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ITERATIVE ALGEBRAS: HOW ITERATIVE ARE THEY?

J. ADÁMEK, R. BÖRGER, S. MILIUS, AND J. VELEBIL*

Abstract. Iterative algebras, defined by the property that every guarded system of recursive equations has a unique solution, are proved to have a much stronger property: every system of recursive equations has a unique strict solution. Those systems that have a unique solution in every iterative algebra are characterized.

1. Introduction

Iterative algebras are those algebras in which every "guarded" system of recursive equations has a unique solution. This concept, introduced by Evelyn Nelson [N] and Jerzy Tiurin [T], is important for the study of Elgot's iterative theories. The condition of guardedness serves to exclude bad guys such as $x = x$. In the present paper we prove that iterative algebras are "very" iterative: *every* system of recursive equations has a solution, in fact, a canonical one. For the latter we need a choice of a global element $\perp$ in the given iterative algebra—then we can introduce the concept of a strict solution. For example, the unique strict solution of $x = x$ is $x \mapsto \perp$. We prove that every recursive equation system has a unique strict solution. We also fully characterize those systems of equations which have unique solutions in all iterative algebras; we call them preguarded.

We prove our results for all finitary endofunctors of well-behaved categories. These are the locally finitely presentable categories of Gabriel an Ulmer in which every object is a coproduct of connected objects. It turns out that each such category is extensive, but not conversely: there are locally finitely presentable, extensive categories in which an equation can have infinitely many strict solutions. We demonstrate this by an example in the category of Jónsson-Tarski algebras. Section 2 is devoted to a discussion of the base categories we need throughout the paper.

In our later research we plan to use the above "stronger iterativity" of iterative algebras for characterizing monadic algebras of the monad of free iteration theories on the category of endofunctors. And we will also use it for the first step in a "reconciliation" of iterative algebras and iteration algebras of Stephen Bloom and Zoltán Ésik [BÉ]. The latter are algebras where all systems of recursive equations have solutions, and a choice of solutions subject to axioms is performed; the motivation stems from continuous algebras on CPO's,

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where recursive equations always have the least solution. The "reconciliation" mentioned above has two steps: one, the subject of the present paper, is to show that every iterative algebra has canonical solutions of all systems of recursive equations. The other step, which we attend to in the paper [ABM], is to show that these canonical solutions satisfy the axioms of iteration algebras.

Related Work. This paper is a (substantially expanded) version of the extended abstract published previously [AMV_2]. The results of Section 2 are new, the concept of strict solution has been simplified, and Example 6.10 and Theorem 5.17 is also new. For endofunctors of **Set** the unique existence of strict solutions was proved by Larry Moss [Mo] and Stephen Bloom *et al.* [BEW_1], [BEW_2]. Our purely categorical proof is independent.

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2. Hyper-Extensive Categories

In this section we introduce the categories for which the main results will be proved: they are the locally finitely presentable categories (see [GU]) which are hyper-extensive. The latter is stronger than extensivity, and for locally finitely presentable categories it states precisely that every object is a coproduct of connected objects. We recall all the concepts needed and illustrate them by examples. Since local presentability is not needed for all our results, we formulate hyper-extensivity more generally first: this turns out to be a slightly stronger condition than extensivity, see [CLW].

2.1. Remark. (i) Recall from [GU] that an object A is called *finitely presentable* if $\hom(A, -)$ preserves filtered colimits.

(ii) We call A *connected* if $\hom(A, -)$ preserves coproducts. For example, in **Set** the finitely presentable objects are the finite sets and the connected ones are the terminal (one-element) sets only. In the category **Gra** of graphs finitely presentable objects are the finite graphs, and the connected ones are the graphs with precisely one (connected) component.

(iii) It is easy to verify that a directed colimit of connected objects is always connected.

(iv) Recall that a category $\mathscr{A}$ is called *locally finitely presentable*, see [GU] or [AR], if it has colimits and a set $\mathscr{A}_{fp}$ of finitely presentable objects such that every object is a filtered colimit of objects in $\mathscr{A}_{fp}$. Examples: **Set**, $\mathbf{Set}^I$, **Gra** and **Pos** (posets and order-preserving maps) are finitely presentable categories.

2.2. Definition. *[CLW]. A category is called* ***extensive*** *if it has finite coproducts which are*

(a) *disjoint, i.e., coproduct injections are monomorphisms and the intersection of two distinct coproduct injections is always* 0 *(the initial object), and*

(b) *universal, i.e., for every morphism* $f\colon C \longrightarrow A_1 + A_2$ *the pullbacks of the coproduct injections along* f *exist and turn* C *into the corresponding coproduct:*

$$
\begin{array}{ccccc}
A_1' & \longrightarrow & C = A_1' + A_2' & \longleftarrow & A_2' \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\
A_1 & \xrightarrow[\text{inl}]{} & A_1 + A_2 & \xleftarrow[\text{inr}]{} & A_2
\end{array}
$$

2.3. Notation. We denote, for every coproduct injection $i\colon A \longrightarrow C$, by $\bar{i}\colon \overline{A} \longrightarrow C$ the complementary coproduct injection, i.e., $C = A + \overline{A}$ with injections i and $\bar{i}$. A subobject $A \hookrightarrow B$ is called *trivial* if A is an initial object.

2.4. Remark. Let $\mathscr{A}$ be an extensive category.

(i) A coproduct $A + B$ is finitely presentable iff A and B are finitely presentable. In fact, sufficiency is trivial, for necessity assume that $c_i\colon C_i \longrightarrow C$ $(i \in I)$ is a filtered colimit and $f\colon A \longrightarrow C$ a morphism. If $A + B$ is finitely presentable, we are to verify that f factors through some c_i, and the factorization is essentially unique (i.e., given $u, v\colon A \longrightarrow C_i$ with $c = c_i{\cdot}u = c_i{\cdot}v$ then u, v are merged by some connecting morphism of the filtered diagram). We know that $f + \mathrm{id}_B\colon A + B \longrightarrow \mathrm{colim}_{i \in I}(C_i + B)$ factors through some $c_i + \mathrm{id}_B$; due to extensivity the factorizing morphism has the form $u + \mathrm{id}_B$ for some $u\colon A \longrightarrow C_i$. It then follows that $f = c_i{\cdot}u$, and the essential uniqueness w.r.t. A follows from that w.r.t. $A + B$.

(ii) An object of $\mathscr{A}$ is connected iff it is non-initial and *indecomposable*, that is, it is not a coproduct of two objects unless one of them is initial.

(iii) In an extensive category given disjoint subobjects $a_i\colon A_i \longrightarrow B$, $i = 1, 2$, each of which is a coproduct injection, then $[a_1, a_2]\colon A_1 + A_2 \longrightarrow B$ is also a coproduct injection. This property does not generalize to countable coproducts, see Example 2.9(3), therefore, we formulate it separately:

2.5. Definition. *A category is called* ***hyper-extensive*** *if it has countable coproducts which are* (a) *disjoint,* (b) *universal, and* (c) *given pairwise disjoint subobjects* $a_i\colon A_i \longrightarrow B$, $i \in \mathbb{N}$, *each of which is a coproduct injection, then* $[a_i]\colon \coprod_{i \in \mathbb{N}} A_i \longrightarrow B$ *is also a coproduct injection.*

2.6. Examples. (1) **Set** is hyper-extensive. The category of finite sets is an example of an extensive category that is not hyper-extensive because it does not have countable coproducts.

(2) Posets, graphs, and unary algebras form hyper-extensive categories.

(3) Free completions Fam $\mathscr{B}$ of categories $\mathscr{B}$ under coproducts (which can be described as the category of families of objects of $\mathscr{B}$) are hyper-extensive. Also free completions under countable coproducts are always hyper-extensive.

(4) If $\mathscr{K}$ is hyper-extensive then so is each functor category $[\mathscr{A}, \mathscr{K}]$, $\mathscr{A}$ small.

(5) The category of compact Hausdorff spaces is extensive but not hyper-extensive: its countable coproducts are not universal.

2.7. THEOREM. *A locally finitely presentable category is hyper-extensive iff every object is a coproduct of connected objects.*

PROOF. (1) Let $\mathscr{A}$ be a locally finitely presentable, hyper-extensive category.

(1a) We prove that in $\mathscr{A}$ all, not necessarily finite, coproducts are universal, i.e., given a morphism $f\colon C \longrightarrow \coprod_{i\in I} A_i$, then pullbacks $c_i\colon C_i \longrightarrow C$ of coproducts injections a_i along f form a coproduct $C = \coprod_{i\in I} C_i$. If C is finitely presentable, then this follows from extensivity because f factors through a finite subcoproduct. If C is arbitrary, express it as a filtered colimit of finitely presentable objects D_t with a colimit cocone $d_t\colon D_t \longrightarrow C$ ($t \in T$). For each t form pullbacks $d_{t,i}\colon D_{t,i} \longrightarrow D_t$ of the coproduct injections a_i along $f\cdot d_t$, then we know that $D_t = \coprod_{i\in I} D_{t,i}$. Since pullbacks commute with filtered colimits, see [AR], 1.59, we have $C_i = \operatorname{colim}_{t\in T} D_{t,i}$ for every $i \in I$. Therefore, we get canonical isomorphisms

$$\begin{aligned}\coprod_{i\in I} C_i &\cong \coprod_{i\in I} \operatorname*{colim}_{t\in T} D_{t,i}\\ &\cong \operatorname*{colim}_{t\in T} \coprod_{i\in I} D_{t,i}\\ &\cong \operatorname*{colim}_{t\in T} D_t\\ &\cong C.\end{aligned}$$

(1b) Every finitely presentable object A is a coproduct of connected objects. In fact, for A initial use empty coproduct. If $A \ncong 0$, we use Remark 2.4(ii):

Assuming that the object A is not a finite coproduct of indecomposable objects, we derive a contradiction. There clearly exists a decomposition $A = A_0 + A_1$ where A_0, A_1 are non-initial objects and A_1 is not indecomposable. Then A_1 has a decomposition $A_1 = A_{10} + A_{11}$ where A_{10}, A_{11} are non-initial and A_{11} is not indecomposable, etc. We get decompositions

$$A = A_0 + A_1 = A_0 + A_{10} + A_{11} = A_0 + A_{10} + A_{110} + A_{111} = \cdots$$

and the coproduct injections

$$a_{1^n0}\colon A_{1^n0} \longrightarrow A = A_0 + A_{10} + A_{110} + \cdots + A_{1^n0} + A_{1^{n+1}}$$

are pairwise disjoint. By assumption, the morphism

$$a = [a_0, a_{10}, a_{110}, \dots]\colon \coprod_{n\in\mathbb{N}} A_{1^n0} \longrightarrow A$$

is a coproduct injection. The "complementary" coproduct injection (see 2.3)

$$b\colon B \longrightarrow A \qquad \text{with} \qquad A = \coprod_{n\in\mathbb{N}} A_{1^n0} + B$$

is disjoint with each a_{1^n0}. However, A is a filtered colimit of the finite coproducts $A_0 + A_{10} + \cdots + A_{1^n0} + B$, and since A is finitely presentable, id_A factors through one of the colimit morphisms $c = [a_0, a_{10}, \ldots, a_{1^n0}, b]$. Thus, c is an isomorphism (being a split epimorphism and a coproduct injection). This is the desired contradiction: $a_{1^{n+1}0}$ factors through c, in spite to being disjoint with each component.

(1c) Every object A is a coproduct of connected objects. In fact, express A as a directed colimit

$$a_i \colon A_i \longrightarrow A \qquad (i \in I)$$

of finitely presentable objects, and let

$$A_i = \coprod_{j \in J_i} B_{i,j} \qquad \text{with injections } b_{ij}$$

be a coproduct of connected objects $B_{i,j}$. Then since coproducts and directed colimits commute, we obtain A_i as a coproduct of directed colimits of the objects $B_{i,j}$; the latter directed colimits are connected by Remark 2.1(ii).

More detailed: for every element of

$$\coprod_{i \in I} J_i = \big\{(i,j); i \in I, j \in J_i\big\}$$

define a diagram

$$\mathscr{D}_{ij} \text{ indexed by all } i' \in I \text{ with } i' \geq i$$

whose object of index i' is the unique object $B_{i',j'}$ $(j' \in J_{i'})$ for which $a_{i,i'} \cdot b_{i,j}$ factors through $b_{i',j'}$:

$$\begin{array}{ccc} B_{i,j} & \xrightarrow{b^{i'}_{i,j}} & B_{i',j'} \\ {\scriptstyle b_{i,j}}\downarrow & & \downarrow{\scriptstyle b_{i',j'}} \\ A_i & \xrightarrow[a_{i,i'}]{} & A_{i'} \end{array}$$

Let

$$C_{ij} = \operatorname*{colim}_{i' \geq j} B_{i',j'}$$

be the directed colimit of $\mathscr{D}_{ij}$. By 2.1(iii), for all i, j

$$C_{ij} \text{ are connected objects.}$$

We denote the colimit cocone by

$$c^{i'}_{ij} \colon B_{i',j'} \longrightarrow C_{i,j} \qquad (i' \geq i).$$

Let $\sim$ be the equivalence relation on $\coprod J_i$ generated by

$$(i,j) \sim (\bar{i}, \bar{j}) \quad \text{iff} \quad \begin{array}{l} \mathscr{D}_{i,j} \text{ and } \mathscr{D}_{i',j'} \text{ have the same object } B_{i',j'} \\ \text{for some upper bound } i' \text{ of } i, \bar{i}. \end{array}$$

Observe that this implies that also all objects indexed by any $i'' \geq i'$ are common. The desired coproduct is

$$A = \coprod_{(i,j)\in S} C_{i,j}$$

for any set $S \subseteq \coprod J_i$ of representatives of $\sim$. In fact, given $(i,j) \in S$ the diagram $\mathscr{D}_{i,j}$ has a cone formed by all $a_{i'} \cdot b_{i',j'} \colon B_{i',j'} \longrightarrow A$, thus, we have a unique morphism

$$f_{i,j} \colon C_{i,j} \longrightarrow A$$

such that all the squares

$$\begin{array}{ccc} B_{i',j'} & \xrightarrow{b_{i',j'}} & A_{i'} \\ {\scriptstyle c^{i'}_{i,j}}\downarrow & & \downarrow{\scriptstyle a_{i'}} \\ C_{i,j} & \xrightarrow[f_{i,j}]{} & A \end{array} \qquad (i' \geq i)$$

commute. It is easy to check that the morphism

$$f = [f_{i,j}] \colon \coprod_{i,j\in S} C_{i,j} \longrightarrow A$$

is an isomorphism.

(2) Let all objects of a locally finitely presentable category $\mathscr{A}$ be coproducts of connected objects.

(2a) Coproduct injections $a_i \colon A_i \longrightarrow A_1 + A_2$ are monomorphisms, and they are disjoint (for $i = 1, 2$). In fact, to prove the former, consider morphisms $p, q \colon B \longrightarrow A_i$ merged by a_i for $i = 1$ or 2. If B is connected, then clearly $p = q$. If B is arbitrary, express it as a coproduct of connected objects and use the individual coproduct injections.

Let $c \colon C \longrightarrow A_1 + A_2$ be the intersection of a_1 and a_2. To prove that C is the initial object, it is sufficient to observe that it is an empty coproduct (of connected objects). In fact, given a morphism $b \colon B \longrightarrow C$ then B cannot be conncted: since $\hom(B, A_i) \neq \emptyset$ for $i = 1, 2$ the coproduct $A_1 + A_2$ is not preserved by $\hom(B, -)$.

(2b) Coproducts are universal. In fact, let $A = \coprod_{i\in I} A_i$ be a coproduct with injections a_i, and let $f \colon B \longrightarrow A$ be a morphism. We first assume that B is connected. Then for the pullbacks of a_i's along f there exists a unique i such that f factorizes though a_i, in other word, there exists a morphism $d \colon B \longrightarrow B_i$ with $b_i \cdot d = \mathrm{id}$.

$$\begin{array}{ccc} B_i & \xrightarrow{f_i} & A_i \\ {\scriptstyle b_i}\downarrow \uparrow{\scriptstyle d} & & \downarrow{\scriptstyle a_i} \\ B & \xrightarrow[f]{} & A \end{array}$$

Since a_i is a monomorphism, so is b_i, consequently, b_i is an isomorphism. Thus, B is (by default) a coproduct of B_j's; as B is connected: we clearly have B_j initial for all $j \neq i$ in I.

If B is not connected, apply the above argument to all connected component of B.

(2c) Suppose that $a_i\colon A_i \longrightarrow B$ $(i \in I)$ are pairwise disjoint coproduct injections. Thus for every i we have a coproduct

$$A_i \xrightarrow{a_i} B = A_i + \overline{A}_i \xleftarrow{\bar{a}_i} \overline{A}_i.$$

Then we are to prove that the morphism $[a_i]_{i\in I}\colon \coprod_{i\in I} A_i \longrightarrow B$ is also a coproduct injection. In fact, express B as a coproduct

$$B = \coprod_{t\in T} B_t \qquad \text{with injections } b_t$$

of connected objects B_t. For every $i \in I$, since $\hom(B_t, -)$ preserves the coproduct $A_i + \overline{A}_i$, we see that b_t factorizes through either a_i or $\bar{a}_i$. Then A_i is the coproduct of all B_t for which b_t factorizes through a_i. Thus, for the set

$$T_0 = \{t \in T; b_t \text{ factorizes through } a_i \text{ for some } i \in I$$

we obviously have

$$\coprod_{i\in I} A_i = \coprod_{t\in T_0} B_t \qquad \text{and} \qquad [a_i]_{i\in I} = [b_t]_{t\in T_0}.$$

The morphism $b = [b_t]_{t\in T_0}$ is a coproduct injection with the complementary injection $\bar{b} = [b_t]_{t\in T-T_0}$. ■

2.8. Remark. It follows rather easily from the above theorem that locally finitely presentable, hyper-extensive categories have the form Fam $\mathscr{B}$ of Example 2.6(3). In fact, they are precisely the categories of the form Fam $\mathscr{B}$ where $\mathscr{B}$ is finitely accessible and has connected colimits.

The proof of this fact is relatively straightforward but rather long and we omit it.

2.9. Examples. (1) All examples of 2.6(1)–(4) are locally finitely presentable.

(2) The category **Vec** of real vector spaces is locally finitely presentable (since it is a variety of algebras) and every vector space is a coproduct of copies of $\mathbb{R}$. But although $\mathbb{R}$ is indecomposable, it is not connected. And, in fact, **Vec** is not extensive.

(3) The category

$$\mathscr{J}\mathscr{T}$$

of Jónsson-Tarski algebras has as objects binary algebras $(A, *)$ such that $*\colon A\times A \longrightarrow A$ is a bijection. This is a variety which is not only extensive, it is a topos, see [J], A1.2.1.11(i). However, $\mathscr{J}\mathscr{T}$ is not hyper-extensive. In fact, let

$$\Phi(1)$$

denote the Jónsson-Tarski algebra on one generator g. There is a unique decomposition $g = g' * g''$. It is easy to see that $\Phi(1)$ is also the the free Jónsson-Tarski algebra on the two generators g' and g''. Consequently,

$$\Phi(1) = \Phi(1) + \Phi(1)$$

where the coproduct injections $l, r\colon \Phi(1) \longrightarrow \Phi(1)$ send g to g' and g'', respectively. This implies that $\Phi(1)$ is not a coproduct of connected objects: since $\Phi(1)$ is finitely presentable, such a coproduct would have to be finite, which is clearly impossible.

3. Equation Morphisms and Solutions

In this section we recall the concepts of algebra, equation morphism and solution for a given endofunctor

$$H\colon \mathscr{A} \longrightarrow \mathscr{A}.$$

We do not require any property of $\mathscr{A}$ besides having finite coproducts.

3.1. Remark. (i) *Algebras* an endofunctor for H are objects A of $\mathscr{A}$ together with morphisms $a\colon HA \longrightarrow A$; the corresponding morphisms are called homomorphisms, they are defined via obvious commutative squares. *Coalgebras* for H are objects A of $\mathscr{A}$ together with morphisms $a\colon A \longrightarrow HA$, coalgebra homomorphisms are also given by obvious commutative squares.

(ii) The functor H is called *finitary* if it preserves filtered colimits. Every finitary functor has free algebras, and as proved by Michael Barr in [B], this yields a monad F of free H-algebras which is a free monad on H. We have

$$FZ = HFZ + Z, \tag{1}$$

where the coproduct injections are the H-algebra structure and the universal arrow.

3.2. Definition. *Given an endofunctor H, a* ***flat equation morphism*** *in an object A is a morphism of the form*

$$e\colon X \longrightarrow HX + A. \tag{2}$$

A ***solution*** *of e in an algebra $a\colon HA \longrightarrow A$ is a morphism $e^\dagger\colon X \longrightarrow A$ such that the square*

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [a,A]} \\ HX + A & \xrightarrow[He^\dagger + A]{} & HA + A \end{array}$$

commutes. The algebra A is called ***completely iterative*** *(or, shortly* **cia***), see [M], if every flat equation morphism has a unique solution.*

3.3. Examples. (i) Let $H = \mathrm{Id}$ be the identity endofunctor of **Set**. Then algebras are the usual algebras on one unary operation α. An algebra is a cia iff α has a fixed point a, and for every infinite path

$$\cdots \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} x_0$$

we have $x_i = a$ for all i. See [M].

(ii) Let $\mathscr{A}$ be an extensive category and consider the constant functor C_1 with the value 1, a terminal object. An algebra is an object A together with a global element $a\colon 1 \longrightarrow A$. Every algebra is a cia. In fact, given a flat equation morphism $e\colon X \longrightarrow 1 + A$ we have a decomposition

$$e = e_l + e_r \colon X_l + X_r \longrightarrow 1 + A.$$

It easy to verify that the unique solution $e^\dagger$ is

$$e^\dagger = \big[a \cdot e_l, e_r\big] \colon X_l + X_r \longrightarrow A.$$

(iii) If H has a terminal coalgebra $\tau\colon T \longrightarrow HT$ then τ is invertible and the algebra $\tau^{-1}\colon HT \longrightarrow T$ is a cia.

More generally: let TZ be the terminal coalgebra for $H(-) + Z$. Then the coalgebra structure

$$\alpha_Z \colon TZ \longrightarrow HTZ + Z \tag{3}$$

is invertible, whence TZ is a coproduct of HTZ and Z

$$TZ = HTZ + Z \tag{4}$$

with injections

$$\tau_Z \colon HTZ \longrightarrow TZ \qquad \text{(“}TZ \text{ is an } H\text{-algebra”)}$$
$$\eta_Z \colon Z \longrightarrow TZ \qquad \text{(“embedding of variables”).}$$

That is, $[\tau_Z, \eta_Z] = \alpha_Z^{-1}$. In fact, TZ is a free cia on Z with η_Z as the universal arrow. We denote by T the monad of free cias for H. Its unit is η and the multiplication μ is given by the unique homomorphism $\mu_Z : TTZ \longrightarrow TZ$ extending identity on TZ. This monad is characterized in [AAMV, M] as a *free completely iterative monad* on H.

3.4. Definition. *[AAMV]. An endofunctor H is called* ***iteratable*** *if TZ, a terminal coalgebra for $H(-) + Z$, exists for every Z.*

3.5. Example. (i) Let Σ be a signature, i.e., a sequence of sets $(\Sigma_n)_{n\in\mathbb{N}}$. Then Σ-algebras in **Set** are H-algebras for the *polynomial* functor $H_\Sigma\colon$ **Set** $\longrightarrow$ **Set**:

$$H_\Sigma Z = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots$$

H_Σ is iteratable, and $T_\Sigma Z$ can be described as the algebra of all Σ-trees on Z, i.e., rooted and ordered trees with leaves labelled in $Z + \Sigma_0$ and nodes with $n > 0$ successors labelled in Σ_n.

A flat equation morphism $e\colon X \longrightarrow H_\Sigma X + A$ with the set $X = \{x_1, x_2, \dots\}$ of variables represents a system of recursive equations, one for every variable x_i, of the form

$$x_i \approx s(x_{i_1}, \dots, x_{i_n}) \qquad \text{for } s \in \Sigma_n$$

or

$$x_i \approx a \qquad \text{for } a \in A.$$

A *solution* in an algebra A is a substitution of elements $x_i^\dagger$ of A for variables x_i such that the formal equations become identities in A. It turns out that in every cia much more general recursive systems have unique solutions: the right-hand sides can be terms (or finite Σ-trees) on $X + A$, or even infinite Σ-trees, i.e., elements of $T_\Sigma(X + A)$:

(ii) Every finitary endofunctor (more generally, every accessible endofunctor) of a locally finitely presentable category is iteratable, see [AAMV].

3.6. Definition. *Let H be an iteratable endofunctor. An* ***equation morphism*** *in a cia A is a morphism of the form*

$$e\colon X \longrightarrow T(X + A).$$

It is called ***guarded*** *if it factors through the coproduct injection of $T(X + A) \cong HT(X + A) + X + A \cong X + [A + HT(X + A)]$:*

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & T(X+A) \\ & \searrow & \uparrow {\scriptstyle [\tau_{X+A}, \eta_{X+A} \cdot \mathrm{inl}]} \\ & & HT(X+A) + A \end{array}$$

3.7. Example. For H_Σ an equation morphism e represents equations

$$x_i \approx t(x_1, x_2, \dots, a_1, a_2, \dots)$$

whose right-hand sides are Σ-trees on $X + A$. And e is guarded if the right-hand sides are not single variables. This excludes trivial cases such as $x_i \approx x_i$ where solutions are almost never unique.

3.8. Notation. If A is a cia, we denote by $\widetilde{a}\colon TA \longrightarrow A$ the unique homomorphism with

$$\widetilde{a} \cdot \eta_A = \mathrm{id}\,.$$

The following theorem (whose proof is a straightforward adaptation of Theorem 3.9 in [M]) states that flat and guarded make no difference:

3.9. Theorem. *[M]. In a cia A every guarded equation morphism $e\colon X \longrightarrow T(X + A)$ has a unique solution, i.e., there exists a unique $e^\dagger\colon X \longrightarrow A$ such that the square*

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e} \downarrow & & \uparrow {\scriptstyle \widetilde{a}} \\ T(X+A) & \xrightarrow[\quad T[e^\dagger, A] \quad]{} & TA \end{array} \tag{5}$$

commutes.

3.10. Definition. *Let H be a finitary endofunctor. A* ***finitary equation morphism*** *in A is a morphism of the form*

$$e\colon X \longrightarrow F(X+A),$$

where X is finitely presentable. It is called ***guarded*** *if it factors through the right-hand coproduct injection of $F(X+A) = X + [A + HF(X+A)]$.*

3.11. Definition. *An H-algebra is called* ***iterative*** *if every finitary flat equation morphism, i.e.,* (2) *with X finitely presentable, has a unique solution.*

3.12. Example. [AMV$_1$]. For $H = \mathrm{Id}_{\mathbf{Set}}$ a unary algebra is iterative iff its operation $\alpha\colon A \longrightarrow A$ has a unique fixed point x and no other cycle. It is a cia if, moreover, the graph of α^{-1} has no infinite path other than $x, x, x, \dots$.

3.13. Example. [N]. For $H = H_\Sigma$ the subalgebra $R_\Sigma Z \subseteq T_\Sigma Z$ of the Σ-tree algebra formed by all *rational trees*, i.e., trees which have up to isomorphism only finitely many subtrees, is iterative. This is a free iterative Σ-algebra on Z.

3.14. Remark. (i) For every finitary functor H free iterative algebras exist and the monad R they form, called the *rational monad* of H, is characterized in [AMV$_1$] as a free iterative monad on H. Analogously to (4) we have

$$R = HR + \mathrm{Id}\,. \tag{6}$$

(ii) In analogy to Definition 3.6 a *rational equation morphism* is a morphism

$$e\colon X \longrightarrow R(X+A) \qquad \text{with } X \text{ finitely presentable.}$$

It is called *guarded* if it factors through the coproduct injection of $HR(X+A)+A$.

(iii) For every iterative algebra (A, a) we denote by

$$\widehat{a}\colon RA \longrightarrow A$$

the unique homomorphism extending the identity on A.

3.15. Theorem. *[AMV$_1$]. In an iterative algebra A every guarded rational equation morphism $e\colon X \longrightarrow R(X+A)$ has a unique solution, i.e., there exists a unique $e^\dagger\colon X \longrightarrow A$ such that the square*

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \widehat{a}} \\ R(X+A) & \xrightarrow[R[e^\dagger, A]]{} & RA \end{array}$$

commutes.

4. Preguarded Equation Morphisms

4.1. Assumption. *Throughout this section H denotes an iteratable endofunctor of a hyper-extensive category, see Definitions 3.4 and 2.5. Coproduct injections of binary coproducts are called* inl *and* inr*.*

4.2. Notation. Let

$$e\colon X \longrightarrow T(X+A)$$

be an equation morphism and

$$i_0 \equiv X \xrightarrow{\text{inl}} X + A \xrightarrow{\eta_{X+A}} T(X+A)$$

be the “standard” embedding of variables. Recall that this is a coproduct injection of $T(X+A) = HT(X+A)+X+A$, see (4). Let us form inverse images of the two coproduct injections, i_0 and $\bar{i}_0$ (see 2.3), along e:

$$\begin{array}{ccccc} X_1 & \xrightarrow{i_1} & X & \xleftarrow{\bar{i}_1} & \overline{X}_1 \\ {\scriptstyle e_1}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \bar{e}_1} \\ X & \xrightarrow[i_0]{} & T(X+A) & \xleftarrow[\bar{i}_0]{} & HT(X+A)+A \end{array}$$

Then the extensivity of our base category implies

$$X = X_1 + \overline{X}_1 \qquad \text{with injections } i_1 \text{ and } \bar{i}_1.$$

Recall that e is called guarded iff $X_1 = 0$, or, equivalently,

$$X \cong \overline{X}_1 \qquad (\bar{i}_1 \text{ an isomorphism}).$$

4.3. Definition. *Let $e\colon X \longrightarrow T(X+A)$ be an equation morphism. A subobject $m\colon M \rightarrowtail X$ is called **ungrounded** provided that e has a restriction to an endomorphism*

$$e'\colon M \longrightarrow M,$$

in other words, the square

$$\begin{array}{ccc} M & \xrightarrow{\quad m \quad} & X \\ {\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle e} \\ M & \xrightarrow[m]{} X \xrightarrow[i_0]{} & T(X+A) \end{array} \tag{1}$$

*commutes. (Example: the trivial subobject $0 \rightarrowtail X$ is ungrounded.) We call an equation morphism e **preguarded** if it has no nontrivial ungrounded subobjects.*

4.4. Example. Let Σ consist of a binary operation A. The system

$$x = y * z$$
$$y = t$$
$$z = x$$
$$t = y$$

has the ungrounded subobject $\{y, t\} \rightarrowtail \{x, y, z, t\}$. Here e' is just the domain-codomain restriction of e.

4.5. Definition. *Given an equation morphism $e\colon X \longrightarrow T(X + A)$ the **derived subobjects** $X_n \rightarrowtail X$, $n = 1, 2, 3, \dots$ are defined by the following pullbacks*

$$\begin{array}{ccccccccc} & X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X & \\ \cdots & \downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e} & \\ & X_2 & \xrightarrow[i_2]{} & X_1 & \xrightarrow[i_1]{} & X & \xrightarrow[i_0]{} & T(X+A) & \end{array} \tag{2}$$

4.6. Remark. (i) Since i_0 is a coproduct injection, so is i_1, and e_1 is a domain-codomain restriction of e. Analogously, since i_1 is a coproduct injection, so is i_2, and e_2 is a domain-codomain restriction of e_1, etc. We denote by $i_n^*\colon X_n \longrightarrow X$ the corresponding composites

$$i_0^* = \mathrm{id}_X \qquad \text{and} \qquad i_{n+1}^* = i_n^* \cdot i_{n+1} \colon X_{n+1} \longrightarrow X. \tag{3}$$

(ii) For every $n \geq 1$ we denote by

$$\bar{i}_n \colon \overline{X}_n \longrightarrow X_{n-1} \qquad (n = 1, 2, 3, \dots)$$

(where $X_0 = X$) the complementary coproduct injection of i_n, thus, $X_{n-1} = X_n + \overline{X}_n$ for $n = 1, 2, 3, \dots$ We consider $\overline{X}_n$ as a subobject of X via

$$\bar{i}_n^* \equiv \overline{X}_n \xrightarrow{\bar{i}_n} X_{n-1} \xrightarrow{i_{n-1}^*} X \qquad (n \geq 1). \tag{4}$$

(iii) In the base category **Set** the variables of $X_1 = e^{-1}(X_0)$ are precisely those x_i where t_i is a single variable in X. That is, those x_i where the corresponding equation has the form $x_i \approx x_{i'}$. We conclude that X_1 are precisely the unguarded variables. To put it positively, $\overline{X}_1$ consists of all the guarded variables. Here we have $e_1\colon X_1 \longrightarrow X$, $x_i \longmapsto x_{i'}$, and thus x_i lies in $X_2 = e_1^{-1}(X_1)$ if and only if $x_{i'}$ is unguarded. Consequently, for every $x_i \in X_2$ we have equations $x_i \approx x_{i'}$ and $x_{i'} \approx x_{i''}$. In other words, $\overline{X}_2$ consists of all variables reaching a guarded variable in one step (of applying e). Analogously, $x_i \in X_3$ if and only if we have equations $x_i \approx x_{i'}$, $x_{i'} \approx x_{i''}$ and $x_{i''} \approx x_{i'''}$ or, equivalently, $\overline{X}_3$ consists of all variables reaching a guarded variable in two steps, etc. To say

$$X = \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots$$

means that every variable reaches a guarded variable in finitely many steps. More generally:

4.7. Proposition. *Every equation morphism has the greatest ungrounded subobject which is* (a) *the intersection of all derived subobjects and* (b) *the complementary subobject of* $\coprod_{n\geq 1} \overline{X}_n$.

Proof. Since $\bar{i}_k^*\colon \overline{X}_k \longrightarrow X$ $(k \geq 1)$ are pairwise disjoint coproduct injections, they form a coproduct injection $[\bar{i}_k^*]\colon \coprod_{k\geq 1} \overline{X}_k \longrightarrow X$, see 2.5. We denote by

$$m\colon M \longrightarrow X$$

the complementary coproduct injection and prove that this is the largest ungrounded subobject of e—this proves the proposition since $X = X_k + \overline{X}_k$ for every $k \geq 1$, implies that the subobject $m\colon M \longrightarrow X$ is just the intersection of all derived subobjects.

(a) $m\colon M \longrightarrow X$ is ungrounded. In fact, denote by

$$m_k\colon M \longrightarrow X_k \qquad \text{with } i_k^* \cdot m_k = m$$

the corresponding embeddings. Then the morphisms $e'_k = e_{k+1} \cdot m_{k+1}\colon M \longrightarrow X_k$ form a compatible cone, i.e., $i_k \cdot e'_k = e'_{k-1}$ for $k = 1, 2, 3, \dots$. Thus, there exists a unique

$$e'\colon M \longrightarrow M \left(= \bigcap_{k\geq 1} X_k\right) \qquad \text{with } e'_k = m_k \cdot e' \ (k \geq 1).$$

The diagram (1) commutes since for $m = i_1 \cdot m_1$ we get

$$e \cdot m = e \cdot i_1 \cdot m_1 = i_0 \cdot e_1 \cdot m_1 = i_0 \cdot e'_0$$

as well as

$$i_0 \cdot m \cdot e' = i_0 \cdot i_1 \cdot m_1 \cdot e' = i_0 \cdot i_1 \cdot e'_1 = i_0 \cdot e'_0.$$

(b) Let $\widehat{m}\colon \widehat{M} \longrightarrow X$ be an ungrounded subobject with

$$e \cdot \widehat{m} = i_0 \cdot m \cdot \widehat{e} \qquad \text{for } \widehat{e}\colon \widehat{M} \longrightarrow \widehat{M}.$$

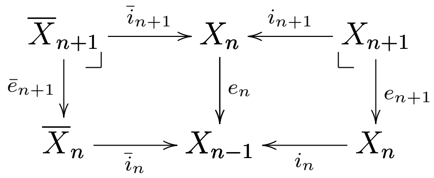
To prove $\widehat{M} \subseteq M$ we need to show $\widehat{M} \subseteq X_k$ for all $k \geq 1$: this is an obvious induction on k using the pullbacks of (2). ∎

4.8. Corollary. *An equation morphism* $e\colon X \longrightarrow T(X + A)$ *is preguarded iff* $X = \coprod_{n\geq 1} \overline{X}_n$.

4.9. Remark. (i) Even if our base category is not hyper-extensive, every ungrounded subobject $m\colon M \longrightarrow X$ factorizes through the intersection of all derived subobjects: use the morphisms e'_k in the proof of Proposition 4.7.

(ii) From Example 3.5 we see that the intuition behind the subobjects $\overline{X}_1, \overline{X}_2, \overline{X}_3, \dots$ is such that $\overline{X}_1$ consists of all guarded variables. If e is a guarded equation morphism, then $X = \overline{X}_1$. If e is preguarded, we always have a passage $\overline{X}_n \longrightarrow \overline{X}_1$, for all $n \geq 1$, which to every variable assigns the guarded variable eventually reached by applying e finitely many times. To formulate this categorically, we need the following

4.10. Notation. We form a pullback of $e_n\colon X_n \longrightarrow X_{n-1}$ along the complement $\bar{i}_n$ of i_n, see Remark 4.6:



$(n \geq 1)$

The canonical passage from $\overline{X}_n$ to $\overline{X}_1$ is the composite $\bar{e}_2 \cdot \cdots \cdot \bar{e}_n$. This defines a morphism

$$u = [\mathrm{id}, \bar{e}_2, \bar{e}_2 \cdot \bar{e}_3, \dots]\colon \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots \longrightarrow \overline{X}_1. \tag{5}$$

4.11. Definition. *For every preguarded equation morphism $e\colon X \longrightarrow T(X+A)$ we define an equation morphism as follows*

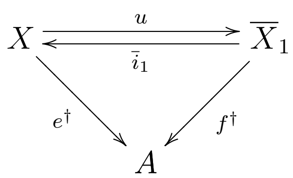
$$f \equiv \overline{X}_1 \xrightarrow{\bar{i}_1} X \xrightarrow{e} T(X+A) \xrightarrow{T(u+A)} T(\overline{X}_1 + A). \tag{6}$$

We call f the ***guarded modification*** *of e.*

4.12. Theorem. *The guarded modification f of an equation morphism e is guarded, and has the same solutions as e. More precisely: for every cia with the underlying object A:*

(a) *If $e^\dagger$ is a solution of e, then $e^\dagger \cdot \bar{i}_1\colon \overline{X}_1 \longrightarrow A$ is a solution of f.*

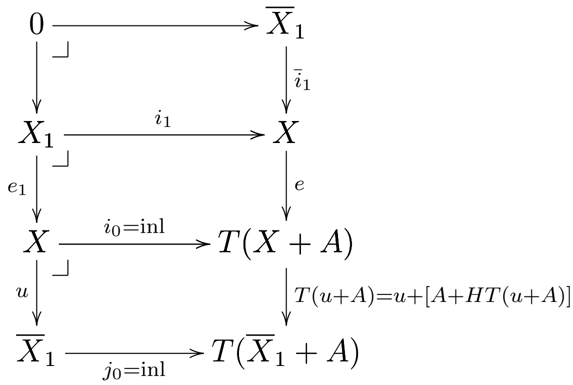
(b) *If $f^\dagger$ is a solution of f, then $f^\dagger \cdot u\colon X \longrightarrow A$ is a solution of e.*



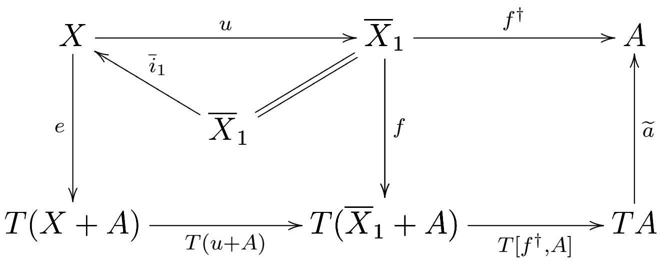
Proof. (1) We verify that f is guarded. Put

$$j_0 = \mathrm{inl}\colon \overline{X}_1 \longrightarrow T(\overline{X}_1 + A) = \overline{X}_1 + A + HT(\overline{X}_1 + A)$$

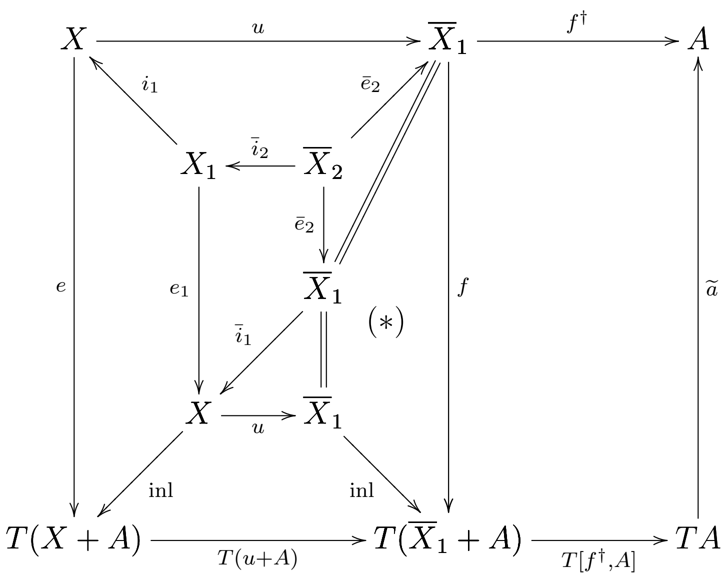
and compute a pullback of f along j_0:



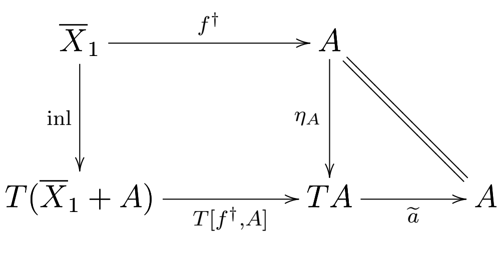
(2) Proof of (b). Given a solution $f^\dagger\colon \overline{X}_1 \longrightarrow A$ of f, we prove that $f^\dagger \cdot u\colon X \longrightarrow A$ is a solution of e, i.e., $f^\dagger \cdot u = \widetilde{a} \cdot T[f^\dagger \cdot u, A] \cdot e\colon X \longrightarrow A$. This equation will be proved by considering the individual components of $X = \coprod \overline{X}_n$, see (4). For $n = 1$ we use the definition (6) of f and obtain the commutative diagram



For $n = 2$, the coproduct injection is $i_1 \cdot \bar{i}_2\colon \overline{X}_2 \longrightarrow X$; thus we consider the diagram

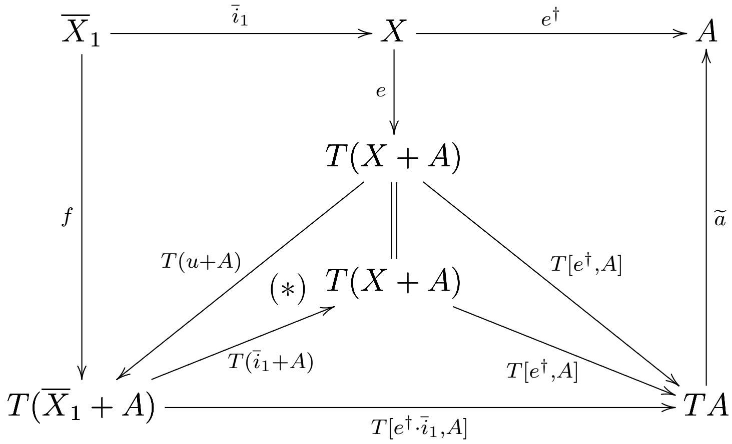


All the inner parts except the one denoted by $(*)$ clearly commute. The part $(*)$ commutes when composed with the passage to A, $\widetilde{a} \cdot T[f^\dagger, A]\colon T(\overline{X}_1 + A) \longrightarrow A$, i.e., this morphism merges the parallel pair f, $\mathrm{inl}\colon \overline{X}_1 \longrightarrow T(\overline{X}_1 + A)$. In fact, by the commutativity of the right-hand square in the above diagram it suffices to observe that $f^\dagger = \widetilde{a} \cdot T[f^\dagger, A] \cdot \mathrm{inl}$:



The cases $n = 3, 4, \ldots$ are analogous to the case $n = 2$.

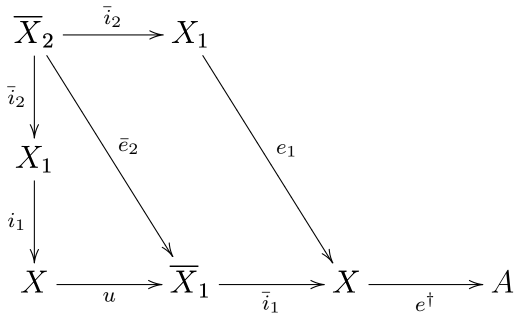
(3) Proof of (a). Let $e^\dagger : X \longrightarrow A$ be a solution of e. We are to prove that the outward square of the following diagram



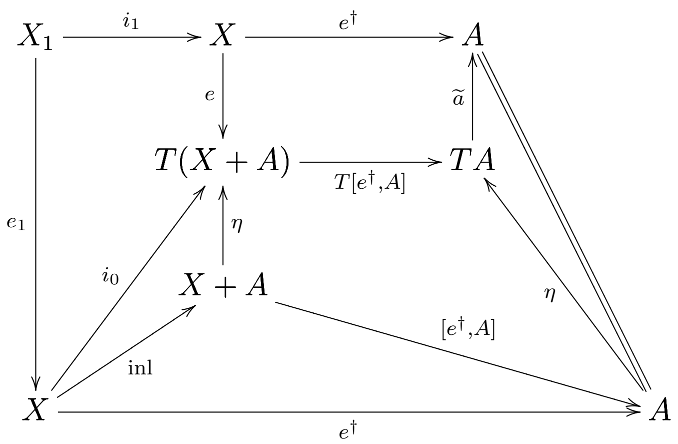
commutes. All the inner parts except that denoted by $(*)$ commute. For $(*)$ it is sufficient to prove that $T[e^\dagger, A]$ merges id and $T(\bar{i}_1 + A) \cdot T(u+A)$. Therefore, the proof of (a) will be finished by proving the equation $e^\dagger = e^\dagger \cdot \bar{i} \cdot u$. We consider the individual components $\overline{X}_n$ of $X = \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots$, see (4):

For $n = 1$ use $u \cdot \bar{i}_1 = \mathrm{id}$ to obtain $e^\dagger \cdot \bar{i}_1 = (e^\dagger \cdot \bar{i}_1 \cdot u) \cdot \bar{i}_1$.

For $n = 2$ we are to prove the equation $e^\dagger \cdot i_1 \cdot \bar{i}_2 = (e^\dagger \cdot \bar{i}_1 \cdot u) \cdot i_1 \cdot \bar{i}_2$. Consider the diagram



from which the right-hand side of the desired equation is expressed as $e^\dagger \cdot e_1 \cdot \bar{i}_2$. It remains to verify $e^\dagger \cdot i_1 = e^\dagger \cdot e_1$ which follows from the next diagram



Cases $n = 3, 4, \ldots$ are analogous. ∎

4.13. Remark. Within the proof of Theorem 4.12 we proved that if $e^\dagger$ is a solution of the preguarded equation morphism $e\colon X \longrightarrow T(X+A)$, then

$$e^\dagger = e^\dagger \cdot \bar{i} \cdot u. \tag{7}$$

4.14. Corollary. *In every cia all preguarded equation morphisms have unique solutions.*

In fact, the morphism u is an epimorphism, due to $u \cdot \bar{i}_1 = \mathrm{id}$, thus the unique existence of $e^\dagger$ follows from the unique existence of $f^\dagger$ via (a) and (b) above.

4.15. Remark. How about the converse: if $e\colon X \longrightarrow T(X+A)$ has unique solutions in all cia's, is e preguarded? The answer is affirmative whenever T satisfies mild side conditions: see Proposition 5.12 below.

5. Strict Solutions

5.1. Assumption. *Throughout this section $\mathscr{A}$ denotes a hyper-extensive category with a terminal object, 1. Moreover, H denotes an iteratable endofunctor of $\mathscr{A}$, see Definition 3.4, for which a morphism*

$$\perp\colon 1 \longrightarrow H0$$

has been chosen.

5.2. Notation. For every equation morphism the intersection of the derived subobjects $i_n^*\colon X_n \rightarrowtail X$ (see (3)) is denoted by

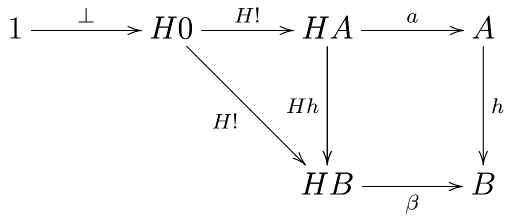
$$i_\infty\colon X_\infty \longrightarrow X.$$

5.3. Remark. By Proposition 4.7 the intersection exists and we have $X = X_\infty + \coprod_{n\geq 1} \overline{X}_n$ (with i_∞ and (4) as injections) and X_∞ is greatest ungrounded subobject.

5.4. NOTATION. $\bot$ is a global constant of H, i.e., every H-algebra $HA \xrightarrow{a} A$ obtains the corresponding global element

$$\bot_A \equiv 1 \xrightarrow{\bot} H0 \xrightarrow{H!} HA \xrightarrow{a} A.$$

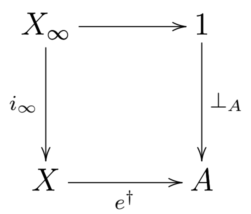
All homomorphisms $h\colon A \longrightarrow B$ preserve this global constant: $h \cdot \bot_A = \bot_B$. In fact, consider the commutative diagram below:



In particular for every object Y we have a global element of TY see Example 3.3(iii) which we denote by $\bot$ for short:

$$\bot \equiv 1 \longrightarrow H0 \xrightarrow{H!} HTY \xrightarrow{\tau_Y} TY$$

5.5. DEFINITION. *Let A be a cia and $e\colon X \longrightarrow T(X + A)$ an equation morphism with a solution $e^\dagger\colon X \longrightarrow A$ (see (5)). We call $e^\dagger$ **strict** if its restriction to every ungrounded subobject $m\colon M \rightarrowtail X$ (see 4.3) is $\bot_A$. Equivalently, the square below commutes:*



(1)

5.6. DEFINITION. *Let A be a cia. For every equation morphism*

$$e\colon X \longrightarrow T(X + A)$$

we define an equation morphism

$$f\colon X \longrightarrow T(X + A)$$

by changing the left-hand component of $e\colon X_\infty + \coprod \overline{X}_n \longrightarrow T(X + A)$ to $\bot$:

$$f \cdot i_\infty \equiv X_\infty \xrightarrow{!} 1 \xrightarrow{\bot} T(X + A)$$
$$f \cdot \bar{i}_\infty = e \cdot \bar{i}_\infty\colon \coprod_{n \geq 1} \overline{X}_n \longrightarrow T(X + A).$$

*We call f the **preguarded modification** of e.*

5.7. Theorem. *The preguarded modification f of an equation morphism e is preguarded and has the same strict solutions. That is:*

(a) *every strict solution of e is a solution of f, and*

(b) *every solution of f is a strict solution of e.*

Proof. (1) f is preguarded. Let $Z_0 = \coprod \overline{X}_n$ and let $j_k \colon Z_k \longrightarrow Z_{k-1}$, $k \geq 1$, denote the derived subobjects of f; we have $j_0 = \overline{i}_\infty$. We will prove that

$$Z_k = \overline{X}_{k+1} + \overline{X}_{k+2} + \cdots \quad \text{and} \quad j_k = \text{inr} \colon Z_k \longrightarrow \overline{X}_k + Z_k,$$

and that the corresponding morphism opposite f_{k-1} is

$$f_k = \bar{e}_{k+1} + \bar{e}_{k+2} + \cdots \colon Z_k \longrightarrow Z_{k-1} \qquad (k \geq 1).$$

This proves obviously that f is preguarded since $\bigcap_{k\in\mathbb{N}} Z_k = 0$.

Case $k = 1$: To find a pullback of $f = [\bot!, e{\cdot}j_0]$ along $i_0 \colon X \longrightarrow T(X + A)$, we just compute a pullback of $e{\cdot}j_0$ along i_0: in fact the component $\bot!$ contributes nothing to the pullback because it factors through $\overline{i}_0$, the complement of i_0, and $\mathscr{A}$ is extensive. Here is the pullback of $e{\cdot}j_0$ along i_0:

$$\begin{array}{ccc}
\overline{X}_2+\overline{X}_3+\cdots = Z_1 \lrcorner & \xrightarrow{\text{inr}} & \overline{X}_1+\overline{X}_2+\overline{X}_3+\cdots \\
\downarrow{\scriptstyle \text{inr}} & & \downarrow{\scriptstyle j_0=\text{inr}} \\
X_\infty+\overline{X}_2+\overline{X}_3+\cdots = X_1 \lrcorner & \xrightarrow{i_1=\text{inr}} & X = X_\infty + \overline{X}_1+\overline{X}_2+\overline{X}_3+\cdots \\
\downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e} \\
X & \xrightarrow[i_0]{} & T(X+A)
\end{array}$$

Consequently, we have $Z_1 = \overline{X}_2 + \overline{X}_3 + \cdots$ with $j_1 = \text{inr} \colon Z_1 \longrightarrow X = X_\infty + \overline{X}_1 + Z_1$, and the corresponding morphism $f_1 \colon Z_1 \longrightarrow X$ is

$$f_1 \equiv Z_1 \xrightarrow{\text{inr}} X_\infty + Z_1 \xrightarrow{e_1} X.$$

Case $k = 2$: We compute a pullback of $f_1 = e_1 \cdot \text{inr}$ along j_1:

$$\begin{array}{ccc}
Z_2 \lrcorner & \longrightarrow & Z_1 \\
\downarrow & & \downarrow{\scriptstyle \text{inr}} \\
\coprod_{n\geq 2} \overline{X}_{n+1} \lrcorner & \longrightarrow & X_1 \\
\downarrow & & \downarrow{\scriptstyle e_1} \\
\coprod_{n\geq 2} \overline{X}_n = Z_1 & \xrightarrow[j_1]{} & X
\end{array}$$

by computing first a pullback P_n of e_1 along the n-th component $\overline{X}_n \longrightarrow X$, $n \geq 2$, of j_1, see (4)

$$\begin{array}{ccccccccc}
P_n = \overline{X}_{n+1} & \xrightarrow{\bar{i}_{n+1}} & X_n & \xrightarrow{i_n} \cdots \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 \\
\Big\downarrow{\scriptstyle \bar{e}_{n+1}} & & \Big\downarrow{\scriptstyle e_n} & & \Big\downarrow{\scriptstyle e_2} & & \Big\downarrow{\scriptstyle e_1} \\
\overline{X}_n & \xrightarrow[\bar{i}_n]{} & X_{n-1} & \xrightarrow[i_{n-1}]{} \cdots \xrightarrow[i_2]{} & X_1 & \xrightarrow[i_1]{} & X
\end{array}$$

The connecting maps are $\bar{e}_n\colon P_n \longrightarrow \overline{X}_n$ and $i_2 \cdot \cdots \cdot i_n \cdot \bar{i}_{n+1}\colon P_n \longrightarrow X_1$. Thus, due to extensivity, a pullback of e_1 along j_1 is $\coprod_{n\geq 2} \overline{X}_{n+1} = Z_2$ with the connecting maps $\coprod_{n\geq 2} \bar{e}_{n+1}\colon Z_2 \longrightarrow Z_1$ and $\mathrm{inr}\colon Z_2 \longrightarrow X_1 = X_\infty + \overline{X}_2 + Z_2$. The pullback of $f_1 = e_1 \cdot \mathrm{inr}$ along j_1 is thus

$$\begin{array}{ccc}
\overline{X}_3 + \overline{X}_4 + \overline{X}_5 + \cdots = Z_2 & \xrightarrow{\mathrm{inr}} & Z_1 = \overline{X}_2 + \overline{X}_3 + \overline{X}_4 + \cdots \\
\Big\| & & \Big\downarrow{\scriptstyle \mathrm{inr}} \\
\overline{X}_3 + \overline{X}_4 + \overline{X}_5 + \cdots = Z_2 & \xrightarrow{\mathrm{inr}} & X_1 = X_\infty + \overline{X}_2 + \overline{X}_3 + \overline{X}_4 + \cdots \\
\Big\downarrow{\scriptstyle \coprod_{n\geq 2} \bar{e}_{n+1}} & & \Big\downarrow{\scriptstyle e_1} \\
Z_1 & \xrightarrow[j_1]{} & X
\end{array}$$

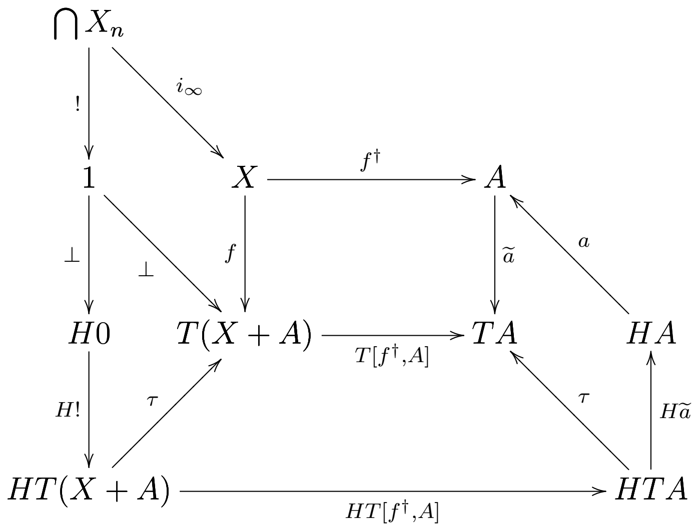
We obtain $Z_2 = \overline{X}_3 + \overline{X}_4 + \overline{X}_5 + \cdots$, $j_2 = \mathrm{inr}$, and $f_2 = \coprod_{n\geq 2} \bar{e}_{n+1}$.

Case $k \geq 3$: Here we use the obvious pullbacks

$$\begin{array}{ccccc}
\cdots\ Z_4 & \longrightarrow & Z_3 & \xrightarrow{\mathrm{inr}} & Z_2 \\
\Big\downarrow{\scriptstyle \coprod_{n\geq 4} \bar{e}_{n+1}} & & \Big\downarrow{\scriptstyle \coprod_{n\geq 3} \bar{e}_{n+1}} & & \Big\downarrow{\scriptstyle \coprod_{n\geq 2} \bar{e}_{n+1}} \\
\cdots\ Z_3 & \longrightarrow & Z_2 & \xrightarrow[\mathrm{inr}]{} & Z_1
\end{array}$$

(2) Proof of (b). If $f^\dagger$ is a solution of f, then $f^\dagger$ is strict. Observe first that the

diagram below commutes:

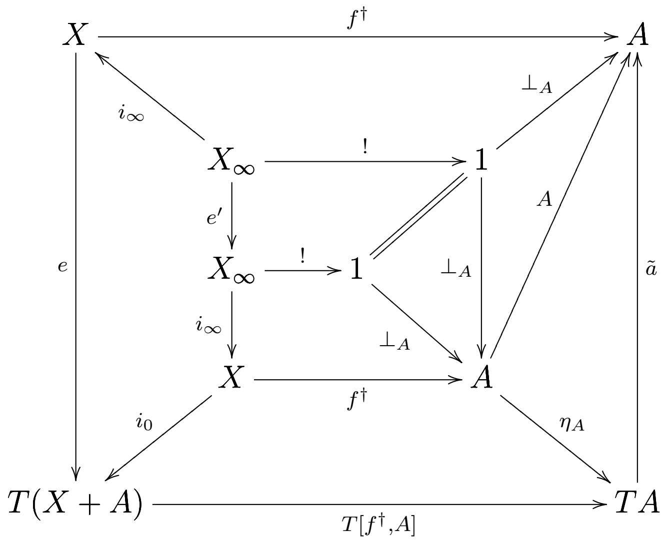


We see that the passage from $H0$ to HA is $H!$ (because $\widetilde{a}{\cdot}T[f^\dagger, A]{\cdot}! = !\colon 0 \longrightarrow A$), thus $f^\dagger{\cdot}i_\infty = a{\cdot}H!{\cdot}\bot{\cdot}! = \bot_A{\cdot}!$ as required.

And $f^\dagger$ is a solution of e, i.e., the equation

$$\widetilde{a}{\cdot}T[f^\dagger, A]{\cdot}e = f^\dagger\colon X_\infty + \coprod_{n\geq 1} \overline{X}_n \longrightarrow A \tag{2}$$

holds (see (5) in the introduction). In fact for the right-hand component $j_0\colon \coprod \overline{X}_n \longrightarrow X$ this follows from $e{\cdot}j_0 = f{\cdot}j_0$. For the left-hand component use the commutative diagram



whose left-hand square commutes for some e' by 5.3.

(3) Proof of (a). If $e^\dagger$ is a strict solution of e, then we are to prove that the equation $\widetilde{a}{\cdot}T[e^\dagger, A]{\cdot}f = e^\dagger$ holds (cf. (5)): for the right-hand component with domain $\coprod \overline{X}_n$ this

follows from the fact that $f\cdot j_0 = e\cdot j_0$. For the left-hand component use the fact that both $e^\dagger$ and f yield $\perp$ (in A and $T(X+A)$, respectively) and that $\widetilde{a}\cdot T[e^\dagger, A]$ preserves $\perp$, being a homomorphism (see Notation 5.4). ■

5.8. Notation. For every equation morphism $e\colon X \longrightarrow T(X+A)$ we denote by

$$\langle e\rangle\colon \overline{X}_1 \longrightarrow T(\overline{X}_1 + A)$$

the guarded modification of the preguarded modification of e.

5.9. Corollary. *In every cia every equation morphism has a unique strict solution, viz., the unique solution of the guarded morphism $\langle e\rangle$.*

5.10. Remark. We will now turn our attention to the question of whether an equation having a unique solution in every cia must be preguarded. In the case of $\mathscr{A} = \mathbf{Set}$, the answer is affirmative whenever $H1$ has at least two elements. In general categories we need the following

5.11. Definition. *We say that the free completely iterative cia monad T is **nontrivial** if it preserves monomorphisms and has at least two global constants:*

$$\operatorname{card} \mathscr{A}(1, T0) \geq 2.$$

5.12. Proposition. *Suppose that morphisms from non-initial objects to 1 are epimorphisms. If the free cia monad is nontrivial, then every equation morphism $e\colon X \longrightarrow T(X+A)$ with a unique solution in TA is preguarded.*

5.13. Remark. We consider e as an equation in TA via $X \xrightarrow{e} T(X+A) \xrightarrow{T(X+\eta)} T(X+TA)$.

Proof. Suppose that e is not preguarded. For every global element $b\colon 1 \longrightarrow T0$ we can find a solution $e_b^\dagger\colon X \longrightarrow TA$ such that

$$e_b^\dagger\cdot i_\infty \equiv X_\infty \longrightarrow 1 \xrightarrow{b} T0 \xrightarrow{T!} TA.$$

The proof is precisely the proof of Theorem 5.7 where $a\colon HA \longrightarrow A$ is the replaced by $\tau_A\colon HTA \longrightarrow TA$ (with $\widetilde{\tau}_A = \mu_A$) and $\perp$ is replaced by b. We will prove that e has more than one solution by showing that $e_b^\dagger$ determines b; for that we just observe that $T!\colon T0 \longrightarrow TA$ is a monomorphism. In fact, $!\colon 0 \longrightarrow A$ is a monomorphism since in every extensive category initial objects are strict, and T preserves monomorphisms. ■

5.14. Example. Suppose that our base category is $\mathscr{A} = \mathbf{Set}$.

(1) Whenever $H1$ has more than one element then H has a nontrivial free completely iterative cia monad. In fact, T preserves monomorphisms by Proposition 6.1 in [AMV$_1$]. And to prove $\operatorname{card} T0 \geq 2$, we decompose $H = H' + H''$ with $H'1 \neq \emptyset$ and $H''1 \neq \emptyset$. This can be done by choosing any $a \in H1$ and defining $H'X$ and $H''X$ as the inverse images of $\{a\}$ and $H1 - \{a\}$, respectively, under $H!\colon HX \longrightarrow H1$. Consider the coalgebras

$$A \equiv 1 \xrightarrow{\text{const}\, a} H'1 \hookrightarrow H1 \qquad \text{and} \qquad B \equiv 1 \xrightarrow{\text{const}\, b} H''1 \hookrightarrow H1$$

($a \in H'1$, $b \in H''1$), and recall that $T0$ is the final coalgebra for H. It is clear that the unique homomorphism $A \longrightarrow T0$ is disjoint with the unique homomorphism $B \longrightarrow T0$. Therefore, $\operatorname{card} T0 \geq 2$.

(2) Conversely, whenever for every equation morphism e the implication

$$e \text{ has unique solution} \implies e \text{ is preguarded}$$

holds, then $H1$ must have more than one element. In fact, $\operatorname{card} H1 = 1$ implies that $T0$ has a unique element. Then the equation $x \approx x$ has a unique solution in $T0$.

5.15. Remark. The previous results of the present section only hold for a functor with a chosen morphism $1 \longrightarrow H0$. The last assumption can be dropped: we now work with an arbitrary iteratable endofunctor H. And we apply the previous results to the endofunctor

$$H' = H(-) + 1 \qquad \text{with injections } \gamma\colon H \longrightarrow H',\ \bot\colon 1 \longrightarrow H'.$$

5.16. Definition. *By a* ***strict H-algebra*** *is meant an H-algebra $\alpha\colon HA \longrightarrow A$ together with a morphism $\bot_A\colon 1 \longrightarrow A$. Let (A, α) be a cia. Given an equation morphism $e\colon X \longrightarrow T(X + A)$, a* ***strict solution*** *is a solution $e^\dagger\colon X \longrightarrow A$ (i.e., (5) commutes) such that its restriction to every ungrounded subobject is $\bot_A$ (i.e., (1) commutes).*

5.17. Theorem. *Let H be an iteratable endofunctor. In every strict cia every equation morphism has a unique strict solution.*

Proof. (1) Since H is iteratable (i.e., $H(-) + X$ has a terminal coalgebra for every X), it follows that H' is iteratable—in fact, substitute $X + 1$ for X. Let T' denote the free completely iterative monad of H'.

(2) If $\alpha\colon HA \longrightarrow A$ is a cia, then for every $\bot_A\colon 1 \longrightarrow A$ the corresponding H'-algebra $[\alpha, \bot_A]\colon HA{+}1 \longrightarrow A$ is a cia. In fact, given a flat equation morphism $e\colon X \longrightarrow HX{+}1{+}A$ denote

$$f \equiv X \xrightarrow{e} HX + 1 + A \xrightarrow{HX + [\bot_A, \mathrm{id}]} HX + A;$$

we prove that a morphism $f^\dagger\colon X \longrightarrow A$ is a strict solution of e w.r.t. H (see 5.16) iff $f^\dagger$ is a strict solution of f w.r.t H' (see 5.5). This follows from the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f^\dagger \quad} & A \\
{\scriptstyle e}\downarrow & (*) & \uparrow{\scriptstyle [\alpha,\perp_A,\mathrm{id}]} \\
HX+1+A & \xrightarrow[Hf^\dagger+1+A]{} & HA+1+A \\
\downarrow{\scriptstyle HX+[\perp_A,\mathrm{id}]} & & \downarrow{\scriptstyle HA+[\perp_A,\mathrm{id}]} \\
HX+A & \xrightarrow[Hf^\dagger+\mathrm{id}]{} & HA+A
\end{array}
\qquad f\colon X \to HX+A,\quad [\alpha,\mathrm{id}]\colon HA+A \to A
$$

More detailed: if $f^\dagger$ is a solution of f, then the outward part commutes, and since all inner parts without denotation commute, it follows that $(*)$ commutes. Thus, $f^\dagger$ is a solution of e. Conversely, if $f^\dagger$ is a solution of e, then $(*)$ commutes—thus the diagram above commutes, proving that $f^\dagger$ is a solution of f.

(3) Let us recall the notation (3)

$$\alpha_Z\colon TZ \longrightarrow HTZ + Z$$

of the structure of a terminal coalgebra for $H(-) + Z$, and let us use the analogous notation for H':

$$\alpha'_Z\colon T'Z \longrightarrow HT'Z + 1 + Z.$$

We obtain a natural transformation

$$\delta\colon T \longrightarrow T'$$

whose components are the unique coalgebra homomorphisms δ_Z for $H'(-) + Z$:

$$
\begin{array}{ccccc}
TZ & \xrightarrow{\alpha_Z} & HTZ+Z & \hookrightarrow & HTZ+1+Z \\
{\scriptstyle \delta_Z}\downarrow & & & & \downarrow{\scriptstyle H\delta_Z+\mathrm{id}} \\
T'Z & & \xrightarrow[\alpha'_Z]{} & & HT'Z+1+Z
\end{array}
$$

For every strict cia A we have the homomorphism

$$\tilde{\alpha}\colon TA \longrightarrow A \qquad \text{with } \tilde{\alpha}\cdot\eta_A = \mathrm{id}$$

of Notation 3.8, as well as the corresponding homomorphism, say,

$$\bar{\alpha}\colon T'A \longrightarrow A \qquad \text{with } \bar{\alpha}\cdot\eta'_A = \mathrm{id}.$$

The uniqueness makes it clear that the triangle

$$\begin{array}{ccc} TA & \xrightarrow{\tilde{\alpha}} & A \\ & \searrow^{\delta_A} & \uparrow{\scriptstyle \bar{\alpha}} \\ & & T'A \end{array}$$

commutes.

(4) To prove the theorem, let A be a strict cia (for H) and let $e\colon X \longrightarrow T(X+A)$ be an equation morphism. For the equation morphism

$$g \equiv X \xrightarrow{e} T(X+A) \xrightarrow{\delta_{X+A}} T'(X+A)$$

there is, due to Corollary 5.9, a unique strict solution $g^\dagger\colon X \longrightarrow A$ in the sense of Definition 5.5. We show that $g^\dagger$ is the unique strict solution of e in the sense of Definition 5.16. It is not difficult to prove that the largest ungrounded subobject of g and e coincide. Thus, the two notions of strictness (according to Definition 5.5 and 5.16, respectively) coincide. Finally, the diagram below shows that solutions of g and e coincide:

$$\begin{array}{ccccc} X & \xrightarrow{g^\dagger} & A & & \\ \downarrow{\scriptstyle e} & & \uparrow{\scriptstyle \tilde{\alpha}} & \nwarrow^{\bar{\alpha}} & \\ T(X+A) & \xrightarrow[T[g^\dagger,\mathrm{id}]]{} & TA & & \\ \downarrow{\scriptstyle \delta_{X+A}} & & & \searrow^{\delta_A} & \\ T'(X+A) & & \xrightarrow[T'[g^\dagger,\mathrm{id}]]{} & & T'A \end{array}$$

■

5.18. Remark. Theorem 5.17 extends from objects to morphisms in the following natural sense. Given cias $\alpha\colon HA \longrightarrow A$ and $\beta\colon HB \longrightarrow B$, every homomorphism of H-algebras

$$\begin{array}{ccc} HA & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle Hh} & & \downarrow{\scriptstyle h} \\ HB & \xrightarrow[\beta]{} & B \end{array}$$

preserves solutions of the flat equation morphisms, and conversely, every solution-preserving morphism is a homomorphism of H-algebras, as proved in [AMV_1]. Preservation of solutions means that given a flat equation morphism $e\colon X \longrightarrow HX + A$ and forming the corresponding equation morphism

$$h \bullet e \equiv X \xrightarrow{e} HX + A \xrightarrow{\mathrm{id}+h} HX + B$$

then the triangle

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ & \searrow^{(h \bullet e)^\dagger} & \downarrow h \\ & & B \end{array} \tag{3}$$

commutes.

Now assume that A and B are strict cias, and f is a *strict homomorphism*, i.e., a homomorphism preserving the chosen global elements ($h \cdot \bot_A = \bot_B$). It follows that h preserves strict solutions of non-flat equation morphisms, too. That is, given an equation morphism

$$e\colon X \longrightarrow T(X + A)$$

and forming

$$h \bullet e \equiv X \xrightarrow{e} T(X + A) \xrightarrow{T(\mathrm{id}+h)} T(X + B)$$

then the following holds: the unique strict solutions $e^\dagger\colon X \longrightarrow A$ and $(h \bullet e)^\dagger\colon X \longrightarrow B$ form a commutative triangle (3). This follows from the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{e^\dagger} & A & \xrightarrow{h} & B \\ \downarrow e & & \uparrow \widetilde{\alpha} & & \\ T(X+A) & \xrightarrow[T[h\cdot e^\dagger, \eta]]{} & TA & & \uparrow \widetilde{\beta} \\ \downarrow T(\mathrm{id}+h) & & & \searrow^{Th} & \\ T(X+B) & & \xrightarrow[T[h\cdot e^\dagger, \eta]]{} & & TB \end{array}$$

and the fact that the largest ungrounded subobjects of e and $h \bullet e$ coincide.

6. Iterative Algebras

6.1. Assumption. *In this section $\mathscr{A}$ is a hyper-extensive locally finitely presentable category and H is a finitary endofunctor for which a morphism*

$$\bot\colon 1 \longrightarrow H0$$

has been chosen. (Or, alternatively, we work with pointed algebras, see Section 4.)

6.2. Definition. *For a rational equation morphism* $e\colon X \longrightarrow R(X+A)$ (*see* Remark 3.14 (ii)), *we define* ***derived subobjects*** $X_n \rightarrowtail X$ *precisely as in Definition* 4.5, *just replacing* T *by* R *everywhere.*

6.3. Remark. We thus have pullbacks

$$\begin{array}{ccccccc}
X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X \\
\downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e} \\
X_2 & \xrightarrow[i_2]{} & X_1 & \xrightarrow[i_1]{} & X & \xrightarrow[i_0=\mathrm{inr}]{} & R(X+A)
\end{array} \quad \cdots$$

We also use the remaining notation $\bar{i}_n\colon \overline{X}_n \longrightarrow X_{n-1}$ and $\bar{e}_n\colon \overline{X}_n \longrightarrow X_{n-1}$ etc. as in Section 3.

6.4. Lemma. *Every rational equation morphism* e *has the greatest ungrounded subobject equal to the least derived subobject: there exists* n *such that* i_n *is an isomorphism, and then* $i_n^*\colon X_n \longrightarrow X$, *see* (3) *is the greatest ungrounded subobject.*

Proof. Let $e\colon X \longrightarrow R(X+A)$ be a rational equation morphism. By assumption, X is a coproduct of k indecomposable objects, $X = Y_1 + \cdots + Y_k$. For every coproduct injection $z\colon Z \longrightarrow X$ we obtain the corresponding morphisms $z_i\colon Z_i \longrightarrow Y_i$ with $Z = Z_1 + \cdots + Z_k$ and $z = z_1 + \cdots + z_k$. Since each z_i is a coproduct injection of Y_i, either $Z_i = 0$ or $Z_i = Y_i$. Consequently, there are (in case $Y_i \not\cong 0$ for every i) precisely 2^k subjects of X which are coproduct injections. Since the subobjects $\overline{X}_n \rightarrowtail X$, $n \in \mathbb{N}$, are pairwise disjoint, it follows that there exists an $m \in \mathbb{N}$ such that $\overline{X}_m \cong 0$. Thus $X_m \cong X_{m+1} + \overline{X}_{m+1} \cong X_{m+1}$. The proof that the intersection of derived subobjects is the greatest ungrounded subobject is as in Proposition 4.7. ∎

6.5. Definition. *A rational equation morphism* e *is called* ***preguarded*** *provided that it has no nontrivial ungrounded subobject, i.e.,* $X_n \cong 0$ *for some* n.

6.6. Remark. This is equivalent to $X_\infty = 0$, thus, e is preguarded iff $X = \coprod_{n\geq 1} \overline{X}_n$. This is analogous to Proposition 4.7.

6.7. Theorem. *In every iterative algebra all preguarded rational equation morphisms have unique solutions.*

Proof. This is completely analogous to the proof in Section 3, see Theorem 4.12 and Corollary 4.14. Given the preguarded rational equation morphism $e\colon X \longrightarrow R(X+A)$, we have n with $X_n = 0$, i.e., $X = \overline{X}_1 + \cdots + \overline{X}_n$ and we define a guarded equation morphism

$$f \equiv \overline{X}_1 \xrightarrow{\bar{i}_1} X \xrightarrow{e} R(X+A) \xrightarrow{R(u+A)} R(\overline{X}_1 + A)$$

where $u\colon X \longrightarrow \overline{X}_1$ has components $\mathrm{id}_{\overline{X}_1}, \bar{e}_1, \bar{e}_1{\cdot}\bar{e}_2, \ldots, \bar{e}_1{\cdot}\bar{e}_2{\cdot}\cdots{\cdot}\bar{e}_n$. Observe that since u is a split epimorphism and X is finitely presentable, so is $\overline{X}_1$. Thus, f is a rational equation morphism. Since f is guarded, it has a unique solution $f^\dagger\colon X \longrightarrow A$, see Remark 5.6. The rest is as in Section 3. ■

6.8. Definition. *Let $e\colon X \longrightarrow R(X+A)$ be a rational equation morphism in an iterative algebra A. A solution $e^\dagger\colon X \longrightarrow A$ of e is called **strict** if its restriction to every ungrounded subobject is $\perp_A$. Equivalently, the square*

$$\begin{array}{ccc} X_n & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle i_n}\big\downarrow & & \big\downarrow{\scriptstyle \perp_A} \\ X & \xrightarrow[e^\dagger]{} & A \end{array}$$

where X_n is the largest ungrounded subobject, commutes.

6.9. Theorem. *In every iterative algebra every finitary equation morphism has a unique strict solution.*

Proof. This is completely analogous to Section 4, see Theorem 5.7 and Corollary 5.9: choose n such that $X_n = X_{n+1}$, see Lemma 6.4, then the role of X_∞ in Section 4 is now played by X_n. ■

6.10. Example. In the category $\mathscr{JT}$ of Jónsson-Tarski algebras we present a preguarded morphism with infinitely many (strict) solutions. This demonstrates that hyper-extensivity is crucial: recall from Example 2.9 that $\mathscr{JT}$ is a locally finitely presentable topos!

We use the constant endofunctor H with value 1. Thus, an H-algebra is a pointed Jónsson-Tarski algebra (given by an object A of $\mathscr{JT}$ and a morphism $a\colon 1 \longrightarrow A$) or a Jónsson-Tarski algebra A with a specified idempotent $a \in A$. Every algebra is a cia, see Example 3.3(ii). The free H-algebra on A is of course $A+1$ and thus equations in an algebra A have the form

$$e\colon X \longrightarrow X + A + 1 \qquad \text{for } X \text{ finitely presentable.}$$

In particular we can use as X the free Jónsson-Tarski algebra $\Phi(1)$ on one generator $g = g' * g''$, see 2.9, and we have, for every algebra A, the following equation morphism

$$e_A\colon \Phi(1) \longrightarrow \Phi(1) + A + 1, \qquad e_A(g) = g' * \perp.$$

(The right-hand side is the result of the operation $*$ applied to $g' \in \Phi(1)$ and $\perp$, the unique element of 1,)

(a) The equation e_A is preguarded. In fact, recall that $\Phi(1) = \Phi(1) + \Phi(1)$ with coproduct injections l, r, see 2.9. The squares

$$\begin{array}{ccccc}
\Phi(1) & \xrightarrow{\;l\;} & \Phi(1) & \xleftarrow{\;r\;} & \Phi(1) \\
 & & & & \downarrow{\scriptstyle !} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle e} & & 1 \\
 & & & & \downarrow{\scriptstyle \mathrm{inr}} \\
\Phi(1) & \xrightarrow[\mathrm{inl}]{} & \Phi(1) + A + 1 & \xleftarrow[\mathrm{inr}]{} & A + 1
\end{array}$$

are pullbacks. This follows from the extensivity of $\mathscr{J}\mathscr{T}$: observe that both squares clearly commute and the lower and upper horizontal parts are coproducts. Thus, $x_0 = \mathrm{inl}$ implies $x_1 = l$. This shows that all derived subobjects are composites of l:

$$\begin{array}{ccccccccc}
\cdots & \Phi(1) & \xrightarrow{\;l\;} & \Phi(1) & \xrightarrow{\;l\;} & \Phi(1) & \xrightarrow{\;l\;} & \Phi(1) \\
 & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle e_A} \\
\cdots & \Phi(1) & \xrightarrow[l]{} & \Phi(1) & \xrightarrow[l]{} & \Phi(1) & \xrightarrow[\mathrm{inl}]{} & \Phi(1) + A + 1
\end{array}$$

The intersection of all of those subobjects is clearly empty. Consequently, e_A has no nontrivial ungrounded subobject: by Remark 4.9 every ungrounded subobject factorizes through the empty (initial) object and is thus empty.

(b) For some algebras A there exist infinitely many solutions of e_A. For example, choose $A = \mathbb{Z}$ and let the binary operation be any bijection

$$*\colon \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \qquad \text{with} \qquad n * 0 = n \text{ for all } n \in \mathbb{N}.$$

Since 0 is an idempotent, we consider $(\mathbb{Z}, *)$ as an H-algebra. Every natural number $n = n * 0$ yields a solution

$$e^\dagger_{\mathbb{Z}}\colon \Phi(1) \longrightarrow \mathbb{Z}, \qquad g \longmapsto n.$$

Conclusions and Future Research

Iterative algebras A are more iterative than what follows immediately from their definition: every equation morphism possesses a unique strict solution in A. This has been known for algebras in **Set**, see the work of L. Moss [Mo] and S. L. Bloom, C. C. Elgot and J. B. Wright [BEW_1, BEW_2]. In the present paper we proved this in a general setting (by using a new technique): we worked in locally presentable categories where objects are coproducts of connected components. If $\mathscr{A}$ is such a category, then so is every presheaf category, thus, also the category

$$\mathrm{Fin}[\mathscr{A}, \mathscr{A}]$$

of all finitary endofunctors on $\mathscr{A}$.

In the future research we will apply our result to the study of monadic algebras of the monad

$$\mathrm{Rat} \qquad \text{on} \qquad \mathrm{Fin}[\mathscr{A}, \mathscr{A}]$$

which assigns to every finitary endofunctor H the free iterative monad $\mathrm{Rat}(H)$ on H (in the sense of Elgot). This monad $\mathrm{Rat}(H)$ exists as proved in [AMV_1]. We hope to describe the monadic algebras for Rat as "Elgot monads" which are analogous to the Elgot algebras studied in [AMV_3].

Another application of the results of the present paper is within the analysis of the relationship of iterative algebras and iteration algebras of Zoltán Ésik, see [ABM].

The assumption that objects are coproducts of their connected components is somewhat restrictive. However, it cannot be completely lifted since in the last example we presented a locally finitely presentable category which, although it is a topos, has non-unique strict solutions of equation morphisms.

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Jiří Adámek and Stefan Milius
Technical University of Braunschweig, Germany

Reinhard Börger
Fernuniversität Hagen, Germany

Jiří Velebil
Technical University of Prague, Czech Republic

Email: `Adamek@iti.cs-tu.bs.de`
`Reinhard.Boerger@Fernuni-Hagen.de`
`mail@stefan-milius.eu`
`Velebil@math.feld.cvut.cz`

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