{H_\odot}{\to} S[x] \overset{v_{\hat{x}}}{\to} S[x] \;=\; T \overset{H_\top}{\to} S[x]$.

Whenever $r < 0$:

$$rx \multimap ry \;\leq\; y \multimap x$$

For each constant $q > r$:

$$q > r\top$$

For each constant $q < r$:

$$q < r\top$$

The free $\mathtt{I}$-scale on n generators is $\mathtt{I}[x_1, x_2, \ldots, x_n]$.

With little effort we may redo the previous two sections to remove all the "dy-"s:

22.1. THEOREM. *The free $\mathtt{I}$-scale on n generators is isomorphic to $\mathcal{C}PA(\mathtt{I}^n)$, the scale of all continuous piecewise affine functions from the standard n-cube, $\mathtt{I}^n$, to $\mathtt{I}$. Given a closed piecewise affine subset X of $\mathtt{I}^n$ we obtain a scale homomorphism $\mathcal{C}PA(\mathtt{I}^n) \to \mathcal{C}PA(X)$. It is onto; $\mathcal{C}PA(X)$ is an f.p. (finitely presented) $\mathtt{I}$-scale; and all f.p. $\mathtt{I}$-scales so arise. The full subcategory of f.p. $\mathtt{I}$-scales is dual to the category of closed piecewise affine sets and continuous piecewise affine maps.*

23. Complete Scales

Consider the smallest full subcategory of scales that includes $\mathtt{I}$ and is closed under the formation of limits. The construction of left-limit–closures of subcategories can be complicated, but the injectivity of $\mathtt{I}$ in the category of scales makes the job easier: it is the full subcategory of scales that appear as equalizers of pairs of maps between powers of $\mathtt{I}$. That is, first take the full subcategory of objects of the form $\prod \mathtt{I}$ and then add all equalizers of pairs of maps between them. It is clear that such are closed under the formation of products. To see that the full subcategory of such is closed under equalizers let X be the equalizer of a pair of maps from $\prod_I \mathtt{I}$ to $\prod_J \mathtt{I}$ and Y the equalizer of a pair of maps from $\prod_K \mathtt{I}$ to $\prod_L \mathtt{I}$. Let f, g be a pair of maps from X to Y. The injectivity of $\mathtt{I}$ allows us to extend f and g to maps from $\prod_I \mathtt{I}$ to $\prod_K \mathtt{I}$. The equalizer of f, g is constructible as the equalizer of the resulting pair of maps from $\prod_I \mathtt{I}$ to $\prod_J \mathtt{I} \times \prod_K \mathtt{I}$.

Any locally small category constructed as the left-limit–closure of a single object is automatically a reflective subcategory. We will call its objects **complete scales**. They have a number of alternative characterizations.

Say that a map of scales, $A \to B$ is a **weak equivalence** (a phrase borrowed from homotopy theory) if it is carried to an isomorphism by the set-valued functor $(-, \mathtt{I})$, or—put another way—if every $\mathtt{I}$-valued map from A factors uniquely through $A \to B$. A scale, S is complete iff $(-, S)$ carries all weak equivalences to isomorphisms. The lluf subcategory of weak equivalences falls apart into connected components, one for each isomorphism type of complete scales. They are precisely the objects that appear as weak terminators in their components.

The most algebraic description of the category of complete scales is as a category of fractions, to wit, the result of formally inverting all the weak equivalences. (All full reflective subcategories are so describable: they are always equivalent to the result of formally inverting all the maps carried to isomorphisms by the reflector functor.)

A quite different description of complete scales—one that appears not to be algebraic—is in terms of a metric structure. In this setting it is useful to take $\mathtt{I}$ to be the unit interval and $\mathbb{I}$ the dyadic rationals therein. The **intrinsic pseudometric** (of diameter one) on a scale S is most easily defined—in the presence of the axiom of choice—by taking the distance from x to y as $\sup_{f:S\to\mathtt{I}} |f(x)-f(y)|$. Such is a metric (not just a pseudometric) iff S is semi-simple. It is, further, a complete metric iff S is a complete scale (indeed, the reflection of an arbitrary scale into the subcategory of complete scales may be described—metrically—as the usual metric completion of the scale viewed as a pseudometric space).

The intrinsic metric may be defined directly without recourse to the axiom of choice. Define the **intrinsic norm** of $x \in S$ as $||x|| = \inf\{\ q \in \mathbb{I}\ :\ x \leq q\ \}$ and the distance between x and y as $||x \multimapdotboth y||$ (which, we recall is the dotting operation applied to $x \multimapboth y$). (It is easy to verify that on the unit interval $x \multimapdotboth y = |x-y|$.) To see that this definition agrees with the previous in the presence of the axiom of choice we need to show that $||x|| = \sup_{f:S\to\mathtt{I}} f(x)$. Clearly $f(x) \leq ||x||$ for all $f : S \to \mathtt{I}$. If $||x|| = 0$ we are done. For the reverse inequality when $||x|| > 0$ it suffices to find a single $f : S \to \mathtt{I}$ such that $f(x) = ||x||$ and for that it suffices—in the presence of the axiom of choice—to find a proper $\top$-face that contains $\{q_n \multimap x\}_n$ for a strictly ascending sequence $\{q_n\}_n$ of dyadic rationals approaching $||x||$. The $\top$-face generated by $\{q_n \multimap x\}_n$ is the ascending union of the principal $\top$-faces $\{\,(\!(q_n \multimap x)\!)\}_n$ hence it suffices to show that each $(\!(q_n \multimap x)\!)$ is proper. It more than suffices to find a linear quotient in which $q_n \leq x$. The linear representation theorem says that if there were no such linear quotient then $x < q_n$, directly disallowed by the choice of the q_ns.

24. Scales vs. Spaces

The main goal of this section is to show that the category of compact-Hausdorff spaces is dual to the category of complete scales. Using the notation already introduced, the equivalence functor from compact Hausdorff spaces to complete scales sends X to $\mathcal{C}(X)$, the scale of continuous $\mathtt{I}$-valued maps on X. The equivalence functor from complete scales to compact Hausdorff spaces sends S to $\mathrm{Max}(S)$, the set of maximal $\top$-faces on S, topologized by the standard "hull-kernel" topology.

As for the duality between f.p. scales and close piecewise dy-affine set, tradition insists that we change notation. Instead of $\mathcal{C}(X)$ we use $X^\star$. Again, the functor that sends X to $X^\star$ may be viewed as an contravariant algebra-valued representable functor, that is, we could also denote $X^\star$ as $(X, \mathtt{I})$ where $\mathtt{I}$ is viewed as a scale algebra in the category of topological spaces. It is clear from the metric characterization that $X^\star$ is a complete scale.

If S is a scale let $S^\star$ denote the set of $\mathtt{I}$-valued scale-homomorphisms on S, topologized by taking as a basis all sets of the form, one for each $s \in S$:

$$U_s = \{\ f \in S^\star\ :\ f(s) < \top\ \}$$

The fact that $\mathrm{Max}(S)$ and $S^\star$ describe the same space rests on the fact that $\mathtt{I}$-valued scale homomorphisms are known by their kernels. (The fact that the hull-kernel topology describes the same space is easily verified: U_s corresponds to the complement of the hull of the principal $\top$-face $(\!(s)\!)$.)

We will find useful the formulas:

$$\begin{aligned} U_s \cap U_t &= U_{s \vee t} \\ U_s \cup U_t &= U_{s \wedge t} \\ U_s &= U_{\hat{s}} \\ U_\top &= \emptyset \\ U_\perp &= S^\star \end{aligned}$$

24.1. Lemma. *Spaces of the form $S^\star$ are compact-Hausdorff.*

For the Hausdorff property let f, g be distinct elements of $S^\star$ and chose $a \in S$ such that $f(a) \neq g(a)$. We may assume without lose of generality that $f(a) < g(a)$. Let $q \in \mathbb{I}$ be such that $f(a) < q < g(a)$. Then $f \in U_{q \multimap a}$ and $g \in U_{a \multimap q}$. The equation of linearity yields $U_{q \multimap a} \cap U_{a \multimap q} = U_{(q \multimap a)\vee(a \multimap q)} = U_\top = \emptyset$. For compactness let S' be a subset of S. The necessary and sufficient condition that the family of sets $\{\ U_s\ :\ s \in S'\ \}$ be a cover of $S^\star$ is that the $\top$-face generated by S' is entire (because the elements $f \in S^\star$ not in any U_s are precisely those such that $S' \subseteq \ker(f)$). A $\top$-face is entire iff it contains $\perp$. Hence there must be $s_1, s_2, \ldots, s_n \in S'$ such that $\perp$ is the result of applying $\top$-zooming a finite number of times to the element $s_1 \wedge s_2 \wedge \ldots \wedge s_n$. And that is enough to tell us that $U_{s_1} \cup U_{s_2} \cup \cdots \cup U_{s_n} = U_{s_1 \wedge s_2 \wedge \cdots \wedge s_n} = U_\perp = S^\star$

For each $s \in S$ we obtain the "evaluation map" from $S^\star$ to $\mathtt{I}$ that sends $f \in S^\star$ to $f(s) \in \mathtt{I}$. Yet another description of the topology on $S^\star$ is as the weakest topology that makes all these evaluation maps continuous: given $q < r \in \mathbb{I}$ the inverse image of the open interval $(q, r) \subseteq \mathtt{I}$ is $U_{(s \multimap q)\vee(r \multimap s)}$ (and, of course U_s is the inverse image of $\mathtt{I} \setminus \top$).

For an arbitrary space X and element $x \in X$ the evaluation map in $X^{\star\star}$ that sends $f \in X^\star$ to $f(x) \in \mathtt{I}$ is clearly a homomorphism. The natural map $X \to X^{\star\star}$ that sends each point in X to its corresponding evaluation map is continuous (the inverse image of $U_f \subseteq X^{\star\star}$ is the f-inverse-image in X of the open subset $\mathtt{I} \setminus \{\top\}$.)

24.2. Theorem. *If X is compact-Hausdorff then $X \to X^{\star\star}$ is a homeomorphism*

It is monic because of the Urysohn lemma. To see that it is onto, let $H : X^\star \to \mathtt{I}$ be an arbitrary scale-homomorphism. Because $\ker(H)$ is maximal it suffices to find x such that $\ker(H)$ is contained in the kernel of the evaluation map corresponding to x, that is, it suffices to find x such that $H(f) = \top$ implies $f(x) = \top$, or put another way, to find x in $\bigcap_{f \in \ker(H)} f^{-1}(\top)$. First note that if $f^{-1}(\top) = \emptyset$ then there is $q \in \mathbb{I}$ such that $f < q < \top$, hence $H(f) < q$ forcing $f \notin \ker(H)$. The compactness of X therefore says that it suffices to show that the finite-intersection property holds for the family of closed sets $\{\ f^{-1}(\top)\ :\ f \in \ker(H)\ \}$. But this family is closed under finite intersection: $f^{-1}(\top)\ \wedge\ g^{-1}(\top) = (f \wedge g)^{-1}(\top)$ and $\ker(H)$ is clearly closed under finite intersection.

The natural map $S \to S^{\star\star}$ that sends each element in S to its corresponding evaluation map is a homomorphism.

24.3. Theorem. *If S is a complete scale then $S \to S^{\star\star}$ is an isomorphism.*

The proof is an immediate consequence of

24.4. Theorem. *If X is compact-Hausdorff then the necessary and sufficient condition for a subscale, S, of $X^\star$ to be dense (under the intrinsic metric) is that S separates the points of X, that is, for every two points $x, y \in X$ there exists $f \in S$ such that $f(x) \neq f(y)$.*

Note that the necessity uses the Urysohn lemma. The proof is much easier than its model, the Stone–Weierstrass theorem (or is it—in this case—the Stone-without-Weierstrass theorem?). We first establish that S has the "two-point approximation property," that is, for every pair $a, b \in \mathbb{I}$ and every pair of distinct points $x, y \in X$ there is $f \in S$ with $f(x) = a$ and $f(y) = b$. If $a = b$ we can, of course, take f to be that constant. Otherwise we can assume without loss of generality that $a < b$. Start with any f such that $f(x) \neq f(y)$, as insured by the hypothesis. If $f(x) > f(y)$ replace f with $\dot{f}$. Let $c \in \mathbb{I}$ be such that $f(x) < c < f(y)$. There exists n such that $(c\triangleleft)^n f(x) = \bot$ and $(c\triangleleft)^n f(y) = \top$. Replace f with $(c\triangleleft)^n f$ to achieve $f(x) = \bot$ and $f(y) = \top$. Finally, replace that f with $a \vee (f \wedge b)$ to achieve $f(x) = a$ and $f(y) = b$.

We can now repeat the Stone argument. Let X be a compact space and S a sublattice in $X^\star$ with the two-point approximation property. Given any $h \in X^\star$ and $\epsilon > 0$ we wish to find $f \in S$ such that the values of f and h are always within ϵ of each other. For each pair of points $x, y \in X$ let $f_{x,y} \in S$ be such that $f_{x,y}(x)$ is within ϵ of $h(x)$ and $f_{x,y}(y)$ is within ϵ of $h(y)$. Define the open set

$$U_{x,y} = \{\, z \in X \;:\; f_{x,y}(z) < h(z) + \epsilon \,\}.$$

It is best to regard $\mathtt{I}$ here as a fixed closed interval in $\mathbb{R}$. Since $y \in U_{x,y}$ we know that for fixed x the family $\{U_{x,y}\}_y$ is an open cover. Let $y_1, y_2, \ldots, y_m$ be such that $U_{x,y_1} \cup U_{x,y_2} \cup \cdots \cup U_{x,y_m} = X$ and define $f_x = f_{x,y_1} \wedge f_{x,y_2} \wedge \cdots \wedge f_{x,y_m}$. Then $f_x(x) > h(x) - \epsilon$ and for all z we have $f_x(z) < h(x) + \epsilon$ Now for each x define the open set

$$U_x = \{\, z \in X \;:\; f_x(z) > h(z) - \epsilon \,\}.$$

Since $x \in U_x$ we know that the family $\{U_x\}_x$ is an open cover. Let $x_1, x_2, \ldots x_n$ be such that $U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n} = X$. Finally define $f = f_{x_1} \vee f_{x_2} \vee \cdots \vee f_{x_n}$. [84]

24.5. Theorem. *The full subcategory of complete scales is dual to the category of compact-Hausdorff spaces.*

For a remarkably algebraic definition of the category of compact Hausdorff spaces: start with the category of scales; chose an object that has a unique endomorphism and is a weak terminator for all objects other than the terminator; call that object $\mathtt{I}$; formally invert all maps that the contravariant functor $(-, \mathtt{I})$ carries to isomorphisms; take the opposite category. One can replace $\mathtt{I}$ with any non-trivial injective object (as will be seen

[84] Yes, the argument of this paragraph fails if X has only one element.

in the next section all injective scales are retracts of cartesian powers of $\mathtt{I}$). Another choice is first to restrict to the full subcategory of semi-simple scales and then formally invert the mono-epis (obtaining what some might call its "balanced reflection").[85]

25. Injective Scales, or: Complete Chromatic Scales

As complete scales are to the study of continuous maps we expect that injective scales will be to measurable functions. As we will see, all injective scales come equipped with a (necessarily unique) chromatic structure but—fortunately for that expectation— maps between them need not preserve that structure. Let us lay out the groundwork.

If an object (in any category) is injective, then it is clearly an **absolute retract**, that is, whenever it appears as a subobject it appears as a retract. The converse need not hold, even for models of an equational theory.[86] As we will see, absolute retracts in the category of scales are, indeed, injective.

25.1. LEMMA. *A scale that is an absolute retract is order-complete (that is, it is not just a lattice but a complete lattice).*

[85] Among semi-simples a map is an epi iff its image is an order-dense subset of the target (and—as always for models of equational theories—monos are one-to-one). But for arbitrary scales the characterization of epimorphisms is more complicated. If we work with $\mathbb{Q}$-scales, that is, scales for which multiplication by each rational in the standard interval is defined, then the only epis are onto. For a proof, it suffices (as usual) to consider "dense subobjects," that is, those whose inclusion maps are epi. Suppose $S' \subseteq S$ is dense. Let V be the enveloping $\mathbb{Q}$-vector space of S (defined, of course, analogously to the enveloping dy-module) with its inherited partial ordering. For any $s_0 \in S \setminus S'$ use the axiom of choice to find a map $f : V \to V$ with S' in its kernel such that $f(s_0) \neq 0$. Partially order $V \oplus V$ lexicographically and define a new scale T as the interval from $\langle \bot, 0\rangle$ to $\langle \top, 0\rangle$ with its induced scale structure. The two scale homomorphisms that send $s \in S$ to $\langle s, 0\rangle$, in the first case, and $\langle s, f(s)\rangle$, in the second, agree on S' and disagree on s_0. For the general scale case, the condition for a subscale to be dense—to use a term from abelian-group theory— is that its "purification" be entire.

[86] The Sierpiński monoid ($\{0, 1\}$ under multiplication) is an absolute retract in the category of commutative monoids: whenever it is a submonoid the "canonical map" from the ambient monoid to the Sierpiński monoid (to wit, the characteristic map of the subgroup of units) is a retraction. But the canonical map from the one-generator monoid (the natural numbers) does not extend to a map from the one-generator group (the integers). Curiously the Sierpiński monoid is injective in the full subcategory of finite commutative monoids, indeed, in the full subcategory of locally finite commutative monoids (which in the commutative case is the same as saying each one-generator submonoid is finite). The full subcategory of idempotent commutative monoids—semi-lattices as they are usually called—is both a reflective and co-reflective subcategory of the category of all commutative monoids. If one restricts to locally finite commutative monoids then not only is it both reflective and co-reflective but the same functor delivers both the reflection and co-reflection: the reflection map sends each element of a locally finite commutative monoid to the unique idempotent element that appears in the sequence of its positive powers. Since the functor also delivers co-reflections it preserves monomorphisms and hence the injectivity of the Sierpiński monoid follows from its injectivity among semi-lattices. If one deems them meet-semi-lattices then maps into the two-element meet-semi-lattice are the characteristic maps of filters and it is easy to see that any such characteristic map extends to any larger meet-semi-lattice. (Note that—quite unusual for injective objects—the axiom of choice is not used in extending maps to the Sierpiński monoid.)

Let E be an absolute retract and $L \subseteq E$ an arbitrary subset, $U \subseteq E$ the set of upper bounds of L. Freely adjoin an element b to E to obtain the "polynomial scale" $E[b]$ and let $\mathcal{F}$ be the $\top$-face generated by the elements $\{ \ell \multimap b : \ell \in L \}$ and $\{ b \multimap u : u \in U \}$. Once we know that $E \to E[b]/\mathcal{F}$ is an embedding we are done because then a retraction $f : E[b]/\mathcal{F} \to E$ necessarily sends b to a least upper bound of L. The fact that $E \to E[b]/\mathcal{F}$ is an embedding is equivalent to the disjointness of $\mathcal{F}$ and E_* (recall that the "lower star" removes the top). If $x \in E_*$ were in $\mathcal{F}$ then a finite number of the generators of $\mathcal{F}$ would account for it. But given $\ell_1 \multimap b, \ell_2 \multimap b, \ldots, \ell_m \multimap b$ and $b \multimap u_1, b \multimap u_2, \ldots, b \multimap u_n$ we may easily obtain a retraction of $E[b]$ back to E by sending b to $u_1 \wedge u_2 \wedge \cdots \wedge u_n$. The kernel of any retraction of E is, of course, disjoint from E_*. But this kernel contains the listed finite number of elements (indeed it contains $\ell \multimap b$ for all $\ell \in L$).

25.2. LEMMA. *Absolute retracts are semi-simple.*

Semi-simplicity is equivalent to $\top$ being the least upper bound of $\mathbb{I}_*$. But a least upper bound of $\mathbb{I}_*$ is necessarily invariant under $\top$-zooming. There is only one such element larger than $\odot$ (check in any linear scale) and that element is $\top$.

25.3. COROLLARY. *Absolute retracts are injective.*

An absolute retract, being semi-simple, can be embedded in a cartesian power of $\mathtt{I}$. A cartesian power of an injective is injective, that is, every absolute retract can be embedded in an injective which, of course, is the necessary and sufficient condition for an absolute retract to be injective (any retract of an injective is injective).

As is the case for any equational theory, a model is an absolute retract iff it has no proper **essential extensions**. We recall the definitions: a monic $A \to B$ is essential if whenever $A \to B \to C$ is monic it is the case that $B \to C$ is monic. For models of an equational theory this translates to the condition that every non-trivial congruence on B remains non-trivial when restricted to A. Note that for any monic $A \to B$ we may use Zorn's lemma to obtain a congruence maximal among those that restrict to the trivial congruence on A thus obtaining a map $B \to C$ such that $A \to B \to C$ is an essential extension. It follows that if A has no essential extensions other than isomorphisms then A is an absolute retract. (The converse is immediate.)

25.4. LEMMA. *A scale B is an essential extension of a subscale $A \subseteq B$ iff A_* is co-final in B_**

Because essentiality is clearly equivalent to every non-trivial $\top$-face in B meeting A non-trivially; and clearly a principal $\top$-face, $(\!(b)\!) \subseteq B$ meets A non-trivially iff there is $a \in A_*$ such that $b < a$. If A_* is co-final in B_* then for any $a \in A$ and $b \in B$ such that $b < a$ there exists $a' \in A$ with $b < a' < a$ because we may use the order-isomorphism $a \multimap (-)$ from $[b, a]$ to $[a \multimap b, \top]$ [87] to obtain $a'' \in A$ such that $a \multimap b < a'' < \top$. The inverse isomorphism thus delivers an element in A (to wit, $a' = a \overset{\wedge}{|} a''$) strictly between b

[87]This was discussed at the end of Section 4.

and a. And we may dualize: if $a < b$ we may find $a' \in A$ such that $a < a' < b$ (simply apply the previous case to $\dot{b} < \dot{a}$).

The converse for the first lemma in this section:

25.5. LEMMA. *If a scale is order-complete then it is an absolute retract.*

Consider an order-complete scale A and essential extension $A \subseteq B$. Given $b \in B$ let $a \in A$ be the greatest lower bound of the A-elements above b. Using the second case above we reach a contradiction from the strict inequality $a < a \vee b$ (because if $a' \in A$ were such that $a < a' < a \vee b$ then a would not be the greatest lower bound. Hence $a = a \vee b$, that is, $b \leq a$. Using the first case above we reach a contradiction from the strict inequality $b < a$ (because if $a' \in A$ were such that $b < a' < a$ then a would not be a lower bound of the A-elements above b. Thus $a = b$. That is, every element in B is in A.[88]

25.6. THEOREM. *An injective scale is of the form $\mathcal{C}(X)$ where X is an extremely disconnected compact Hausdorff space (that is, one in which the closure of every open set is open).*

Note first that any retract of a (metrically) complete scale is complete, hence injective scales, being retracts of cartesian powers of $\mathtt{I}$, are necessarily of the form $\mathcal{C}(X)$. We need to show that the order-completeness implies that X is extremely disconnected. Let $V \subseteq X$ be open. Define $\mathcal{U} \subseteq \mathcal{C}(X)$ to be the set of all continuous functions from X to $\mathtt{I}$ that are constantly equal to $\top$ on V. The Urysohn lemma says that for every point x in the exterior of V (the complement of the closure of V) there is a function in $\mathcal{U}$ that sends x to $\bot$ hence we know that any lower bound of $\mathcal{U}$ is constantly equal to $\bot$ on the exterior of V. Dually, we know that for every $x \in V$ there is a lower bound of $\mathcal{U}$ that is constantly equal to $\bot$ on the complement of V but sends x to $\top$ hence the greatest lower bound, f, of $\mathcal{U}$ must be constantly equal to $\top$ on V, indeed, on the closure of V. But f (as is every lower bound of $\mathcal{U}$) is constantly equal to $\bot$ on the complement of the closure of V. It follows that the only values of f are $\top$ and $\bot$. It is the characteristic function of the closure of V which means, of course, that the closure of V is open.

25.7. COROLLARY. *Order-complete scales are precisely those (metrically) complete scales that are chromatic.*

Given $f : X \to \mathtt{I}$ where X is extremely disconnected, construct its support, $\overline{f}$, as the characteristic map of the closure of $\{\, x \in X \,:\, f(x) > \bot \,\}$.

[88]In great generality—in particular for the models of any equational theory—a maximal essential extension of an object is injective, indeed, it is minimal among injective objects in which the object can be embedded. Such maximal-essential/minimal-injective extensions are called **injective envelopes**. (If an object can be embedded in an injective then each of its essential extensions must appear therein and we may find one that is maximal.)

For scales we have noted that $A \subseteq B$ is essential iff A_* is cofinal in B_*. If a scale is semi-simple we know that it can be embedded in an injective (to wit, a cartesian power of $\mathtt{I}$), hence that it has an injective envelope. If it is not semi-simple it will have essential extensions of unbounded cardinality. (In Section 12 we saw this phenomenon for the Richter scale when we saw that it appears co-finally in every SDI.)

Perhaps the most conceptual proof that complete chromatic scales are order-complete is as follows: given a subset, $\mathcal{U}$, of $\mathcal{C}(X)$ where X is extremely disconnected, then for any "grid" $\bot = q_0 < \cdots < q_n = \top$ we may define $\emptyset = U_0 \subseteq U_1 \subseteq U_1 \subseteq \cdots \subseteq U_n$ by taking U_j to be the largest open set such that $f(x) < q_j$ for all $x \in U_j$ and $f \in \mathcal{U}$ and let $h \in \mathcal{C}(X)$ be the step-function such that $f(\overline{U}_j \setminus \overline{U}_{j-1}) = q_{j-1}$. By taking finer and finer grids we obtain an ascending Cauchy sequence whose limit is easily checked to be the greatest lower bound of $\mathcal{U}$.

Since we have identified the injective objects in the full category of complete scales it follows that we have identified the projectives in its dual category, the category of compact Hausdorff spaces.[89] Given such a space X let Y be the set of ultrafilters on the (discrete) set X. Y is, of course, the "compactification" of that discrete set, that is, the reflection of the discrete space in the full subcategory of compact Hausdorff spaces. The canonical map from Y back to X is a retraction of X and, hence, $\mathcal{C}(X)$ is a retract of $\mathcal{C}(Y) = \prod_X \mathtt{I}$ which, being a cartesian product of injective objects, is, of course, injective.

In the category of compact Hausdorff spaces Gleason constructed the minimal projective cover of a space X as the Stone space of the Boolean algebra of regular closed sets of X (those closed sets that are the closures of their interiors). That Stone space may be constructed as the set of ultrafilters of regular closed sets. The covering map is clear: send such an ultrafilter to the one point in its intersection. It may be described also as the scale of "adjoint pairs" of semi-continuous $\mathtt{I}$-valued functions on X, that is pairs consisting of a lower- and an upper-semi-continuous function where the lower-semi-continuous function is the largest such less than the upper-semi-continuous function, and dually.

26. Diversion: Finitely Presented Chromatic Scales

If we move to chromatic scales we remove the word "continuous" to obtain "piecewise dy-affine map" and the word "closed" to obtain "piecewise dy-affine set." The definitions are no longer as simple (we can not, for example, get by just with the existence of a dy-affine certification). The proofs of the parallel theorems, however, are easier. The category of finitely presented chromatic scales is dual to the category of piecewise dy-affine maps between piecewise dy-affine sets. The non-empty spaces are determined up to isomorphism by just two invariants—dimension and Euler characteristic.[90]

[89] First done by Andrew Gleason, Projective topological spaces. *Illinois J. Math.* 2 1958 482–489. But this result is an easy consequence of the fact that the injective objects in the category of boolean algebras are the complete boolean algebras: Roman Sikorski, A theorem on extension of homomorphisms. *Ann. Soc. Polon. Math.* 21 (1948), 332–335.

[90] Any piecewise dy-affine set is a disjoint union of "open simplices"—ordinary closed simplex with all the boundary points removed (note that the 0-dimensional open simplex is not empty but a single point). Such a disjoint union may be described with an $(n+1)$-tuple of natural numbers, $\langle s_0, s_2, \ldots, s_n\rangle$ where n is the set's dimension (hence $s_n > 0$); its Euler characteristic is $s_0 - s_1 + \cdots + (-1)^n s_n$. We show that any such set is piecewise dy-affine isomorphic to one described by an $(n+1)$-tuple where $s_0 = s_1 = \cdots = s_{n-2} = 0$ and either $s_{n-1} = 0$ or $(s_n = 1) \wedge (s_{n-1} > 0)$ (the first possibility occurs precisely when the Euler characteristic is non-zero with signature $(-1)^n$). First, any k-dimensional open

The chromatic scale corresponding to an n-dimensional space can not be generated with fewer than n generators. If $\chi = 1$ the corresponding chromatic scale may be taken to be the free chromatic scale on n generators.

If the Euler characteristic is greater than 1 then the corresponding scale may be constructed as $\mathbb{I}[x_1, x_2, \ldots, x_n]/ (\!(\underline{t})\!)$ where $t = (x_1 \multimap q_1) \vee (x_1 \circ\!\!-\!\!\circ q_2) \vee \cdots \vee (x_1 \circ\!\!-\!\!\circ q_\chi)$ and $\perp < q_1 < q_2 < \cdots < q_\chi = \top$ (one $\multimap$, the rest $\circ\!\!-\!\!\circ$ s). As always for finitely presented chromatic scales this is isomorphic to an interval in the free algebra, to wit, $[\perp, \underline{t}]$. The free algebra splits as the product $[\perp, \underline{t}] \times [\underline{t}, \top]$ and we may dispatch the case of negative characteristic with the observation that the second factor is of the same dimension and the characteristics of the two factors add to one.

For Euler characteristic 0 we can use $\mathbb{I}[x_1, x_2, \ldots, x_n]/ (\!(\overline{x_1})\!)$. Finally, when $n = 0$ the only corresponding chromatic scales are the cartesian powers of $\mathbb{I}$ (bear in mind that χ is necessarily non-negative and that the empty product is the one-element terminal scale).

The only time that we need more generators than dimension is when $n = 0$ and $\chi > 1$. (The corresponding chromatic scale is constructible with one generator: $\mathbb{I}[x]/ (\!(\underline{v})\!)$ where $v = (x \circ\!\!-\!\!\circ q_1) \vee (x \circ\!\!-\!\!\circ q_2) \vee \cdots \vee (x \circ\!\!-\!\!\circ q_\chi)$ and $q_1 < q_2 < \cdots < q_\chi$.)

27. Appendix: Lattice-Ordered Abelian Groups

By a **lattice-ordered abelian group**, or **LOAG** for short, is meant, of course, an object with both an abelian-group and a lattice structure in which the lattice ordering is preserved by addition (that is, $x + (y \diamond z) = (x + y) \diamond (x + z)$ for either lattice-operation $\diamond$). We simplify matters by noting that we need only **truncation at zero**, $0 \vee x$, as a primitive. We will denote this truncation here as $\lfloor \boldsymbol{x} \rfloor$.

The axioms:

TRUNC-1:

$$x \;=\; \lfloor x \rfloor - \lfloor -x \rfloor$$

TRUNC-2:

$$\lfloor x - \lfloor y \rfloor \rfloor \;=\; \lfloor \lfloor x \rfloor - \lfloor y \rfloor \rfloor$$

.

(Trunc-1 is justified by $x + (0 \vee -x) = (x + 0) \vee (x + (-x)) = x \vee 0$. For Trunc-2 it suffices to justify $\lfloor y \rfloor + \lfloor x - \lfloor y \rfloor \rfloor = \lfloor y \rfloor + \lfloor \lfloor x \rfloor - \lfloor y \rfloor \rfloor$. But $\lfloor y \rfloor + (0 \vee (x - \lfloor y \rfloor)) = (\lfloor y \rfloor + 0) \vee (\lfloor y \rfloor + (x - \lfloor y \rfloor)) = \lfloor y \rfloor \vee x = (y \vee 0) \vee x$ and $\lfloor y \rfloor + (0 \vee (\lfloor x \rfloor - \lfloor y \rfloor)) = (\lfloor y \rfloor + 0) \vee (\lfloor y \rfloor + (\lfloor x \rfloor - \lfloor y \rfloor)) = \lfloor y \rfloor \vee \lfloor x \rfloor = (v \vee 0) \vee (x \vee 0)$.)

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simplex with $k > 0$ is the disjoint union of two k-dimensional and one $(k-1)$-dimensional open simplices, which means that we may, without changing isomorphism type, increment by 1 any two adjacent s_is provided the right-hand one is positive. A sequence of such increments, working from the top down, can guarantee that all the s_is are positive, indeed as big as we want them. We can then minimize the number of positive s_is by successively performing the reverse of such increments, working from the bottom up, until $s_0 = s_2 = \cdots = s_{n-2} = 0$. We then perform as many such reverse increments as we can on the pair s_{n-1}, s_n. The result is as advertised.

Use the truncation operator to define:

$$x \vee y = x + \lfloor y - x \rfloor$$

Its idempotence is easily seen to be equivalent to:

Trunc-0:

$$\lfloor 0 \rfloor \;=\; 0$$

which can be proven by $\lfloor 0\rfloor = \lfloor\lfloor 0\rfloor\rfloor - \lfloor 0 - \lfloor 0\rfloor\rfloor = \lfloor\lfloor 0\rfloor\rfloor - \lfloor\lfloor 0\rfloor - \lfloor 0\rfloor\rfloor = \lfloor\lfloor 0\rfloor\rfloor - \lfloor 0\rfloor = \lfloor\lfloor\lfloor 0\rfloor\rfloor - \lfloor 0\rfloor\rfloor - \lfloor\lfloor 0\rfloor - \lfloor\lfloor 0\rfloor\rfloor\rfloor = \lfloor\lfloor 0\rfloor - \lfloor 0\rfloor\rfloor - \lfloor\lfloor\lfloor 0\rfloor\rfloor - \lfloor\lfloor 0\rfloor\rfloor\rfloor = \lfloor 0\rfloor - \lfloor 0\rfloor = 0$.

The fact that addition distributes with $\vee$ is immediate. For commutativity note that $x \vee y = y \vee x$ translates to $x + \lfloor y - x\rfloor = y + \lfloor x - y\rfloor$ and that rearranges to $x - y = \lfloor x - y\rfloor - \lfloor y - x\rfloor$, an instance of Trunc-1.

For associativity note that by adding $\lfloor y\rfloor$ to both sides of Trunc-2 we obtain $\lfloor y\rfloor \vee x = \lfloor y\rfloor \vee \lfloor x\rfloor$, making it clear that $\lfloor y\rfloor \vee x$ is commutative. We may rewrite this last fact (using the commutativity of $\vee$) as $x \vee (0 \vee y) = (x \vee 0) \vee y$ which, together with distributivity with addition, easily yields full associativity.[91]

The induced ordering, that is, the one obtained by defining $x \le y$ iff $x \vee y = y$, is, of course, preserved under addition. And from that we may infer that it is reversed by negation: $x \le y$ iff $x-(x+y) \le y-(x+y)$ iff $-y \le -x$. Hence negation must convert least upper bounds into greater lower bounds, yielding what can only be called De Morgan's law: $-(x \wedge y) = (-x) \vee (-y)$. For a direct formula we have

$$x \wedge y = x - \lfloor x - y \rfloor$$

(because $x \wedge y = -((-x) \vee (-y)) = -((-x) + \lfloor (-y) - (-x)\rfloor\,) = x - \lfloor x - y\rfloor$).

This is a complete equational theory, that is, every equation on these operators (zero, addition, negation and truncation) is either inconsistent or a consequence. This result is, we trust, somewhere in the literature. But note here that every consistent equation holds for the LOAG of integers, $\mathbb{Z}$, because every consistent equation has a non-trivial model, every non-trivial model has a positive element (e.g. $\lfloor x\rfloor + \lfloor -x\rfloor$ for any $x \neq 0$) and any positive element generates a sub-LOAG isomorphic to $\mathbb{Z}$, all of which says that the maximal consistent equational extension of the theory of LOAGs is—precisely—the theory of $\mathbb{Z}$. To verify that the equations in hand already provide that maximal consistent extension it thus suffices to show that every equation not a consequence of those axioms has a counterexample in $\mathbb{Z}$. It clearly suffices to find a counterexample in the rationals, $\mathbb{Q}$, because multiplying by a suitable positive integer would then yield a counterexample in $\mathbb{Z}$ and because the operations are continuous, it clearly suffices for that to find a

[91] Trunc-2 is stronger than associativity (as seen, it has Trunc-0 built into it). It was chosen as an axiom not for its strength but for its simplicity: the truncation equation equivalent to associativity is $\lfloor x + \lfloor y - x\rfloor\rfloor = \lfloor x\rfloor + \lfloor y - \lfloor x\rfloor\rfloor$. For a separating example take the positive rationals under multiplication and define the associative "join" operation to be ordinary addition. The truncation operator is then just shifting by 1. Trunc-1, when rewritten, becomes $x = (1+x)(1+x^{-1})^{-1}$ which is satisfied and Trunc-2 becomes $1 + x(1+y^{-1}) = 1 + (1+x)(1+y^{-1})$ which is not.

counterexample in the reals, $\mathbb{R}$. The previous proof for the theory of scales can be easily replicated for this case. Or, if one wishes, we can reduce this case to that previous case. Given an equation with a counterexample in some LOAG we can first tensor with the dyadic rationals, $\mathbb{D}$, to obtain a dy-module and then chose an element, $\top$, large enough so that the computation of the terms in the counterexample all lie in the interval $[-\top, \top]$. Replacing 0 with $\odot$, $-x$ with $\dot{x}$, $x + y$ with $\odot \triangleleft (x|y)$ and $\lfloor x \rfloor$ with $\top | \hat{x}$ we obtain a scale with a counterexample for the given equation and from that we know that there is a counterexample in $\mathbb{R}$.

The linear representation theorem for LOAGs:

27.1. THEOREM. *Every lattice-ordered abelian group can be embedded (as such) in a product of linearly ordered abelian groups.*

The usual name for the latter objects is **TOAG** for **totally ordered abelian groups.** The proof that all LOAGs can be embedded in a product of TOAGs is—as is to be expected—essentially the same as it was for scales. It is necessary and sufficient to show that every subdirectly irreducible LOAG is a TOAG. We need a handle on congruences and for that we will need the triangle inequality for the truncation operator, that is, $\lfloor a + b \rfloor \leq \lfloor a \rfloor + \lfloor b \rfloor$ (most easily obtained by noting that truncation is covariant, hence $\lfloor a + b \rfloor \leq \lfloor \lfloor a \rfloor + \lfloor b \rfloor \rfloor$, and that positive elements are closed under addition, hence $\lfloor \lfloor a \rfloor + \lfloor b \rfloor \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$.)

The kernel of a LOAG congruence is, of course, defined as the set of elements congruent to zero. Kernels turn out not to be the most convenient subsets for identifying congruences but nonetheless we need to find out just which subgroups are kernels. Besides being subgroups, LOAG kernels are, first, closed under the action of the truncation operator and, second, satisfy the **betweenness condition:** if $x \leq y \leq z$ and if x and z are in the kernel then so is y. We will say that a subgroups satisfying these conditions are **closed subgroups**.

To see that an arbitrary closed subgroup, C, is the kernel of a congruence we need only show that whenever $a - b \in C$ it is the case that $\lfloor a \rfloor - \lfloor b \rfloor \in C$. From the triangle inequality we know that $\lfloor a \rfloor - \lfloor b \rfloor \leq \lfloor a - b \rfloor$ (because $\lfloor a \rfloor = \lfloor (a-b) + b \rfloor \leq \lfloor a - b \rfloor + \lfloor b \rfloor$). Transposing a and b we obtain $\lfloor b \rfloor - \lfloor a \rfloor \leq \lfloor b - a \rfloor$ which gives us the other inequality needed to apply the betweenness condition: $-\lfloor b - a \rfloor \leq \lfloor a \rfloor - \lfloor b \rfloor \leq \lfloor a - b \rfloor$.

By the **closed positive cone** in A, **Cone(A)** $\subset A$, we will mean $\{ a \in A : a \geq 0 \}$. The closed subgroups of A are determined by their intersection with Cone(A) (since $x = \lfloor x \rfloor - \lfloor -x \rfloor$) and it is these subsets that turn out to be most useful in determining congruences. We define a **LOAG ideal** of A to be a non-empty subset of Cone(A) closed under addition and hereditary downwards (that is, if x is in the ideal then so is y whenever $0 \leq y \leq x$). Note that an ideal is automatically closed under join (because if x and y are positive then $x + y$ is a common upper bound). Clearly intersection of any closed subgroup with Cone(A) is an ideal. And every ideal so arises: given an ideal, $\mathfrak{A}$, define $H = \{ x - y : x, y \in \mathfrak{A} \}$; it is clearly a subgroup and closed under the truncation operator (because $0 \leq \lfloor x - y \rfloor \leq \lfloor x \rfloor = x$); for the betweenness condition note that if

$u - v \leq x \leq w - y$ where $u, v, w, y \in \mathfrak{A}$ then $x = \lfloor x \rfloor - \lfloor -x \rfloor$ where $\lfloor x \rfloor \leq \lfloor w - y \rfloor \leq w$ and $\lfloor -x \rfloor \leq \lfloor v - u \rfloor \leq v$.

There is—as is to be expected—a great similarity between the theory of LOAGs and scales but it often comes with a twist. For example we used the fact that a scale is linear iff it satisfies the coalgebra condition, $\overset{\vee}{x} = \top$ or $\hat{x} = \bot$ for all x. A LOAG is a TOAG iff $\lfloor x \rfloor = 0$ or $\lfloor -x \rfloor = 0$ for all x. The internalization of the coalgebra condition was the coalgebra equation $\overset{\vee}{x} \vee \dot{\hat{x}} = \top$. The corresponding "coalgebra equation for LOAGs" is $\lfloor x \rfloor \wedge \lfloor -x \rfloor = 0$ directly obtained by $\lfloor x \rfloor \wedge \lfloor -x \rfloor = \lfloor x \rfloor - \lfloor \lfloor x \rfloor - \lfloor -x \rfloor \rfloor = \lfloor x \rfloor - \lfloor x \rfloor = 0$. For scales we have a binary operation that detects the ordering: $x \multimap y = \top$ iff $x \leq y$. For LOAGs we have that $\lfloor x - y \rfloor = 0$ iff $x \leq y$. For scales the internalization of linearity is the equation of linearity $(x \multimap y) \vee (y \multimap x) = \top$. For LOAGs it is $\lfloor x - y \rfloor \wedge \lfloor y - x \rfloor = 0$. For scales we used the fact that linearity is equivalent with the disjunction property, that is, $x \vee y = \top$ iff $x = \top$ or $y = \top$. For LOAGs it is equivalent with what we have to call the "conjunction property": $a \wedge b = 0$ iff $a = 0$ or $b = 0$.

We will need one more lemma: $\lfloor 2x \rfloor = 2\lfloor x \rfloor$. For its verification note that the distributivity of addition over join yields $2\lfloor x \rfloor = (0 \vee x) + (0 \vee x) = 0 \vee x \vee x \vee 2x$. For $0 \vee x \vee x \vee 2x = 0 \vee 2x = \lfloor 2x \rfloor$ it suffices to show that $x \leq 0 \vee 2x$ which is equivalent to $0 \leq -x \vee x$. We verify (using the coalgebra equation) its De Morgan dual, $0 \geq -x \wedge x$ as follows: $0 \wedge (-x \wedge x) = (\lfloor -x \rfloor \wedge \lfloor x \rfloor) \wedge (-x \wedge x) = (\lfloor -x \rfloor \wedge -x) \wedge (\lfloor x \rfloor \wedge x) = (-x \wedge x)$ (clearly $y \leq \lfloor y \rfloor$ for all y).

We can now prove the linear representation theorem for LOAGs be showing that every subdirectly irreducible LOAG satisfies the conjunction condition. Let G be an SDI LOAG and $s \in G$ be a non-trivial element such that every non-trivial ideal includes s. Suppose there were a counterexample: $a \neq 0$, $b \neq 0$ and $a \wedge b = 0$. The principal ideal generated by a is the set of all $x \geq 0$ such that $x \leq na$ for almost all n, hence $s \leq na$ for almost all n and, similarly, $s \leq nb$ for almost all n which, of course, would force $s \leq na \wedge nb$. If we replace n with 2^n in order to apply the lemma $2a \wedge 2b = 2(a \wedge b)$, and its iterations $2^n a \wedge 2^n b = 2^n(a \wedge b)$, we obtain the contradiction $0 < s \leq 2^n(a \wedge b) = 0$ for almost all $n > 0$ (and one such n is enough).

One immediate consequence of the linear representation theorem is that $m\lfloor x \rfloor = \lfloor mx \rfloor$ for every positive integer m (not just 2) and that extends to every term: if $A\langle x_1, x_2, \ldots, x_n \rangle$ is a term in the signature of LOAGs then $mA\langle x_1, x_2, \ldots, x_n \rangle = A\langle mx_1, mx_2, \ldots, mx_n \rangle$ for all positive integers m.

The single most important consequence is that every LOAG is a distributive lattice. And that yields a proof that the lattice of congruences is distributive. Given two ideals, $\mathfrak{A}$ and $\mathfrak{B}$ the smallest ideal that contains them both is $\{\, a \vee b \,:\, a \in \mathfrak{A},\ b \in \mathfrak{B} \,\}$: down-closure is the observation that distributivity says $x \leq a \vee b$ implies $x = (x \wedge b) \vee (x \wedge c)$; additive closure is the observation that given $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ in any TOAG either $a \leq b$ or $b \leq a$ hence that either $a + b \leq 2a$ or $a + b \leq 2b$ and in either case $a + b \leq 2a \vee 2b$. The proof that $(\mathfrak{A} \vee \mathfrak{B}) \cap \mathfrak{C} \;=\; (\mathfrak{A} \cap \mathfrak{C}) \vee (\mathfrak{B} \cap \mathfrak{C})$ is dual to the same assertion for filters in a distributive lattice.

Define Spec(A) to be the lattice of congruences of a LOAG A. As for any equational theory the distributivity of Spec(A) implies that it is a spatial locale: to prove $\mathfrak{A} \cap \bigvee \mathfrak{B}_i = \bigvee(\mathfrak{A} \cap \mathfrak{B}_i)$ first close the family $\{\mathfrak{B}_i\}$ under finite join to obtain $\bigvee \mathfrak{B}_i = \bigcup \mathfrak{B}_i$; to prove the existence of enough "points" verify that for $a \neq b \in A$ any congruence maximal among those that do not identify a and b is a point.

Since Spec(A) is spatial the alternative definition presents itself: it may be defined as the space of **prime ideals** with the hull-kernel topology; that is, its elements are the ideals that are not the intersection of two larger ideals with the closure of any set of such defined as the set (the "hull") of all prime ideals that contain the intersection (the "kernel") of the prime ideals in the given set. It is convenient to know that an ideal, $\mathfrak{A}$, is prime iff $a \wedge b \in \mathfrak{A}$ implies either $a \in \mathfrak{A}$ or $b \in \mathfrak{A}$. The proof is most easily found if one first notes that $\mathfrak{A} \cap B$ is $\{ a \wedge b : a \in \mathfrak{A},\ b \in \mathfrak{B} \}$.

When we restrict to the subspace of maximal ideals Max(A) there is an alternative description. $\mathfrak{A}$ is maximal iff $A/\mathfrak{A}$ is simple. A simple LOAG, being an SDI, is linear. Moreover, for each $a > 0$ it is the case that every element is in its principal ideal, $(\!(a)\!)$ and as we noted, $(\!(a)\!)$ is the set of all $x \geq 0$ such that $x \leq na$ for almost all n, hence for any pair of elements $s, a > 0$ it is the case that $s \leq na$ for almost all n. All of which says that simple LOAGs are precisely the Archimedean TOAGs. It is clear that any Archimedean TOAG can be embedded into the TOAG of reals, $\mathbb{R}$. The embedding is unique up to a positive scalar multiple.

So, the alternative description of Max(A) has its elements the rays in the set of maps $(A, \mathbb{R})$ (a ray is a non-trivial set of the form $\{ rf : r \in \mathbb{R}^+ \}$). Note that a ray is known by the elements in A that are sent to the positive half of $\mathbb{R}$.

The topology on Max(A) has a neighborhood basis, $\{U_a\}_{a \in \mathrm{Cone}(A)}$, where $U_a = \{ f : f(a) > 0 \}$. We will find useful the formulas:

$$\begin{aligned} U_a \cap U_b &= U_{a \wedge b} \\ U_a \cup U_b &= U_{a \vee b} \\ U_0 &= \emptyset \end{aligned}$$

Max(A) is always Hausdorff because if $f, g : A \to \mathbb{R}$ represent different elements in Max(A) they engender different maximal ideals and we may choose $a \in \mathrm{Cone}(A)$ in the kernel of f but not g and choose $b \in \mathrm{Cone}(B)$ in the kernel of g but not f. Then $g \in U_{\lfloor a-b \rfloor}$, $f \in U_{\lfloor b-a \rfloor}$ and $U_{\lfloor a-b \rfloor} \cap U_{\lfloor b-a \rfloor} = U_{\lfloor a-b \rfloor \wedge \lfloor b-a \rfloor} = U_0 = \emptyset$.

If A is finitely generated then Max(A) is compact: first note that a family $\{U_{a_i}\}$ covers all of Max(A) iff the family $\{a_i\}$ is contained in no proper ideal and that is equivalent with $\{a_i\}$ generating the entire ideal, Cone(A); if $x_1, x_2, \ldots, x_n$ generate A define $s = \lfloor x_1 \rfloor + \cdots + \lfloor x_n \rfloor + \lfloor -x_1 \rfloor + \cdots + \lfloor -x_n \rfloor$; note that $f(s) = 0$ iff f is constantly 0; hence a family $\{U_{a_i}\}$ covers all of Max(A) iff s is less then equal to some sum of the a_is and the finite collection of such a_is yields a finite subcover.[92]

[92] Clearly the existence of an element $s \in \mathrm{Cone}(A)$ such that $(f(s) = 0) \Rightarrow (f = 0)$ guarantees that Max(A) is compact. Conversely, consider in any LOAG the cover $\{U_a\}_{a \in \mathrm{Cone}(A)}$. If $U_{a_1}, U_{a_2}, \ldots, U_{a_n}$ is a

$\text{Max}(F_n)$, where F_n is the free LOAG on n generators, is—remarkably enough—homeomorphic to the (n–1)-sphere. First, regard the free abelian group, $\mathbb{Z}^n$ as a subgroup of F_n and note that every map $\mathbb{Z}^n \to \mathbb{R}$ extends uniquely to $F_n \to \mathbb{R}$. The maps $\mathbb{Z}^n \to \mathbb{R}$ are, of course, in natural correspondence with $\mathbb{R}^n$. It will be useful to use inner-product notation: $u \in \mathbb{R}^n$ corresponds to the map that sends $w \in \mathbb{Z}^n$ to $(u, w) \in \mathbb{R}$. We will denote the unique extension of $(u, -) : \mathbb{Z}^n \to \mathbb{R}$ as $f_u : F_n \to \mathbb{R}$. It sends the sequence of generators $\langle x_1, x_2, \ldots, x_n\rangle$ to the vector $\langle u_1, u_2, \ldots, u_n\rangle$. The vectors u and v are in the same ray iff the maps f_u and f_v are. Hence the rays in $\mathbb{R}^n$ are in natural correspondence with the elements in $\text{Max}(F_n)$.

A proof that the standard topology on the set of rays, the one that makes it the (n–1)-sphere, is the same as $\text{Max}(F_n)$ is—finally—remarkably simple. Recall that a continuous bijection between compact Hausdorff spaces is necessarily a homeomorphism, equivalently, given two topologies, $\mathcal{O}_1 \subseteq \mathcal{O}_2$, on the same set where $\mathcal{O}_1$ is Hausdorff and $\mathcal{O}_2$ is compact then necessarily $\mathcal{O}_1 = \mathcal{O}_2$.[93] So take $\mathcal{O}_2$ to be $\text{Max}(F_n)$ and $\mathcal{O}_1$ to be the topology induced by the sub-basis of open sets of the form $U_{\lfloor a\rfloor}$ where a is of the form $m_1x_1 + m_2x_2 \cdots + m_nx_n$. Let u and v be distinct elements in the unit sphere of $\mathbb{R}^n$. To find members of $\mathcal{O}_1$ to serve as disjoint neighborhoods of $f_u, f_v \in \text{Max}(F)_n$ approximate $u - v$ with a rational n-tuple $w = \langle m_1/d, m_2/d, \ldots, m_n/d\rangle$. Then $(u, w) \sim (u, u - v) = 1 - (u, v)$ and $(v, w) \sim (v, u - v) = (u, v) - 1$. If the approximation is close enough we obtain $(u, w) > 0$ and $(v, w) < 0$. Let $s = m_1x_1 + m_2x_2 \cdots + m_nx_n \in F_n$. Then $f_u \in U_{\lfloor s\rfloor}$ and $f_v \in U_{\lfloor -s\rfloor}$. The Us are disjoint because (quite generally) $U_{\lfloor s\rfloor} \cap U_{\lfloor -s\rfloor} = U_{\lfloor s\rfloor\wedge\lfloor -s\rfloor} = U_0 = \emptyset$. Hence $\mathcal{O}_1 = \mathcal{O}_2$. We finish, then, by showing that $\mathcal{O}_1 \subseteq \mathcal{O}'_2$ where $\mathcal{O}'_2$ is the standard topology: it suffices to show that $U_{\lfloor\pm s\rfloor}$ is open in the standard topology; easily enough, $f_u \in U_{\lfloor\pm s\rfloor}$ iff $(u, w) > 0$; hence $\mathcal{O}'_2 = \mathcal{O}_1 = \mathcal{O}_2$.[94]

We will say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is **radial** if $f(tv) = tf(v)$ whenever $t \geq 0$.

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subcover then $U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n} = U_{a_1\vee a_2\cdots\vee a_n} = \text{Cone}(A)$ and we may take $s = a_1 \vee a_2 \cdots \vee a_n$. There are non-finitely generated LOAGs with such an s. An irrelevant example is one with a finitely generated semi-simple reflection ($\text{Max}(A)$ depends only on such). (Let B be any ordered set with a bottom. Consider the set of functions from B to $\mathbb{Z}$ with "finite support." Lexicographically order these functions to obtain a LOAG and take s to be the characteristic function of $-\infty$.) For a relevant example start with the free LOAG on the sequence of generators $s, x_1, x_2, \ldots,$ and reduce by the ideal generated by the sequences $\{\lfloor x_i - s\rfloor\}, \{\lfloor -x_i\rfloor\}$.

[93]Given $x \in U \in \mathcal{O}_2$ we need to find $V \in \mathcal{O}_1$ such that $x \in V \subseteq U$. For each $y \notin U$ choose $V_y, W_y \in O_1$ such that $x \in V_y$, $y \in W_y$ and $V_y \cap W_y = \emptyset$. The W_ys together with U form a cover. Let $U, W_{y_1}, W_{y_2}, \ldots, W_{y_n}$ be a finite subcover. Then $V_{y_1} \cap V_{y_2} \cap \cdots \cap V_{y_n}$ is the $\mathcal{O}_1$-member being sought.

[94]It is not just the topology. To measure the distance between maximal ideals use the probability that the two orderings they induce disagree whether an element of $\mathbb{Z}^n \subset F_n$ is positive. To be precise, for each $k \in \mathbb{N}$ compute the proportion, D_k, of elements $\{\ \langle a_0, a_1, \ldots, a_n\rangle : a_0^2 + a_1 + \cdots + a_n^2 < k^2\ \}$ on which they disagree. The probability of disagreement is $\lim_{k\to\infty} D_k$. Or—to remove any hint of roundness—use the central limit theorem: define a k-walk to be a sequence of k+1 elements in $\mathbb{N}^n$ such that the difference of each element and the next lies in $\{+1, -1\}^n$; let D_k to be the proportion of all k-walks starting at the origin that end on an element on which the two orderings disagree. (If we incorporate the law of large numbers then with probability one we can compute the distance as the limiting frequency of disagreement on an endless random walk and it doesn't matter where we start.) If one insists on radians, multiply the probability of disagreement by π.

27.2. THEOREM. *The free* LOAG *on* n *generators is isomorphic to the* LOAG *of all continuous piecewise linear radial functions from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *where "linear" is understood to mean linear with integer coefficients.*

There is an obvious map from F_n to the LOAG of such functions. It is faithful because of the semisimplicity of F_n; the task is to show that the map is onto. Given such a function, $f : \mathbb{R}^n \to \mathbb{R}$, note first that since $f = \lfloor f \rfloor - \lfloor -f \rfloor$ it suffices to show that any $f \geq 0$ is given by an F_n term. Thus we assume, without lose of generality that $f \geq 0$. We will find it convenient to prove the case, first, for the free lattice-ordered dy-module. Among other things we can then, without loss of generality, assume that $f \leq 1$.

We understand the "standard cube" in $\mathbb{R}^n$ to be the set of vectors, $\langle x_1, x_2, \ldots, x_n \rangle$ such that $-1 \leq x_i \leq +1$ for each $i = 1, 2, \ldots, x_n$. Because of the radial assumption it clearly suffices to construct a term that names a function that agrees with f on the standard cube, indeed, on the surface of the standard cube, that is, those vectors such that for some i it is the case that $x_i = \pm 1$. The surface is the union of $2n$ faces. We will name them with a pair, σ, i where $\sigma \in \{+1, -1\}$ and $i \in \{1, 2, \ldots, n\}$. It suffices to solve the problem for $2n$ functions, $f_{\sigma,i}$, such that

$$\begin{array}{lll} f_{\sigma,i}(x) = f(x) & \text{if} & x_i = \sigma \\ f_{\sigma,i}(x) \leq f(x) & \text{if} & x_i = -\sigma \\ f_{\sigma,i}(x) \leq f(x) & \text{if} & x_j = \pm 1 \;\text{ for }\; j \neq i \end{array}$$

Then necessarily

$$f = \bigvee_{\sigma,i} f_{\sigma,i}$$

So let us fix σ and i. Define

$$g\langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle = f\langle x_1, \ldots, x_{i-1}, \sigma, x_{i+1}, \ldots, x_n \rangle$$

Use the representation theorem for free scales to find a scale-term h that gives the same function as g from $\mathtt{I}^{n-1}$ to $\mathtt{I}$ where $\mathtt{I}$ is understood to be the unit interval. Then translate h into a LOAG term k by replacing each $\bot$ with 0 and each $\top$ with σx_i. Finally define

$$f_{\sigma,i}\langle x_1, \ldots, x_n \rangle = k\langle x_1, \ldots, x_n \rangle - m \sum_{\tau,j} \lfloor \tau x_j - \sigma x_i \rfloor$$

where m is a positive integer to be determined.

For the 1[st] of the three conditions on $f_{\sigma,i}$ above, note that if $x_i = \sigma$ and $-1 \leq x_j \leq +1$ then $\lfloor \tau x_j - \sigma x_i \rfloor = \lfloor \tau x_j - \sigma^2 \rfloor = \lfloor \tau x_j - 1 \rfloor = 0$ for all τ and j. For the 2[nd] condition, if $x_i = -\sigma$ and τ is taken to be $-\sigma$ and $j = i$ then $\lfloor \tau x_j - \sigma x_i \rfloor = \lfloor -2\sigma x_i \rfloor = \lfloor -2\sigma(-\sigma) \rfloor = \lfloor 2 \rfloor = 2$ and $f_{\sigma,i} \leq k\langle x_1, \ldots, x_n \rangle - 2 \leq 0$.

For the 3[rd] condition let $\tau = \pm 1$ and $j \neq i$. For each of the polytopes in the canonical dissection of the $\langle \tau, j \rangle$-face we need that $f_{\sigma,i} \leq f$ on that polytope. Because $f_{\sigma,i}$ and f are linear on that polytope it suffices for the inequality to hold on its extreme points. We thus need the inequality for only a finite number of points on the $\langle\tau, j\rangle$-face. For those points in which the ith coordinate is σ we know that $f_{\sigma,i} = f$. For those points in which the ith coordinate is different from σ we have $\lfloor\tau x_j - \sigma x_i\rfloor = \lfloor\tau(\tau) - \sigma x_i\rfloor = \lfloor 1 - \sigma x_i\rfloor > 0$ (since we are on the $\langle\tau, j\rangle$-face we know that $x_j = \tau$ and since $x_i \neq \sigma$ we know that $\sigma x_i < 1$). Hence $\sum_{\tau,j}\lfloor\tau x_j - \sigma x_i\rfloor$ is positive and we obtain the inequality for all the extreme points if m is chosen large enough.

We repeat his construction for the case of the free $\mathbb{T}$-module where $\mathbb{T}$ is the ring of rational numbers whose denominators are powers of 3 (in the translation into scale-terms use the lattice operations as if they were primitives). Finally, then, given a continuous piecewise integral-linear radial function, f, from $\mathbb{R}^n$ to $\mathbb{R}$ let f_2 be an ordered dy-module term and f_3 an ordered $\mathbb{T}$-module term that describe f. Let r be a positive integer such that $2^r f_2$ and $3^r f_3$ are both integral. Let a and b be integers such that $a2^r - b3^r = 1$. Then $a2^r f_2 - (b3^r f_3)$ describes the same function as $a2^r f - (b3^r f) = (a2^r - b3^r)f = f$

If we move to lattice-ordered dy-modules, we obtain a family of functions uniformly dense in the family of all continuous radial functions. All of this can be, of course, translated to statements not about radial functions on $\mathbb{R}^n$ but continuous functions on S^{n-1}.

The rays in the set of epimorphic maps from F_n to F_m are in natural one-to-one correspondence with the rays of continuous piecewise integer-linear radial maps from $\mathbb{R}^m$ to $\mathbb{R}^n$.(If $A \to B$ is not epimorphic it might induce only a partial map from $\mathrm{Max}(B)$ to $\mathrm{Max}(A)$.) Among the immediate consequences are that there are no epimorphisms from F_n to F_m if $n < m$ and that all epimorphisms from F_n to itself are automorphisms.[95]

One of the biggest differences between the theories of LOAGs and scales is the plenitude of simple algebras. Consider the TOAG of integral polynomials in one variable "ordered at $+\infty$," that is, $f \le g$ iff $f(n) \le g(n)$ for almost all natural numbers n. The only proper ideals are of the form $\{\ f\ :\ \mathrm{degree}(f) < d\ \}$, one for each natural number d. None is maximal. There is no simple quotient.

Another huge difference is that $\mathrm{Max}(-)$ is not a functor into the category of spaces; a LOAG map $A \to B$ does not induce a function $\mathrm{Max}(B) \to \mathrm{Max}(A)$, only a partial function (with open support).[96]

28. Appendix: Computational Complexity Issues

Finding which equations can be counterexampled in the theory of either scales or LOAGs is NP-complete. There is no substantive difference between the equational and universal

[95] There are a number of ways of defining whether an automorphism of a free LOAG is orientation-preserving. Not the easiest to prove, but perhaps the nicest, is to use the fact that the abelian reflection of the group of all automorphisms of F_n has just two elements. We will call its kernel—the commutator subgroup—the "special automorphism group" (it's simple, by the way). Given epimorphisms from F_{n+k} to F_n, with $k \neq 2$, there is always an automorphism on F_{n+k} that converts one to the other. When $k > 2$ there is always a special automorphism that will do it. When $k = 2$ it is more complicated. The classification of the orbits on the set of epis from F_{n+3} to F_{n+1} under the action of the special automorphism group is usually called the theory of knotted n-spheres in $(n+2)$-space.

[96] We could obtain a functor by replacing $\mathrm{Max}(-)$ with its scone.

Horn theory problems. Indeed, if we change scales to *linear* scales and LOAGs to TOAGs we obtain the result for the full first-order universal theory.

First, we observe that the "satisfaction" problem for Boolean algebras can be easily converted to a satisfaction problem of roughly the same size in scales. The proof would be straightforward if it were not the case that one of the variables in the formula $x \vee y = x \overset{\vee}{|} \widehat{\dot{x}\,|y}$ appears twice. An uncaring use of this formula would lead to an exponential growth in the length of the translation. Let $\mathcal{B}$ denote the Boolean algebra of extreme points in a scale. We avoid the problem by using:

28.1. LEMMA. *If $x, y \in \mathcal{B}$ then $x \wedge y = x \overset{\wedge}{|} y$ and $x \vee y = x \overset{\vee}{|} y$.*

(Check in any linear scale.)

Using these translations (plus the translation of negation into dotting) any Boolean term may be converted to a scale term of the same size. We are not done until we restrict the variables to $\mathcal{B}$. Given a scale term A on variables $v_1, v_2, \ldots, v_n$ any solution of the equation:

$$(v_1 \multimap \hat{v}_1)|(v_2 \multimap \hat{v}_2)|\cdots|(v_n \multimap \hat{v}_n)|A = \top$$

(associate at will) is a solution of $A = \top$ that necessarily lies entirely in $\mathcal{B}$.

The complementary problem is also convertible. Every equation in the theory of Boolean algebras is equivalent to an equation—of manageable size—in the theory of scales. (As in the satisfaction problem the proof would be straightforward if it were not the case that one of the variables in the formula for $x \vee y$ appears twice.)

For the conversion we will understand that $x \vee y$ is the term $x \overset{\vee}{|} \widehat{\dot{x}\,|y}$, the one in which y appears once and, dually, $x \wedge y$ is the term $x \overset{\wedge}{|} (\dot{x} \overset{\vee}{|} y)$, again the one in which y appears once. We will need a term $M\langle v_1, v_2, \ldots, v_n\rangle$ defined recursively by:

$$(n = 0) \Rightarrow (M = \top)$$

$$M\langle v_1, v_2, \ldots, v_n\rangle = (v_n \vee \dot{v}_n) \wedge M\langle v_1, v_2, \ldots, v_{n-1}\rangle$$

Given terms A and B in the signature of Boolean algebras we produce terms $\langle\!\langle A\rangle\!\rangle$ and $\langle\!\langle B\rangle\!\rangle$ such that $A = B$ is true for Boolean algebras iff $\langle\!\langle A\rangle\!\rangle = \langle\!\langle B\rangle\!\rangle$ is true for scales. The length of $\langle\!\langle A\rangle\!\rangle$ will be bound by a constant multiple of the length of A times the number of variables in A and B.

The conversion is defined recursively by the following rules in which $v_1, v_2, \ldots, v_n$ are the variables and M denotes $M\langle v_1, v_2, \ldots, v_n\rangle$.

$$\begin{aligned}
\langle\!\langle \top \rangle\!\rangle &= M \\
\langle\!\langle \bot \rangle\!\rangle &= \dot{M} \\
\langle\!\langle v_i \rangle\!\rangle &= \dot{M} \vee (v_i \wedge M) \\
\langle\!\langle \neg A \rangle\!\rangle &= \langle\!\langle A \rangle\!\rangle^{\bullet} \\
\langle\!\langle A \vee B \rangle\!\rangle &= M \wedge [\dot{M} \triangleleft (\, \langle\!\langle A \rangle\!\rangle \mid \langle\!\langle B \rangle\!\rangle\,)] \\
\langle\!\langle A \wedge B \rangle\!\rangle &= \dot{M} \vee [M \triangleleft (\, \langle\!\langle A \rangle\!\rangle \mid \langle\!\langle B \rangle\!\rangle\,)]
\end{aligned}$$

Note that in a linear scale if any one of the variables is instantiated as $\odot$ then $M = \odot$, otherwise M and $\dot{M}$ are distinct. In either case, an inductive argument shows that $\langle\!\langle A \rangle\!\rangle$ is either M or $\dot{M}$ for every term A. Moreover $\langle\!\langle A \diamond B \rangle\!\rangle = \langle\!\langle A \rangle\!\rangle \diamond \langle\!\langle B \rangle\!\rangle$ where $\diamond$ is either lattice operation.

We may redo this construction for LOAGs instead of scales.

Define M by

$$(n = 0) \Rightarrow (M = \top)$$

$$M\langle v_1, v_2, \ldots, v_n\rangle = (\lfloor v_n \rfloor \vee \lfloor -v_n \rfloor) \vee M\langle v_1, v_2, \ldots, v_{n-1}\rangle$$

and:

$$\begin{aligned}
\langle\!\langle \top \rangle\!\rangle &= M \\
\langle\!\langle \bot \rangle\!\rangle &= -M \\
\langle\!\langle v_i \rangle\!\rangle &= (-M) \vee (v_i \wedge M) \\
\langle\!\langle \neg A \rangle\!\rangle &= -\langle\!\langle A \rangle\!\rangle \\
\langle\!\langle A \vee B \rangle\!\rangle &= M \wedge (\,\langle\!\langle A \rangle\!\rangle + \langle\!\langle B \rangle\!\rangle + M\,) \\
\langle\!\langle A \wedge B \rangle\!\rangle &= (-M) \vee (\,\langle\!\langle A \rangle\!\rangle + \langle\!\langle B \rangle\!\rangle - M\,)
\end{aligned}$$

29. Appendix: Independence

The independence of all but the first two scale axioms is easy:

For the independence of the medial axiom, consider the set $\{-1, 0, +1\}$ with $x|y$ defined as "truncated addition," that is

$$x|y = -1 \vee (x + y) \wedge 1$$

We take $\dot{x} = -x$, $\hat{x} = x$ and $\top = 0$. All defining laws of scales hold except for the medial law $((+1|0) \bigm| (+1|-1) \neq (+1|+1) \bigm| (0|-1))$.

For the independence of the unital and constant laws consider the set $\{0, 1\}$ with ordinary multiplication for $x|y$ and the identity function for $\hat{x}$. If we take $\dot{x} = 1 - x$ and $\top = 1$ then every equation is satisfied except for $\bot \overset{\vee}{|} x = x$. If, instead, we take $\top = 0$ then every equation is satisfied except for $\top \overset{\wedge}{|} x = x$. If we take $\dot{x} = x$ and $\top = 1$ then every equation is satisfied except for the constancy of $\dot{x}\,|x$.

For the independence of the scale identity consider $\mathtt{I} \times \mathtt{I}$ with the standard product-algebra structure except for $\top$-zooming. The unital laws determine $\widehat{\langle x, y\rangle}$ only when x and y are both nonnegative. We maintain all the laws except for the scale identity, therefore, if $\top$-zooming is standard on just that top quadrant.

As promised in a Section-3 footnote, we can do better. The absorbing laws determine $\widehat{\langle x, y\rangle}$ only when x and y are both non-positive. We can maintain the minor-scale equations, therefore, by keeping the standard definition of $\widehat{\langle x, y\rangle}$ just on the top and bottom quadrants, (that is, the pairs $\langle x, y\rangle$ such that $xy \geq 0$).

The first three uses of the scale identity were for $\overset{\vee}{\top} = \top$, $x = \overset{\vee}{x} \mid \hat{x}$ and $\widehat{x|\odot} = \hat{x} \mid\bot$ (the absorbing law, $\widehat{\bot|x} = \bot$, is a consequence of these). We may maintain the law of compensation by stipulating $\widehat{\langle x, y\rangle} = \langle x, y\rangle$ for $xy < 0$. Central distributivity requires a recursive definition. Given $\langle x, y\rangle$ such that $xy < 0$ let n be the largest integer such that there exist u, v with $\langle x, y\rangle = (\odot|)^n\langle u, v\rangle$. If $n = 0$ then define $\widehat{\langle x, y\rangle} = \langle x, y\rangle$. For $n > 0$ recursively define $\widehat{\langle x, y\rangle} = ((\odot|)^{n-1}\widehat{\langle u, v\rangle})|\langle\bot, \bot\rangle$. The scale identity itself fails (most easily seen by noting that $\top$-zooming no longer preserves order).

Also easy is the independence of the axioms for chromatic scales: if the support operation is constantly $\top$ then only the 1st equation, $\overline{\bot} = \bot$, fails; if it is the identity function then only the 2nd equation, $\overset{\wedge}{\overline{x}} = \overline{x}$, fails; if it is constantly $\bot$ then only the 3rd equation, $x \wedge \overline{x} = x$, fails; if $\overline{x} = \bot$ when $x = \bot$ else $\overline{x} = \top$ then the 4th equation, $\overline{x \wedge y} = \overline{x} \wedge \overline{y}$, fails when the scale is non-linear but only it fails.[97] As promised, we can eliminate the 2nd equation by strengthening the 3rd equation to $x \wedge \dot{\overline{x}} = \bot$. Show first that $\dot{\overline{x}}$ is the Heyting negation, that is, $y \leq \dot{\overline{x}}$ iff $y \wedge x = \bot$: if $y \leq \dot{\overline{x}}$ then $y \wedge x \leq \dot{\overline{x}} \wedge x = \bot$; if $y \wedge x = \bot$ then $y \vee \dot{\overline{x}} = (y \vee \dot{\overline{x}}) \wedge \top = (y \vee \dot{\overline{x}}) \wedge \dot{\overline{\bot}} = (y \vee \dot{\overline{x}}) \wedge \dot{\overline{y \wedge x}} = (y \vee \dot{\overline{x}}) \wedge (\dot{\overline{y}} \vee \dot{\overline{x}}) = (y \wedge \dot{\overline{y}}) \vee \dot{\overline{x}} = \bot \vee \dot{\overline{x}} = \dot{\overline{x}}$. This implies, in particular, that $\dot{\overline{x}}$ is an extreme point because $x \wedge \overset{\vee}{\dot{\overline{x}}} \leq \overset{\vee}{x} \wedge \overset{\vee}{\dot{\overline{x}}} = (x \wedge \dot{\overline{x}})^{\vee} = \overset{\vee}{\bot} = \bot$ hence $\dot{\overline{x}} \leq \overset{\vee}{\dot{\overline{x}}}$ and, consequently, $\dot{\overline{x}} = \overset{\vee}{\dot{\overline{x}}}$. Since $\dot{\overline{x}}$ is an extreme point, so is $\overline{x}$. We obtain the original 3rd equation by $x \wedge \overline{x} = (x \wedge \overline{x}) \vee (\dot{\overline{x}} \wedge \overline{x}) = (x \wedge \dot{\overline{x}}) \vee \overline{x} = \bot \vee \overline{x} = \overline{x}$.

We have not yet established the independence of the idempotence and commutative laws. We already proved in a Section-2 footnote that the commutative law may be replaced with the single instance $\bot|\top = \top|\bot$ and we promised in an earlier footnote that we could remove the commutative law entirely by replacing the first unital law with $\overset{\wedge}{\dot{\bot} | x} = x$. To do so, first establish the left cancellation law using a different construction for dilatation, $(((\dot{a} \,|\, \dot{\bot})|x)^{\wedge})^{\vee}$. Then $(((\dot{a} \,|\, \dot{\bot}) \;\big|\; (a|x))^{\wedge})^{\vee} = (((\dot{a} \,|a) \;\big|\; (\dot{\bot} \,|x))^{\wedge})^{\vee} = (((\dot{\bot} \,|\bot) \;\big|\; (\dot{\bot} \,|x))^{\wedge})^{\vee} = (((\dot{\bot} \;\big|\; (\bot|x))^{\wedge})^{\vee} = (\bot|x)^{\wedge} = x$. If $a|x = a|y$ then the first term in this sequence of five equations remains the same and we may conclude $x = y$. Just as in the derivation of full commutativity from the commutativity of $\top$ and $\bot$ we obtain dot-distributivity. Then obtain the involutory law from the second unital law written in full: $x = \overset{\vee}{\bot|x} = (((\bot|x)^{\bullet})^{\wedge})^{\bullet} = ((\dot{\bot} \,|\, \dot{x})^{\wedge})^{\bullet} = (\dot{x})^{\bullet}$. (Note in passing that we now have the original first unital law.) Finish as before by first showing: that $\dot{x} = x$ implies the centrality of the center; that $\odot \;\big|\; (x|y) = \odot \;\big|\; (y|x)$; and, finally, that $x|y = y|x$.

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[97]Similarly, the 1st, 3rd and 4th for support operations on rings are independent: if the support operation is constantly 1 then only $\overline{0} = 0$ fails; if it is constantly 0 then only $x\overline{x} = x$ fails; if $\overline{x} = 0$ when $x = 0$ else $\overline{x} = 1$ then $\overline{xy} = \overline{x}\;\overline{y}$ fails when the ring is not a domain but only it fails. (For the redundancy of the 2nd equation note first that it was not used to show that $x^2 = 0$ implies $x = 0$, hence $(1 - \overline{x})x$ is 0 since its square is 0; finish with $(1 - \overline{x})\overline{x} = (1 - \overline{x})\overline{(1 - \overline{x})x} = (1 - \overline{x})\overline{(1 - \overline{x})x} = (1 - \overline{x})\overline{0} = 0$.)

30. Appendix: $\mathtt{I}$ as the Final Interval Coalgebra

In Section 1 there appeared a quick and dirty procedure for computing the binary expansion of $f(x)$ where f is the unique interval coalgebra-map from a given interval X to the unit interval, to wit:

`If` $\overset{\vee}{x} = \top$ `then`

  `emit 1;`

  `replace` x `with` $\hat{x}$;

  `return.`

`else`

  `emit 0;`

  `replace` x `with` $\overset{\vee}{x}$;

  `return.`

A numerical analyst will object to the very first line: how does one determine when an equality holds? There may be procedures that are guaranteed to detect when things are not equal (assuming, of course, that they are, indeed, not equal) but in analysis there tend not to be definitive proofs of equality.[98]

Before considering computationally more realistic settings let us prove (in the classical setting) that the unit interval is the final interval coalgebra. Using binary expansions the interval coalgebra on $[0, 1]$ is described with an automaton with three states L, U, and initial state, I. It takes $\{0,1\}$-streams as input and produces $\{0,1\}$-streams as output:

Next State

	L	I	U
0	L	L	U
1	L	U	U

$\vee$-Output

	L	I	U
0	0		1
1	1		1

$\wedge$-Output

	L	I	U
0	0		0
1	0		1

The blanks in the output tables will be called **stammers**. The output streams will always be one digit behind the number of input digits.

We need a (dual) pair of definitions: define $\top \ll \boldsymbol{x}$ if a finite iteration of $\bot$-zooming carries x to $\top$, and $\boldsymbol{x} \ll \bot$ if a finite iteration of $\top$-zooming carries it to $\bot$. In $\mathtt{I}$ these are unneeded properties: $\ll$ coincides with $<$. [99] For any $\mathtt{I}$-valued coalgebra map, f, they tell us how $f(x)$ is situated with respect to the center, $\odot$:

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[98] For just one example, suppose X is, itself, the unit interval but that we know an element x only by listening to its binary expansion. If that expansion happens to be 0 followed by all 1s we will never have enough information to conclude that $\overset{\vee}{x} = \top$. But this is not the most telling example (we know in this case that $\hat{x} = \bot$). Suppose that we know x only as the midpoint of a pair of binary expansions. If one sequence is the expansion of an arbitrary element y in the open unit interval and the other is the expansion of $1 - y$ then we will never have enough information to know either $\overset{\vee}{x} = \top$ or $\hat{x} = \bot$).

[99] In any non–semi-simple scale $<$ is weaker than $\ll$.

$$\begin{array}{lll} \odot < f(x) & \text{iff} & \bot << \hat{x} \\ f(x) = \odot & \text{iff} & \text{neither } \bot << \hat{x} \text{ nor } \overset{\vee}{x} << \top \\ f(x) < \odot & \text{iff} & \overset{\vee}{x} << \top \end{array}$$

Note that $\bot << \hat{x}$ implies $\overset{\vee}{x} = \top$ (since $\hat{x} \neq \bot$) hence in the 1st case the procedure produces a stream of binary digits starting with 1 followed by the stream for $\hat{x}$ which is precisely what is demanded by $f(x) = (fx)^{\vee}|(fx)^{\wedge} = \top|f(\hat{x})$. The 3rd case is dual. In the 2nd case if $\overset{\vee}{x} = \top$ the procedure will produce a 1 followed by all 0s and if $\overset{\vee}{x} \neq \top$ a 0 followed by all 1s. That it produces one of these two streams (and it doesn't matter which) is just what is demanded by $f(x) = \odot$. [100]

For a computationally more realistic setting we are handed a guide to the needed modifications. Experience tells us that when working in $\boldsymbol{Sh(X)}$, the category of sheaves on a space X, it is clear that, the real numbers are best taken as the sheaf of continuous $\mathbb{R}$-valued functions on X, that is, the sheaf whose stalks are germs of continuous functions (as defined in the topos of sets) from X to $\mathbb{R}$. That experience leads us to view the sheaf of continuous $\mathtt{I}$-valued functions as the best candidate for the closed interval. We note immediately that the disjunctive coalgebra condition fails. Given continuous $g : X \to \mathtt{I}$ the "truth value" of the equation $\overset{\vee}{g} = \top$ (necessarily an open subset of X) is the interior of $g^{-1}[\odot, \top]$ and the value of $\hat{g} = \bot$ is the interior of $g^{-1}[\bot, \odot]$. Their union is not, in general, all of X. But it is a dense subset. Hence we replace the **discrete coalgebra condition**:

$$\bot = \hat{x} \quad \text{or} \quad \overset{\vee}{x} = \top$$

with the weaker **continuous coalgebra condition**:

$$\neg\neg[\bot = \hat{x} \quad \text{or} \quad \overset{\vee}{x} = \top]$$

We will usually use this in the intuitionistically equivalent form:

$$\neg[\bot \neq \hat{x} \quad \text{and} \quad \overset{\vee}{x} \neq \top]$$ [101] [102]

(We will understand that both the discrete and continuous coalgebra conditions entail that $\top$ and $\bot$ are zooming fixed points.)

[100] This argument works even in the intuitionistic setting if we hold on to the computationally unrealistic coalgebra condition $\overset{\vee}{x} = \top$ or $\hat{x} = \bot$ (as in the Cantor—rather than Dedekind—closed interval).

[101] Informally: one will never detect both $\overset{\vee}{x} \neq \top$ and $\hat{x} \neq \bot$.

[102] In the Section-1 footnote on the modal operators $\Diamond$ and $\Box$ it was the continuous coalgebra condition we invoked when we wrote "No one allows simultaneously both $\Diamond\Phi \neq \top$ and $\Box\Phi \neq \bot$ (less than possible but somewhat necessary)." The discrete condition would have been the stronger "All insist that $\Diamond\Phi = \top$ or $\Box\Phi = \bot$ (either totally possible or completely unnecessary)." It's worth checking the other mentioned interpretations: "Nothing is less than tenable but somewhat certain," as opposed to "Everything is either totally tenable or completely uncertain"; "Nothing is less than conceivable but somewhat known," as opposed to "Everything is totally conceivable or completely unknown."

We must, however, capture the detectability of inequality. Hence we replace the **apartness condition**:

$$\bot \neq \top$$

with this stronger **separation condition**:

$$\bot \neq x \text{ or } x \neq \top \text{ [103]}$$

Keeping in mind that the truth values in $Sh(X)$ are open subsets of X, the extent to which any characteristic map, χ_A, is different from $\bot$ is the interior of A, the extent to which it is different from $\top$ is the "exterior" of A (the interior of its complement), hence $\bot \neq \chi_A \;\vee\; \chi_A \neq \top$ holds iff A is both open and closed.[104]

The **continuously ordered wedge** of X and Y is $\{\ \langle x, y\rangle : \neg[x \neq \top \;\wedge\; y \neq \bot]\ \}$. Its top is $\langle \top, \top\rangle$, its bottom $\langle \bot, \bot\rangle$. If either X or Y satisfies the separation condition, so does their ordered wedge. A continuous coalgebra structure on X is a map from X to the continuously ordered wedge of X with itself.[105]

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[103] In the presence of De Morgan's law the two conditions are, of course, equivalent. In a topos the top and bottom of the subobject classifier Ω are always apart but their separation implies that De Morgan holds throughout the topos.

[104] One may show that if $f :\to \mathtt{I}$ and all of its descendants under arbitrary application of the zooming functions continue to satisfy the separation condition, then necessarily f is continuous. Put another way, the sheaf of germs of continuous $\mathtt{I}$-valued functions is the largest separated subobject invariant under the zooming operation inside the sheaf of germs of all functions from X to $\mathtt{I}$. Put still another way, if f is not continuous at x, then there is a zooming sequence α such that it is not the case that $\bot \neq (f(x))^\alpha$ or $(f(x))^\alpha \neq \top$. To find α we start with the fact that there are values of f on arbitrarily small neighborhoods of x that are bounded away from $f(x)$. We may assume without loss of generality that those values are below $f(x)$. Let $\ell \in \mathbb{I}$ be such that $\ell < f(x)$ and for all neighborhoods of x there are values of f below ℓ. Let $u \in \mathbb{I}$ be such that $\ell < u < f(x)$. Let α be such that $\ell^\alpha = \bot$ and $u^\alpha = \top$. Then for any open $U \subseteq X$ such that $f \neq \top$ on U it must be the case that $x \notin U$. And for any open $V \subseteq X$ such that $f \neq \bot$ it must be the case, again, that $x \notin V$. That is $[\bot \neq (f(x))^\alpha] \;\vee\; [(f(x))^\alpha \neq \top]$ fails.

[105] In opening section footnote treatment of limits we can replace the scale-algebras with interval coalgebras by adopting these intuitionistic modifications of the coalgebra definition to the classical setting. Given a set X with constants $\top$ and $\bot$ and unary operations whose value are denoted $\hat{x}$ and $\overset{\vee}{x}$ impose the conditions:

$$\begin{array}{ccc} \neg[\bot << \hat{x} & \text{and} & \overset{\vee}{x} << \top] \\ \bot << x & \text{or} & x << \top \end{array}$$

The set, A, of sequences, $\mathtt{I}^{\mathbb{N}} = \prod_{\mathbb{N}} \mathtt{I}$, reduced by almost-everywhere equality does not satisfy either condition (consider, for example a sequence that is equal infinitely often to $\top$ and to $\bot$). But there is a largest subset that does, to wit, the set of sequences s such that for all zooming sequences α it is the case that

$$\begin{array}{ccc} \neg[\bot << s^{\alpha\wedge} & \text{and} & s^{\alpha\vee} << \top] \\ \bot << s^\alpha & \text{or} & s^\alpha << \top \end{array}$$

The resulting set is precisely the set of convergent sequences. If we now collapse to a point the set of its members such that $\neg(s << \top)$ and dually for $\neg(\bot << s)$ we obtain an interval coalgebra. Its unique coalgebra map to $\mathtt{I}$—as one must now have surely learned to expect—is *Lim*. And the same modifications work for defining limits of functions at a point in a space and for defining derivatives.

31. Appendix: Signed Binary Expansions

On July 31, 2000, I posted a note on the category net on how to obtain *signed* binary expansions for elements from such coalgebras. Five days later Peter Johnstone posted a note pointing out that I was implicitly using the axiom of dependent choice and that a Dedekind-cut construction is necessary in the general case. (The disjunction property on truth values also suffices.) Because so many "infinite-precision" programmers are quite happy (wittingly or not) with dependent choice let me first briefly describe that approach.

Every element of the standard interval has a representation of the form

$$\sum_{n=1}^{\infty} a_n 2^{-n}$$

where $a_n \in \{-1, 0, +1\}$. The sequence of a_ns is, of course, not unique: everything other than the two endpoints has infinitely many expansions—indeed, everything not a dyadic rational has uncountably many.

Given an object C with separated elements $\bot$ and $\top$ and self-maps whose values are denoted $\overset{\vee}{x}, \hat{x}$ satisfying the continuous coalgebra condition we seek a procedure that delivers for each $c \in C$ a signed binary expansion. Starting with the initial pair $\langle x, y\rangle = \langle \overset{\vee}{c}, \hat{c}\rangle$ iterate (forever) the non-deterministic parallel procedure:

If $x \neq \top$ `then emit -1;`
  `replace` $\langle x, y\rangle$ `with` $\langle \overset{\vee}{x}, \hat{x}\rangle$.
If $\hat{x} \neq \bot$ `and` $\overset{\vee}{y} \neq \top$ `then emit 0;`
  `replace` $\langle x, y\rangle$ `with` $\langle \hat{x}, \overset{\vee}{y}\rangle$.
If $y \neq \bot$ `then emit +1;`
  `replace` $\langle x, y\rangle$ `with` $\langle \overset{\vee}{y}, \hat{y}\rangle$.

It is to be understood that there are three simultaneous attempts to detect $[x \neq \top]$ or $[(\hat{x} \neq \bot) \wedge (\overset{\vee}{y} \neq \top)]$ or $[y \neq \bot]$. (We will see, assuming that $\neq$s can be detected, at least one of these attempts will succeed.

In this procedure we are working actually in $C \vee C$. For any endofunctor T and coalgebra $C \to TC$ we obtain another coalgebra $TC \to T^2C$ just by applying T. The given map from C to TC is automatically a coalgebra homomorphism. The coalgebra structure on $C \vee C$ is therefore defined by the two maps on pairs $\overset{\vee}{\langle x, y\rangle} = \langle \overset{\vee}{x}, \hat{x}\rangle$ and $\overset{\wedge}{\langle x, y\rangle} = \langle \overset{\vee}{y}, \hat{y}\rangle$. What we gain is another zoom, to wit, the **mid-zoom** defined by $\overline{\boldsymbol{\langle x, y\rangle}} = \langle \hat{x}, \overset{\vee}{y}\rangle$. (If we could interpret $C \vee C$ as a scale algebra then mid-zooming would be the same as the central dilatation.) We are obliged to check that $\overline{\langle x, y\rangle}$ lies in the ordered wedge. In the discrete version that is simply the assertion $[x = \top \;\vee\; y = \bot] \;\Rightarrow\; [\hat{x} = \top \;\vee\; \overset{\vee}{y} = \bot]$ which is, of course, an immediate consequence of the fact that $\top$ and $\bot$ are zoom-invariants. The continuous version is barely different: one needs only observe that double-negation is a logically covariant operation to obtain $\neg\neg[x = \top \;\vee\; y = \bot] \;\Rightarrow\; \neg\neg[\hat{x} = \top \;\vee\; \overset{\vee}{y} = \bot]$.

The procedure may be rewritten:

`If` $x \neq \top$ `then emit -1;`
 `replace` $\langle x, y\rangle$ `with` $\overset{\vee}{\langle x, y\rangle}$.
`If` $\hat{x} \neq \bot$ `and` $\overset{\vee}{y} \neq \top$ `then emit 0;`
 `replace` $\langle x, y\rangle$ `with` $\overline{\langle x, y\rangle}$.
`If` $y \neq \bot$ `then emit +1;`
 `replace` $\langle x, y\rangle$ `with` $\overset{\wedge}{\langle x, y\rangle}$.

The three branching alternatives correspond to whether the element $\langle x, y\rangle \in C \vee C$ lies in (what we will call) the

bottom open half if $x \neq \top$;
middle open half if $\hat{x} \neq \bot$ and $\overset{\vee}{y} \neq \top$;
top open half if $y \neq \bot$.

(Note that the continuous coalgebra condition is precisely the condition that no element lies in both the top and bottom open half.) The procedure emits the appropriate signed binary digit to record which of these open halves and the appropriate zoom operation is then applied to continue the resolution. The fact that at least one of the three branching conditions holds is the fact that the three open halves jointly cover—it is right here that the strengthening of the apartness condition to the separation condition is invoked; it yields the two facts

$$\hat{x} \neq \bot \;\vee\; \hat{x} \neq \top$$
$$\overset{\vee}{y} \neq \bot \;\vee\; \overset{\vee}{y} \neq \top$$

Their conjunction distributes itself as the disjunction of the four terms:

$$\hat{x} \neq \bot \;\wedge\; \overset{\vee}{y} \neq \bot$$
$$\hat{x} \neq \bot \;\wedge\; \overset{\vee}{y} \neq \top$$
$$\hat{x} \neq \top \;\wedge\; \overset{\vee}{y} \neq \top$$
$$\hat{x} \neq \top \;\wedge\; \overset{\vee}{y} \neq \bot$$

Using that negation is logically contravariant and, once again, that $\top$ and $\bot$ are zoom invariants we have that the 1^{st} alternative implies $y \neq \bot$, that is, $\langle x, y\rangle$ is in the top open half; the 2^{nd} alternative is the definition of the middle open half; the 3^{rd} alternative implies $\langle x, y\rangle$ is in the bottom open half. The 4^{th} alternative is vacuous (the continuous coalgebra condition is its negation).

There are a number of things to be verified but they turn out to be most easily accomplished as a corollary of the more general Dedekind-cut approach developed in the next appendix. But first let us apply the signed binary digit approach inside $Sh(X)$ for arbitrary X (dependent choice holds only when X is totally disconnected). Given a continuous coalgebra C we seek continuous $f : X \to \mathtt{I}$, in particular, given a partial section $U \to C$ and $x \in U$ we seek an element of $\mathtt{I}$. Having fixed x we are prompted to pass to the category of "micro-sheaves" at x, to wit, the result of identifying things when they agree when restricted to some neighborhood of x. In that category we are entitled to view the partial section as global, a map from 1 to C. What we gain is the disjunction property: if the disjunction of two (or, even better, three) sentences is true then one of them is already true. We may now repeat the above procedure to obtain an unending stream of signed binary digits. It is the $\mathtt{I}$-element being sought. Continuity is left as an exercise.

Before embarking on the Dedekind approach we will take this occasion to describe the scale-algebra structure in the signed binary digit setting. A signed binary digit expansion may be viewed as an encoding of a scale-term of the form "$t_1|(t_2|(t_3|(\cdots$" where the t_is are from the set $\{\bot, \odot, \top\}$. A three-state automaton is easily constructible for computing zooming. We will denote the states as L, M (the initial state) and U. We will denote the signed binary digits as $-, 0, +$ (suppressing the 1s):

Next State

	L	M	U
−	L	L	U
0	L	M	U
+	L	U	U

∨-Output

	L	M	U
−	−		+
0	0	+	+
+	+	+	+

∧-Output

	L	M	U
−	−	−	−
0	−	−	0
+	−		+

In these machines at most one stammer occurs; restated, the output is never more than one digit behind the input.[106]

It's worth noting that no automaton, finite or not, can compute midpoints (or, as usually pointed out, sums) in the context of streams of standard (unsigned) binary digits. If at any point only heterogenous pairs of standard digits (one 0, one 1) have been heard then we do not know whether the result will be in the upper or lower open half and if we are restricted to the digits 0 and 1 we can not specify its first digit. If, perchance, all digit-pairs are heterogenous we will never be able to compute the first digit.

[106] There are four-state automata with strict initial states, I, that always stammer at the first input digit and never thereafter:

Next State

	I	L	M	U
−	L	L	L	U
0	M	L	M	U
+	U	L	U	U

∨-Output

	I	L	M	U
−		−	+	+
0		0	+	+
+		+	+	+

∧-Output

	I	L	M	U
−		−	−	−
0		−	−	0
+		−	−	+

There is a remarkably simple automaton, on the other hand, for midpointing in the context of streams of signed binary digits. Returning to the scale-algebra notation, given another such term in the form "$u_1|(u_2|(u_3|(\cdots$" its midpoint with "$t_1|(t_2|(t_3|(\cdots$" can be expressed in the form "$v_1|(v_2|(v_3|(\cdots$" where $v_i = t_i|u_i$. We can easily construct an instantaneous automaton that produces a stream of symbols from the set $\{\bot, \mathrm{Q}_1, \odot, \mathrm{Q}_3, \top\}$ where the "quarter symbols" are defined by $\mathrm{Q}_1 = \bot|\odot$ and $\mathrm{Q}_3 = \odot|\top$. The problem, then, is to remove the quarter symbols.

Consider the following computations:

$$\begin{array}{rcl}
\mathrm{Q}_1|(\bot|x) & = & \bot|(\odot|x) \\
\mathrm{Q}_1|(\mathrm{Q}_1|x) & = & \bot|(\mathrm{Q}_3|x) \\
\mathrm{Q}_1|(\odot|x) & = & \odot|(\bot|x) = \bot|(\top|x) \\
\mathrm{Q}_1|(\mathrm{Q}_3|x) & = & \odot|(\mathrm{Q}_1|x) \\
\mathrm{Q}_1|(\top|x) & = & \odot|(\odot|x) \\
\mathrm{Q}_3|(\bot|x) & = & \odot|(\odot|x) \\
\mathrm{Q}_3|(\mathrm{Q}_1|x) & = & \odot|(\mathrm{Q}_3|x) \\
\mathrm{Q}_3|(\odot|x) & = & \odot|(\top|x) = \top|(\bot|x) \\
\mathrm{Q}_3|(\mathrm{Q}_3|x) & = & \top|(\mathrm{Q}_1|x) \\
\mathrm{Q}_3|(\top|x) & = & \top|(\odot|x)
\end{array}$$

The significance of these computations[107] is that each term may be replaced with one whose left-most symbol is not a quarter symbol and that allows us to build an automaton that systematically removes all the quarter symbols from the output streams. A remarkably simple three-state finite automaton results for midpointing signed binary expansions. The nine pairs of digits behaviourly group themselves into five classes defined by the sum of the two digits, hence the inputs will be denoted with $-2, -1, 0, +1+2$. The states will be $\mathsf{S}_-, \mathsf{S}_0$ (the initial state) and S_+. Whenever we leave S_0 a stammer will occur. Whenever we return to S_0 a **stutter** will occur, to wit, two output digits. The machine is never more than one output digit behind the number of input pairs (that is, between every pair of stammers there is a stutter):

[107]For the first five rows (the last five are, of course, dual):

$$\mathrm{Q}_1\Big|(\bot|x) = (\bot|\odot)\Big|(\bot|x) = \bot\Big|(\odot|x)$$

$$\mathrm{Q}_1\Big|(\mathrm{Q}_1|x) = (\odot|\bot)\Big|(\mathrm{Q}_1|x) = (\odot|\mathrm{Q}_1)\Big|(\bot|x) = ((\bot|\top)\Big|(\bot|\odot))\ \Big|\ (\bot|x) = (\bot\Big|(\top|\odot))\ \Big|\ (\bot|x) = (\bot|\mathrm{Q}_3)\Big|(\bot|x) = \bot\Big|(\mathrm{Q}_3|x)$$

$$\mathrm{Q}_1\Big|(\odot|x) = (\odot|\bot)\Big|(\odot|x) = \odot\Big|(\bot|x) = (\bot|\top)\Big|(\bot|x) = \bot\Big|(\top|x)$$

$$\mathrm{Q}_1\Big|(\mathrm{Q}_3|x) = (\odot|\bot)\Big|(\mathrm{Q}_3|x) = (\odot|\mathrm{Q}_3)\Big|(\bot|x) = ((\top|\bot)\Big|(\top|\odot))\ \Big|\ (\bot|x) = (\top\Big|(\bot|\odot))\ \Big|\ (\bot|x) = (\top|\mathrm{Q}_1)\Big|(\bot|x) = (\top|\bot)\Big|(\mathrm{Q}_1|x) = \odot\Big|(\mathrm{Q}_1|x)$$

$$\mathrm{Q}_1\Big|(\top|x) = (\bot|\odot)\Big|(\top|x) = (\bot|\top)\Big|(\odot|x) = \odot\Big|(\odot|x)$$

Next State

	S_-	S_0	S_+
-2	S_0	S_0	S_0
-1	S_+	S_-	S_+
0	S_0	S_0	S_0
$+1$	S_-	S_+	S_-
$+2$	S_0	S_0	S_0

Output

	S_-	S_0	S_+
-2	-0	$-$	00
-1	$-$		0
0	$-+$	0	$+-$
$+1$	0		$+$
$+2$	00	$+$	$+0$

The states may be interpreted as follows: S_- corresponds to having a Q_1 to contend with and dually for S_+. The stammer occurs because we do not yet have enough information to resolve the quarter-symbol. Note that whenever returning to S_0 from either S_- or S_+ a stutter occurs and the number of output digits then equals the number of input pairs. S_0 thus occurs when the output digits fully describe the midpoint of the numbers described by the input digits so far.

(A non-stuttering machine is available. It needs six states, a strict initial state, I, and five others.

Next State

	I	S_{-2}	S_{-1}	S_0	S_{+1}	S_{+2}
-2	S_{-2}	S_{-2}	S_0	S_{-2}	S_0	S_{-2}
-1	S_{-1}	S_{-1}	S_{+1}	S_{-1}	S_{+1}	S_{-1}
0	S_0	S_0	S_{+2}	S_0	S_{-2}	S_0
$+1$	S_{+1}	S_{+1}	S_{-1}	S_{+1}	S_{-1}	S_{+1}
$+2$	S_{+2}	S_{+2}	S_0	S_{+2}	S_0	S_{+2}

Output

	I	S_{-2}	S_{-1}	S_0	S_{+1}	S_{+2}
-2		$-$	$-$	0	0	$+$
-1		$-$	$-$	0	0	$+$
0		$-$	$-$	0	$+$	$+$
$+1$		$-$	0	0	$+$	$+$
$+2$		$-$	0	0	$+$	$+$

A stammer occurs at the very beginning, Thereafter it is always exactly one output digit behind the number of input pairs:)

The lattice operations require, curiously, a larger automaton than the midpoint operation. First—for reasons to become clear—consider an automaton with seven states denoted L_3, L_2, L_1, M (the initial state),U_1, U_2, U_3. We imagine an "upper" stream of signed binary digits and a "lower" one. The nine input pairs again group themselves into five classes for the next-state behavior. It is determined by their difference: the upper digit minus the lower digit.

Next State

	L_3	L_2	L_1	M	U_1	U_2	U_3
-2	L_3	L_3	L_3	L_2	M	U_2	U_3
-1	L_3	L_3	L_3	L_1	U_1	U_3	U_3
0	L_3	L_3	L_2	M	U_2	U_3	U_3
$+1$	L_3	L_3	L_1	U_1	U_3	U_3	U_3
$+2$	L_3	L_2	M	U_2	U_3	U_3	U_3

Before considering the output let us interpret the states: M occurs when the streams presently describe the same number; the U-states occur when the upper stream presently describes a number larger than the lower; U_1 when it is possible that the upper stream will end up smaller;[108] U_2 and U_3 when it is known that the upper stream will henceforth always describe a larger number; U_2 when it is possible that the numbers will converge, that is, even though the upper will always be larger the difference will go to zero; U_3 when it is known that the numbers will not converge, that is, the difference is bounded away from zero.

For the "max" operation define the output so that it echos the upper digit whenever moving to or from a U-state and echos the lower digit whenever moving to or from an L-state. Note that the only times when the automaton stays in state M is when the upper and lower digits are equal—in that case echo that unique digit. (There is no direct motion between the L- and U-states.) For the "min" operation just reverse, obviously, these output rules.

For the lattice operations one can conflate the outer pairs of states, that is, we can replace every subscript 3 with the subscript 2 to obtain a five-state machine.[109] But if we wish for a machine that can tell us when the two streams are describing necessarily unequal numbers we can not do that. It is possible in states U_2 and L_2 that the numbers are equal (e.g., in state U_2 if when all subsequent upper digits are -1 and lower digits $+1$). Six states are required for an "apartness" machine (realized by conflating L_3 and U_3). All seven states are required if a machine is desired that remembers—when the streams are describing unequal numbers—which number is larger and which smaller.

32. Appendix: Dedekind Cuts

If we wish to avoid the axiom of dependent choice, one approach is to use Dedekind cuts. As previously noted, if experience with topoi is any guide we know in advance what $\mathtt{I}$ should turn out to be in the category of sheaves over a topological space X, to wit, the sheaf of continuous $\mathtt{I}$-valued functions on X, that is, the sheaf whose stalks are germs of

[108] If we view the automaton as a Markov process with two absorbing states and if we take the digits as randomly equidistributed then the odds are 13 to 1 that the machine will eventually reach U_3 starting in U_1.

[109] For streams of standard (unsigned) binary digits a very simple automaton suffices. Its states are L, I (the initial state) and U. Its inputs are columns of binary digits:

Next State

	L	I	U
$\begin{pmatrix}0\\0\end{pmatrix}$	L	I	U
$\begin{pmatrix}0\\1\end{pmatrix}$	L	L	U
$\begin{pmatrix}1\\0\end{pmatrix}$	L	U	U
$\begin{pmatrix}1\\1\end{pmatrix}$	L	I	U

Output

	L	I	U
$\begin{pmatrix}0\\0\end{pmatrix}$	0	0	0
$\begin{pmatrix}0\\1\end{pmatrix}$	1	1	0
$\begin{pmatrix}1\\0\end{pmatrix}$	0	1	1
$\begin{pmatrix}1\\1\end{pmatrix}$	1	1	1

continuous functions (as defined in the topos of sets) from X to $\mathtt{I}$. There are a number of ways of define a Dedekind cut—all of them equivalent in the category of sets—but not all give give the right answer in a category of sheaves.

When working with $\mathbb{I}$ rather than $\mathbb{D}$ it is convenient to define Dedekind cuts as subsets not of $\mathbb{I}$ but of its "interior," $\mathring{\mathbb{I}}$, the result of removing the two endpoints.[110]

We say that $L \subseteq \mathring{\mathbb{I}}$ is a **downdeal** if:

$$\ell \in L \Rightarrow \forall_{\ell' < \ell}\, \ell' \in L$$

and it is an **open downdeal** if, further:

$$\ell \in L \Rightarrow \exists_{\ell' > \ell}\, \ell' \in L$$

We say that U is an **updeal** if

$$u \in U \Rightarrow \forall_{u' > u}\, u' \in U$$

and it is an **open updeal** if, further

$$u \in U \Rightarrow \exists_{u' < u}\, u' \in U$$

In the case of sheaves on X we may reinterpret L and U as a families of open subsets of X indexed by elements of $\mathring{\mathbb{I}}$ (that is, L_ℓ gives the "extent to which $\ell \in L$"). The conditions then rewrite to:

$$L_\ell = \bigcup_{\ell' > \ell} L_{\ell'}$$

$$U_u = \bigcup_{u' < u} U_{u'}$$

Note that for any downdeal L, open or not, the largest open downdeal contained therein is

$$\mathring{L} = \{\ \ell \ :\ \exists_{\ell' \in L}\, \ell' > \ell\ \}$$

[110] Bear in mind that $\mathbb{I}$ and $\mathring{\mathbb{I}}$ are isomorphic—as objects—to the natural numbers $\mathbb{N}$. Another structure that we will need is the **free action** generated by a two-element set (which will here usually be the two zooming operations). It comes equipped with one generator and two self-functions and is the initial object in the category of such structures (it may be defined as the initial algebra for the functor that sends X to $1+X+X$). One may easily prove in topoi that for any object A the free A-action, $A^\star$, has a canonical monoid structure and as such it is the free monoid generated by A (the free A-action can be defined as the initial algebra for the functor that sends X to $1+A\times X$). When $A = 1{+}1$ we have that $(1+1)^\star$ is also isomorphic to $\mathbb{N}$. Just in case one demands actual isomorphisms take $(1+1)^\star \xrightarrow{\sim} \mathbb{N}$ to be the canonical map that sends the generator to the generator (i.e.,the empty word to zero) where the action of $(1+1)$ on $\mathbb{N}$ is given by the pair of maps, the first of which sends n to $2n{+}1$ and the second sends n to $2n{+}2$. Take $(1+1)^\star \xrightarrow{\sim} \mathring{\mathbb{I}}$ to be the canonical map that sends the empty word to $\odot$ where the action is given by the pair of maps, the first of which sends x to $\bot|x$ and the second sends x to $\top|x$. (And, of course, $\mathbb{N}$ is isomorphic to $1+1+\mathbb{N}$, hence $\mathring{\mathbb{I}}$ to $\mathbb{I}$.)

and the largest open updeal contained in an updeal U is

$$\overset{\circ}{U} = \{\ u\ :\ \exists_{u' \in U}\ u' < u'\ \}$$

When working in sheaves these correspond to

$$\overset{\circ}{L}_\ell = \bigcup_{\ell' > \ell} L_{\ell'}$$

$$\overset{\circ}{U}_u = \bigcup_{u' < u} L_{u'}$$

For any lower semi-continuous map $f : X \to \mathtt{I}$ we obtain an open downdeal by defining $L_\ell \ = f^{-1}(\ell, \top] \ \cap \overset{\circ}{\mathbb{I}}$ and for any upper semi-continuous g we obtain an open updeal with $U_u \ = g^{-1}[\bot, u) \ \cap \overset{\circ}{\mathbb{I}}$. Conversely, given an open downdeal, L, define $f(x)$ to be $\sup\{\ \ell\ :\ x \in L_\ell\ \}$ and given an open updeal, U, define $g(x) = \inf\{\ u\ :\ x \in U_u\ \}$. It is easy to check that these assignments establish a correspondence between upper/lower semi-continuous functions and open up/down-deals.

In the classical setting there are several ways of defining $\mathtt{I}$: as the set of open downdeals; as the set of open updeals; as the set of pairs $\langle L, U\rangle$ where L is maximal among those downdeals disjoint from U and U is maximal among those disjoint from L. In the more general setting, none of these are guaranteed to satisfy the separation condition: on any space X and open set V take L_ℓ to be constantly equal to V. The extent to which L is $\top$ is V and the extent to which it is different from $\top$ is $\neg V$, to wit, its exterior (defined as the largest open set disjoint from V, the interior of its complement). The extent to which L is not $\bot$ is $\neg\neg V$ (the interior of the closure of V). The union $\neg V \cup \neg\neg V$ fails, in general, to be all of X. Note that the maximal open updeal disjoint from L is U where U_u is constantly $\neg V$. The maximal open downdeal disjoint from U is constantly $\neg\neg V$. If we take any adjoint pair of semi-continuous maps $\langle f, g\rangle$ and $x \in X$ such that $f(x) < g(x)$ we may let $r = f(x)|g(x)$ and apply a suitable power of the dilatation at r to obtain an adjoint pair $\langle (r\triangleleft)^n f, (r\triangleleft)^n g\rangle$ that will recreate the same sort of pathology. We need, in other words. a condition on $\langle L, U\rangle$ that will force $f = g$.

Given any function $h : X \to \mathtt{I}$ there is a maximal lower semi-continuous $f : X \to \mathtt{I}$ below h and a minimal upper semi-continuous g above h. (To obtain f first define a downdeal L by first taking L_ℓ to be the interior of $h^{-1}(\ell, \top]$ and then replacing L_ℓ with $\overset{\circ}{L}_\ell$.) Of course a function both upper and lower semi-continuous is plain continuous.

A **Dedekind cut** on $\overset{\circ}{\mathbb{I}}$ is a pair of subsets $\langle L, U\rangle$ such that:

L is an open downdeal:

$$\ell \in L \Rightarrow [\forall_{\ell' < \ell}\ \ell' \in L] \wedge [\exists_{\ell' > \ell}\ \ell' \in L]$$

U is an open updeal:

$$u \in U \Rightarrow [\forall_{u' > u}\ u' \in U] \wedge [\exists_{u' < u}\ u' \in U]$$

L and U disjoint:

$$\neg[q \in L \wedge q \in U]$$

L and U almost cover:

$$\ell < u \Rightarrow [\ell \in L] \vee [u \in U]$$ [111]

In the case of sheaves on X the conditions rewrite to:

$$\begin{aligned} U_u &= \bigcup_{u'<u} U_{u'} \\ L_\ell &= \bigcup_{\ell'>\ell} L_{\ell'} \\ \ell < u &\Rightarrow L_\ell \cup U_u = X \\ L_q \cap U_q &= \emptyset \end{aligned}$$

For any continuous $f : X \to \mathtt{I}$ we obtain such a cut be defining $L_\ell = f^{-1}(\ell, \top)$ and $U_u = f^{-1}[\bot, u)$. All Dedekind cuts so arise: given the L_ℓs and U_us define $f : X \to \mathtt{I}$ by $f(x) = \inf\{\, u : x \in U_u \,\}$ and verify that f is continuous (the key observation is that the closure of U_u is contained in $U_{u'}$ whenever $u < u'$, hence $\{\, x : f(x) \leq u \,\} = \bigcap_{u'>u} U_{u'} = \bigcap_{u'>u} \overline{U}_{u'}$ is closed and, dually, $\{\, x : f(x) \geq u \,\}$ is open).

We define $\mathtt{I}$ to be the set of such Dedekind cuts. The bottom cut is $\langle \emptyset, \mathring{\mathbb{I}} \rangle$; the top cut is $\langle \mathring{\mathbb{I}}, \emptyset \rangle$.

Note that $\langle L, U \rangle \neq \bot$ iff L is non-empty and, dually, $\langle L, U \rangle \neq \top$ iff U is non-empty. The almost-cover condition for the case $\bot < \top$ is precisely the separation condition for the set of Dedekind cuts.

Define

$$\langle L, U \rangle^{\vee} = \langle \{\, \overset{\vee}{\ell} : \odot > \ell \in L \,\}, \{\, \overset{\vee}{u} : \odot > u \in U \,\} \rangle$$

$$\langle L, U \rangle^{\wedge} = \langle \{\, \overset{\wedge}{\ell} : \odot < \ell \in L \,\}, \{\, \hat{u} : \odot < u \in U \,\} \rangle$$

The Dedekind-cut conditions for $\langle L, U \rangle^{\vee}$ and $\langle L, U \rangle^{\wedge}$ are pretty routine. (For the almost-cover condition, given $\ell < u$ use the scale structure on $\mathbb{I}$: since $(\top|\ell) < (\top|u)$ we know that either $(\top|\ell) \in L$ or $(\top|u) \in U$. In the first case $\odot \leq (\top|\ell)^{\wedge} \in L$ and in the second case $\odot < (\top|u)^{\wedge} \in U$.)

The disjointness condition for the case $\odot$ is precisely the continuous coalgebra condition for these operations.

Given an interval coalgebra, C, satisfying the separation and continuous coalgebra condition, and given $c \in C$, we wish to construct a Dedekind cut $\langle L(c), U(c) \rangle$. By a **zooming sequence** is meant an element of the free monoid on two generators, $\top$-zooming and $\bot$-zooming.

$\ell \in L(c)$ iff there is a zooming sequence α such that $\ell^\alpha = \bot$ and $\neg\neg[c^{\alpha} = \top]$.

$u \in U(c)$ iff there is a zooming sequence, α such that $u^\alpha = \top$ and $\neg\neg[c^{\alpha} = \bot]$.

[111] L and U each determine the other: given L then $U = \{\, u : \exists_{u'<u} u' \notin L \,\}$. The axioms are, in fact, redundant: the 3rd and 4th conditions imply that L is a downdeal and U an updeal (but not the openness condition).

We need to verify the conditions for a Dedekind cut.

Before doing so let us pause to collect a few easily verified observations in the intuitionistic setting. For any function, f, it is, of course, trivial that $(x = y) \Rightarrow (fx = fy)$. Because negation is contravariant we also have $(fx \neq fy) \Rightarrow (x \neq y)$ and $\neg\neg(x = y) \Rightarrow \neg\neg(fx = fy)$. We will apply these trivial observations to the case when f is a zooming sequence and incorporate the fact that $\top$ and $\bot$ are fixed points. Hence

$$\begin{array}{rcl} x = \top & \Rightarrow & x^{\alpha} = \top \\ x = \bot & \Rightarrow & x^{\alpha} = \bot \\ x^{\alpha} \neq \top & \Rightarrow & x \neq \top \\ x^{\alpha} \neq \bot & \Rightarrow & x \neq \bot \\ \neg\neg(x = \bot) & \Rightarrow & \neg\neg(x^{\alpha} = \bot) \\ \neg\neg(x = \bot) & \Rightarrow & \neg\neg(x^{\alpha} = \bot) \end{array}$$

We will also use these trivial consequences of the apartness of $\top$ and $\bot$:

$$\begin{array}{rcl} \neg\neg(x = \bot) & \Rightarrow & x \neq \top \\ \neg\neg(x = \bot) & \Rightarrow & x \neq \bot \end{array}$$

And we will freely use all sorts of nice properties enjoyed by $\mathbb{I}$ (including the discrete coalgebra condition).

Not so trivial is this critical lemma:

32.1. Lemma. *For $c \in C$ and $u \in \mathbb{I}$ the following conditions on $c \in C$ and $u \in \mathbb{I}$ are equivalent:*

$$\begin{array}{rl} \alpha: & \exists_{\alpha}[\neg\neg(c^{\alpha} = \bot) \;\;\wedge\;\; (u^{\alpha} = \top)] \\ \beta: & \exists_{\beta}[\neg\neg(c^{\beta} = \bot) \;\;\wedge\;\; (u^{\beta} \neq \bot)] \\ \gamma: & \exists_{\gamma,v}[(c^{\gamma} \neq \top) \;\;\wedge\;\; (v < u) \;\;\wedge\;\; (v^{\gamma} = \top)] \end{array}$$

(Note that the γ-condition will tend be much more computationally feasible than the other two.)

<u>$\alpha \Rightarrow \gamma$:</u>

Given α let v be the unique element in $\mathbb{I}$ such that $v^{\alpha} = \odot$ and γ the result of following α with a $\bot$-zooming (and use $(\neg\neg(c^{\alpha} = \bot) \Rightarrow \neg\neg(c^{\gamma} = \bot) \Rightarrow (c^{\gamma} \neq \top)$.)

<u>$\gamma \Rightarrow \beta$:</u>

Given γ and v, We may assume that γ is the minimal zooming sequence for the task. We know that it is non-empty since $v < \top$ and $v^{\gamma} = \top$. If γ ends with an $\top$-zooming then the sequence obtained by removing that final $\top$-zooming would work as well. Thus from its minimality we may infer that γ ends with a $\bot$-zooming. Let γ' be the zooming sequence obtained by removing that final $\bot$-zooming. Since $v^{\gamma'\vee} = \top$ we know that $v^{\gamma'} \geq \odot$, hence $u^{\gamma'} > \odot$ and $u^{\gamma'\wedge} \neq \bot$. Since $c^{\gamma'\vee} \neq \top$ the continuous zooming condition says that $\neg\neg[c^{\gamma'\wedge} = \bot]$. Thus we finish by defining β to be the result of following γ' with an $\top$-zooming.

$\underline{\beta \Rightarrow \alpha:}$

Define α to be the result of following β with a sufficient number of $\bot$-zoomings to insure $u^{\alpha} = \top$.

Now for the Dedekind-cut conditions.

$U(c)$ is an open updeal:

Suppose $u \in U(c)$. Let γ, v be such that $c^{\gamma} \neq \top$, $v < u$ and $v^{\gamma} = \top$. Then for any $u' > v$ we have the same three conditions with u' instead of u, hence $u' \in U(c)$ for all $u' > v$.

$L(c)$ and $U(c)$ almost cover:

Given $\ell < u$ choose $\ell < k < v < u$. [112] Let γ be a zooming sequence (say the shortest one) such that $k^{\gamma} = \bot$ and $v^{\gamma} = \top$. The separation condition on C says that either $c^{\gamma} \neq \top$ or $c^{\gamma} \neq \bot$. In the first case we have $u \in U(c)$ and, dually, in the second case $\ell \in L(c)$.

$L(c)$ and $U(c)$ disjoint:

We wish to reach a contradiction from the assumption that there is $c \in A$, $q \in \mathbb{I}$ and zooming sequences σ, τ such that $\neg\neg(c^{\sigma} = \bot)$, $q^{\sigma} = \top$, $\neg\neg(c^{\tau} = \top)$ and $q^{\tau} = \bot$. We will settle for a weaker condition: $\neg\neg(c^{\sigma} = \bot)$ implies $c^{\sigma} \neq \top$ and $\neg\neg(c^{\tau} = \top)$ implies $c^{\tau} \neq \bot$. That is, we will reach a contradiction just from $c^{\sigma} \neq \top$, $q^{\sigma} = \top$, $c^{\tau} \neq \bot$ and $q^{\tau} = \bot$.

If σ were empty then $q = \top$ and it would not be possible for $q^{\tau} = \bot$. Dually, τ is non-empty. If σ were to start with $\top$-zooming we would know that $\hat{q} \neq \bot$ forcing $\overset{\vee}{q} = \top$ (the discrete coalgebra condition holds in $\mathbb{I}$) and thus $q^{\tau} = \top$, contradicting $q^{\tau} = \bot$. Hence σ starts with $\bot$-zooming and, dually, τ with $\top$-zooming. From $c^{\sigma} \neq \top$ we may infer $\overset{\vee}{c} \neq \top$ and from $c^{\tau} \neq \bot$ we infer $\hat{c} \neq \bot$. But the conjunctions of these two $\neq$s is precisely what the continuous coalgebra condition says can not happen.

We must show that this assignment of Dedekind cuts preserves the coalgebra structure. There is no difficulty in showing that $\langle L(\bot), U(\bot)\rangle$ and $\langle L(\top), U(\top)\rangle$ are what they should be. What we must show is $\langle L(\hat{c}), U(\hat{c})\rangle = \overset{\wedge}{\langle L(c), U(c)\rangle}$ (the other equation, of course, is dual). Restated: we must show

$$\overset{\wedge}{\ell} \in L(\hat{c}) \ \text{ iff } \ \odot \leq \ell \in L(c)$$

$$\hat{u} \in U(\hat{c}) \ \text{ iff } \ \odot < u \in U(c)$$

The forward directions are immediate: if α is a zooming sequence such that $\neg\neg(\hat{c}^{\alpha} = \top)$ and $\overset{\wedge}{\ell}^{\alpha} = \bot$ then if α' is the result of following an $\top$-zooming with α we have $\neg\neg(c^{\alpha'} = \top)$

[112] For example, $k' = \ell|u$, $v = k|u$.

and $\ell^{\alpha'} = \bot$ (and if $\ell < \odot$ then replace it with $\odot$). The same argument works when $\hat{u} \in U(\hat{c})$ (and since $\hat{u}^{\alpha} = \top$ we know that $u > \odot$).

For the reverse direction, suppose $\ell \geq \odot$, $\neg\neg(c^{\alpha} = \top)$ and $\ell^{\alpha} = \bot$. Then α is necessarily non-empty (that is, $\ell \neq \bot$) and it can not start with a $\bot$-zooming (since $\ell \geq \odot$ implies $\check{\ell} = \top$). Let α' be the rest of the sequence after the initial $\top$-zooming. Then $\neg\neg(\hat{c}^{\alpha'} = \top)$ and $\hat{\ell}^{\alpha'} = \bot$ forcing $\hat{\ell} \in L(\hat{c})$

Finally, suppose $u > \odot$, $\neg\neg(c^{\alpha} = \bot)$ and $u^{\alpha} = \top$. If α is empty then the empty sequence also establishes $\hat{u} \in U(\hat{c})$. If α starts with a $\top$-zooming we use the same sort of argument just above to establish $\hat{u} \in U(\hat{c})$. If α starts with a $\bot$-zooming then for any $u > \odot$ it is the case that $u^{\alpha} = \top$, hence we need to show that everything in $\mathbb{I}$ other than $\bot$ is in $U(\hat{a})$. But we may infer $\neg\neg(c^{\alpha} = \bot) \Rightarrow (c^{\alpha} \neq \bot) \Rightarrow (\check{c} \neq \bot)$ and the continuous coalgebra condition says that $\neg\neg(\hat{c} = \top)$. For β the empty sequence we thus have $\neg\neg(\hat{c}^{\beta} = \top)$ and $\hat{u}^{\beta} \neq \bot$ forcing $\hat{u} \in U(\hat{c})$.

33. Appendix: The Peneproximate origins

I always disliked analysis. Algebra, geometry, topology, even formal logic, they captivated me; analysis was different.

My attitude, alas, wasn't improved when I was supposed to tell a class of Princeton freshmen about numerical integration. I was expected to tell them that the trapezoid rule was better than Riemann and that Simpson was better than trapezoids. I was not expected to prove any of this.

I was appalled by the gap between applied mathematical experience and what we could even imagine proving. How does one integrate over all continuous functions to arrive at the expected error of a particular method?

Of course one can carve out finite dimensional vector spaces of continuous functions and compute an expected error thereon. But all continuous functions? It's easy to prove that there is no measure—not even a finitely additive measure—on the set of all continuous functions assuming at least that we ask for even a few of the most innocuous of invariance properties. Yet experience said that there was, indeed, such a measure on the set of functions one actually encounters.

But it wasn't just a problem in mathematics: I learned from physicists that they succeed in coming to verifiable conclusions by pretending to integrate over the set of all paths between two points. Again it is not hard to prove that no such "Feynman integral" is possible once one insists on a few invariance properties.

Even later I learned (from the work of David Mumford) about "Bayesian vision": in this case one wants to integrate over all possible "scenes" in order to deduce the most probable interpretations of what is being seen. A scene is taken to be a function from, say, a square to shades of gray. It would be a mistake to restrict to continuous functions—sharp contour boundaries surely want to exist. Quite remarkable "robotic vision" machines had been constructed for specific purposes by judiciously cutting down to appropriate finite-dimensional vector spaces of scenes. But once again, there is no possible measure on sets of all scenes which enjoy even the simplest of invariance conditions.

Thus three examples coming with quite disparate origins—math, science, engineering—were shouting that we need a new approach to measure theory.

One line of hope arose from the observation that the non-existence proofs all require a very classical foundation. There's the enticing possibility that a more computationally realistic setting—as offered, say, by effective topoi—could resolve the difficulties. A wonderful dream presents itself: the role of foundations in mathematics—and its applications—could undergo a transformation similar to the last two centuries' transformation of geometry.

Geometry moved from fixed rigidity to remarkable flexibility and—in the last century—that liberal view became a critical tool in physics. We learned that there was a trade-off between physics and geometry; we could still insist on classical (Euclidean) geometry but only at the expense of a cumbersome physics. We no longer even view most questions on the nature of geometry to be well put unless first the nature of physics be stipulated and—of course—vice versa.

Could we now learn the same about foundations? Elementary topoi provide a general setting for shifting foundations in a manner similar to the role of Riemannian manifolds in geometry. Might the trade-off between physics and geometry be replicated for physics and foundations? Two hundred years ago there was only one geometry. It was more than taken for granted; it was deemed to be certain knowledge.

Of course the geometry we now call Euclidean was certain; it may not be innate but it is inevitable. I have no doubt that if we lived in a universe with a visibly non-zero curvature we would get around to building our blackboards with zero curvature.[113]

It must be deemed remarkable that we learned to think—and make correct predictions—in non-Euclidean geometry. We learned to imagine living in a 3-sphere, in spaces of higher genus, even in projective space. The representation theorems of Riemannian manifolds (long before they were all proved) played a critical role in that process; and so it is with foundations. With a few topoi on hand for comparison, people do learn to shift their foundations between what's called classical and what's called (alas) intuitionistic. Again, the representation theorems play a critical role: in a fully classical setting a category of sheaves on a site can support a fully intuitionistic logic—change the topology on the cite and you can revert to the classical.

Coming back to earth: I must confess that the perfectly obvious idea that one should first establish ordinary integration in the right way on something as simple as the closed interval, that simple idea took longer than it should have (it had to await a day's boat trip in Alaska, of all places). For some years I preached this doctrine to the category/topos

[113]In this neighborhood, of course, Euclidean geometry is the natural geometry from the very beginning. When I was a kid a friend had measured the distance around a giant tree. We estimated the tree's width by solving the same problem on a little fruit-juice glass. We never questioned that the same ratio would hold for trees and glasses.

crowd and some trace of those preachings can be found scattered in the literature.[114] In September 1999 at an invited talk at the annual CTCS meeting (held that year in Edinburgh) I even characterized the mean value of real-valued continuous functions on the closed interval as an order-preserving linear operation that did the right thing to constants and had the property that the mean value on the entire interval equaled the midpoint of the mean-values on the two half intervals. I described it with a diagram that used (twice) the canonical equivalence between $\mathtt{I}$ and $\mathtt{I} \vee \mathtt{I}$.

But one equivalence, even used twice, doesn't bring forth the general notion: it doesn't prompt one to invent ordered wedges; without ordered wedges one doesn't define zoom operators nor discover the theorem on the existence of standard models. One doesn't learn how remarkably algebraic real analysis can become.

What I needed was someone to kick me into coalgebra mode. Three months later two guys did just that and on December 22 I wrote to the category list:

> There's a nice paper by Duško Pavlović and Vaughan Pratt. It's entitled
> On Coalgebra of Real Numbers [115] and it has turned me on.

A solution, alas, for the three motivating problems still awaits; but, at least, now I like analysis.

$$\int$$

Department of Mathematics, University of Pennsylvania
Philadelphia, PA, 19104
Email: `pjf@cis.upenn.edu`

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[114]e.g. Abbas Edalat and Martin Hötzel Escardó, Integration in real PCF
LICS 1996, Part I (New Brunswick, NJ). *Information and Computation* 160 (2000), no. 1-2, 128–166).
[115]CMCS'99 *Coalgebraic Methods in Computer Science (Amsterdam, 1999)*, 15 pp. (electronic)
Electron. Notes Theor. Comput. Sci., 19, *Elsevier, Amsterdam*, 1999.