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EXPONENTIABILITY IN HOMOTOPY SLICES OF **TOP** AND PSEUDO-SLICES OF **CAT**

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Abstract. We prove a general theorem relating pseudo-exponentiable objects of a bicategory $\mathcal{K}$ to those of the Kleisli bicategory of a pseudo-monad on $\mathcal{K}$. This theorem is applied to obtain pseudo-exponentiable objects of the homotopy slices $\mathbf{Top}//B$ of the category of topological spaces and the pseudo-slices $\mathbf{Cat}//\mathbf{B}$ of the category of small categories.

1. Introduction

The 2-slice $\mathbf{Top}/B$ is the 2-category whose objects are continuous maps $p\colon X \to B$, morphisms are commutative triangles

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

and 2-cells are equivalence classes $\{F\}$ of homotopies $F\colon f \to f'$ over B, i.e., commutative triangles

$$\begin{array}{ccc} X \times I & \xrightarrow{F} & Y \\ & {\scriptstyle p\pi_1}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

such that $F|_{X\times 0} = f$ and $F|_{X\times 1} = f'$, where $F \sim F'$ if there is a homotopy $\Phi\colon F \to F'$ over B such that $\Phi|_{X\times 0\times I} = f$ and $\Phi|_{X\times 1\times I} = f'$. Note that the use of "fiberwise homotopies" here makes $X \times_B Y$ into the 2-product of $p\colon X \to B$ and $q\colon Y \to B$ in $\mathbf{Top}/B$.

Exponentiability results for the 1-category $\mathbf{Top}/B$ (see [14]) easily generalize to dimension 2 (and higher). In particular, given an exponentiable map $q\colon Y \to B$, the natural bijections

$$\theta_{X,Z}\colon \mathbf{Top}/B(X \times_B Y, Z) \longrightarrow \mathbf{Top}/B(X, Z^Y) \tag{1}$$

are 2-natural isomorphisms of categories, or equivalently, the adjunction $- \times_B Y \dashv (\ )^Y$ is a 2-adjunction, in the sense of [10].

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Working with commutative triangles over B is at times too rigid, since constructions are made in the fibers of maps over B. One can relax this restriction by taking morphisms to be triangles

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \ \overrightarrow{\varphi}\ \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

which commute up to specified homotopy φ. However, composition of triangles is then neither associative nor unital, since composition of homotopies is only associative and unital up to homotopy. This can be rectified by imposing an equivalence relation on the triangles, as is done in [9], but one loses the distinct homotopies. An alternative is to move to the realm of weak 2-categories, i.e., bicategories, in the sense of Benabou [1]. But, what is a suitable choice of 2-cell?

Given a triangle

$$\begin{array}{ccc} X \times I & \xrightarrow{F} & Y \\ & {\scriptstyle p\pi_1}\searrow \ \overrightarrow{\Phi}\ \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

restricting to $X \times t$, we get a triangle

$$\begin{array}{ccc} X & \xrightarrow{F_t} & Y \\ & {\scriptstyle p}\searrow \ \overrightarrow{\Phi_t}\ \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

and hence, a continuous family of homotopies from $(f, \varphi) = (F_0, \Phi_0)$ to $(f', \varphi') = (F_1, \Phi_1)$. We will see that taking suitable equivalence classes of these families gives rise to a bicategory $\mathbf{Top}//B$, which we call a *homotopy slice* of **Top**.

For exponentiability of $q\colon Y \to B$ in $\mathbf{Top}//B$, the role of the fiber product in $\mathbf{Top}/B$ will be played by the homotopy pullback $X \times_B B^I \times_B Y$, where the map $B^I \to B$ is evaluation at 0 (denoted by ev_0) when B^I appears on the right of $\times_B$, and evaluation at 1 (denoted by ev_1) when B^I appears on the left. The existence of a right adjoint to the functor

$$(X \xrightarrow{p} B) \ \mapsto \ (X \times_B B^I \times_B Y \xrightarrow{p\pi_1} B)$$

is rare when considered as an endofunctor of $\mathbf{Top}/B$ (see [15]). However, in the context of bicategories, it is more appropriate to consider pseudo-adjoints (or equivalently, biadjoints in the sense of Street [17]), and thus to replace the isomorphisms $\theta_{X,Z}$ in (1) by pseudo-natural equivalences of categories. We will see that there are many pseudo-exponentiable maps $q\colon Y \to B$ in $\mathbf{Top}//B$ for which $ev_0\pi_1\colon B^I \times_B Y \to B$ is pseudo-exponentiable in $\mathbf{Top}/B$.

When considering $\mathbf{Top}//B$, to avoid cumbersome verification of details, it is useful to work in a more general setting and call upon an analogy with pseudo-slices of the 2-category **Cat** of small categories. Since these pseudo-slices are themselves 2-categories, and not merely bicategories, it will be necessary to generalize some of the concepts involved.

Recall that $\mathbf{Cat}/\mathbf{B}$ is the 2-category whose objects are functors $p\colon \mathbf{X} \to \mathbf{B}$, morphisms are commutative triangles

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathbf{B} & \end{array}$$

and 2-cells are natural transformations $F\colon f \to f'$ such that $qF = id_p$. The pseudo-slice $\mathbf{Cat}/\!/\mathbf{B}$ has the same objects but the morphisms are triangles

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \\ & {\scriptstyle p}\searrow \overset{\to}{\varphi} \swarrow{\scriptstyle q} & \\ & \mathbf{B} & \end{array}$$

which commute up to a specified natural isomorphism, and 2-cells from (f, φ) to (f', φ') are natural transformations $F\colon f \to f'$ such that the following diagram commutes

$$\begin{array}{ccc} & p & \\ {\scriptstyle \varphi}\swarrow & & \searrow{\scriptstyle \varphi'} \\ qf & \xrightarrow[qF]{} & qf' \end{array}$$

Using a variation of a construction by Street [16], one can show that $\mathbf{Cat}/\!/\mathbf{B}$ is the Kleisli 2-category of a 2-monad on $\mathbf{Cat}/\mathbf{B}$. This construction cannot be applied to the 2-category $\mathbf{Top}$, since $\mathbf{Top}$ it is not representable and the 2-cells of $\mathbf{Top}/B$ differ from those arising in [16]. Moreover, changing the 2-cell would result in a loss of 2-products in $\mathbf{Top}/B$. However, using an analogous construction, we will see that $\mathbf{Top}/\!/B$ is the Kleisli bicategory of a pseudo-monad on $\mathbf{Top}/B$.

Pseudo-exponentiability in $\mathbf{Cat}/\!/\mathbf{B}$ was considered by Johnstone in [6], where it is stated that $q\colon \mathbf{Y} \to \mathbf{B}$ is pseudo-exponentiable if and only if q satisfies a certain factorization lifting property, called FPL in Section 5 below. Only a sketch of the sufficiency proof is given in [6], since it is analogous to that of Conduché [3] and Giraud [5] for $\mathbf{Cat}/\mathbf{B}$, and the necessity proof is completely omitted, as it is not relevant to the paper. Taking $\mathbf{I}$ to be the category with objects 0 and 1 and a single isomorphism between them, it turns out that $q\colon \mathbf{Y} \to \mathbf{B}$ is pseudo-exponentiable in $\mathbf{Cat}/\!/\mathbf{B}$ if and only if $\mathbf{B}^{\mathbf{I}} \times_{\mathbf{B}} \mathbf{Y} \xrightarrow{ev_0\pi_1} \mathbf{B}$ is 2-exponentiable in $\mathbf{Cat}/\mathbf{B}$, and the latter is easily seen to be equivalent to the relevant factorization lifting property. Moreover, the sufficiency of FPL can be established via a general theorem about Kleisli bicategories of pseudo-monoids, which will also be applied to obtain examples of pseudo-exponentiable objects in $\mathbf{Top}/\!/B$.

We begin with a presentation of $\mathbf{Top}/\!/B$ and $\mathbf{Cat}/\!/\mathbf{B}$ as the Kleisli bicategories of the related 2-slice categories. In section three, we show that if T is a pseudo-monad on a bicategory $\mathcal{K}$ (satisfying certain properties which hold in the examples under consideration), and TY is pseudo-exponentiable in $\mathcal{K}$, then Y is pseudo-exponentiable in the Kleisli bicategory $\mathcal{K}_T$. We conclude in sections four and five with applications yielding pseudo-exponentiable of objects of $\mathbf{Top}//B$ and $\mathbf{Cat}//\mathbf{B}$. In particular, we show that every exponentiable (Hurewicz) fibration is pseudo-exponentiable in $\mathbf{Top}//B$ and every FPL functor is pseudo-exponentiable in $\mathbf{Cat}//\mathbf{B}$.

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2. The Kleisli bicategory of a pseudo-monad

In this section, we exhibit $\mathbf{Top}//B$ and $\mathbf{Cat}//\mathbf{B}$ as the Kleisli bicategories of pseudo-monads on $\mathbf{Top}/B$ and $\mathbf{Cat}/\mathbf{B}$, respectively.

Recall that a *2-monad* on a 2-category $\mathcal{K}$ consists of a 2-functor $T\colon\mathcal{K}\to\mathcal{K}$ together with 2-natural transformations $\eta\colon id_{\mathcal{K}}\to T$ and $\mu\colon T^2\to T$ such that

$$\mu(T\eta)=id_T \qquad \mu(\eta T)=id_T \qquad \mu(T\mu)=\mu(\mu T) \tag{2}$$

The *Kleisli 2-category* $\mathcal{K}_T$ of T is the 2-category whose objects are the same as those of $\mathcal{K}$, and $\mathcal{K}_T(X,Y)=\mathcal{K}(X,TY)$ with $id_X=\eta_X$ and composition induced by μ. Moreover, the 2-functor $U\colon\mathcal{K}\to\mathcal{K}_T$, given by the identity on objects and composition with η_Y on morphisms and 2-cells, has a right 2-adjoint $T\colon\mathcal{K}_T\to\mathcal{K}$ given by $X\mapsto TX$ and

$$\mathcal{K}_T(X,Y)=\mathcal{K}(X,TY)\longrightarrow\mathcal{K}(TX,T^2Y)\xrightarrow{\mathcal{K}(TX,\mu_Y)}\mathcal{K}(TX,TY)$$

For example, take $\mathcal{K}=\mathbf{Cat}/\mathbf{B}$, and let $\mathbf{I}$ be as in the introduction. Then there is an internal category (in the sense of [7])

$$\mathbf{B}^{\mathbf{I}}\times_{\mathbf{B}}\mathbf{B}^{\mathbf{I}}\xrightarrow{c}\mathbf{B}^{\mathbf{I}}\underset{ev_1}{\overset{ev_0}{\underset{\xleftarrow{i}}{\rightrightarrows}}}\mathbf{B}$$

in $\mathbf{Cat}$, where ev_0 and ev_1 denote the evaluation functors at 0 and 1, respectively, i is the functor $B\mapsto id_B$, and c is the composition functor. Note that, as in $\mathbf{Top}$, we write $\mathbf{B}^{\mathbf{I}}$ on the left of $\times_{\mathbf{B}}$ when $ev_1\colon\mathbf{B}^{\mathbf{I}}\to\mathbf{B}$, and on the right when $ev_0\colon\mathbf{B}^{\mathbf{I}}\to\mathbf{B}$. Define

$$T(\mathbf{X}\xrightarrow{p}\mathbf{B})=\mathbf{B}^{\mathbf{I}}\times_{\mathbf{B}}\mathbf{X}\xrightarrow{ev_0\pi_1}\mathbf{B}$$

with the induced maps on morphisms and 2-cells, and η and μ given by

$$\mathbf{X}\xrightarrow{<i_p,id_{\mathbf{X}}>}\mathbf{B}^{\mathbf{I}}\times_{\mathbf{B}}\mathbf{X}\quad\text{and}\quad\mathbf{B}^{\mathbf{I}}\times_{\mathbf{B}}\mathbf{B}^{\mathbf{I}}\times_{\mathbf{B}}\mathbf{X}\xrightarrow{c\times id_{\mathbf{X}}}\mathbf{B}^{\mathbf{I}}\times_{\mathbf{B}}\mathbf{X}$$

Then it is not difficult to show that **Cat**//**B** is (isomorphic to) the Kleisli 2-category of T.

To obtain the homotopy slices **Top**$//B$, a slight modification is necessary. We can start with $\mathcal{K} = \mathbf{Top}/B$ and consider

$$B^I \times_B B^I \xrightarrow{c} B^I \underset{ev_1}{\overset{ev_0}{\rightrightarrows}} B \quad (i \colon B \to B^I)$$

Since composition is associative and unital only up to homotopy, this is not an internal category in **Top**. We still get a 2-functor $T\colon \mathbf{Top}//B \to \mathbf{Top}//B$ given by

$$T(X \xrightarrow{p} B) = B^I \times_B X \xrightarrow{ev_0\pi_1} B$$

and 2-natural transformations

$$\eta_p\colon X \xrightarrow{<i_p, id_X>} B^I \times_B X \quad \text{and} \quad \mu_p\colon B^I \times_B B^I \times_B X \xrightarrow{c \times id_X} B^I \times_B X$$

but this is not a 2-monad since the equations in (2) do not hold. Instead, we have invertible modifications

$$\begin{array}{ccccc} T & \xrightarrow{T\eta} & T^2 & \xleftarrow{\eta T} & T \\ & {\scriptstyle id_T}\searrow \overset{l}{\Leftarrow} & \downarrow{\scriptstyle \mu} & \overset{r}{\Rightarrow} \swarrow{\scriptstyle id_T} & \\ & & T & & \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & \overset{t}{\Rightarrow} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \tag{3}$$

which satisfy the coherence conditions in the definition of a pseudo-monad on a bicategory (c.f., [12, 2]). Note that the definition of a pseudo-monad does not require that T, η, μ be strict, as they are here, just pseudo-functors and pseudo-natural transformations.

The Kleisli construction $\mathcal{K}_T$ is defined for any pseudo-monad on a bicategory $\mathcal{K}$, but it is merely a bicategory (even when $\mathcal{K}$ is a 2-category and T, η, μ are strict) since the equations in (2) have been replaced by the modifications in (3). Moreover, we get a pseudo-adjoint pair

$$\mathcal{K} \underset{T}{\overset{U}{\rightleftarrows}} \mathcal{K}_T$$

defined as above.

3. Exponentiability in Kleisli bicategories

In this section, we discuss pseudo-exponentiability of the Kleisli bicategory of a pseudo-monad T on a bicategory $\mathcal{K}$. Through a series of lemmas (whose hypotheses are satisfied by the relevant pseudo-monads on **Top**$//B$ and **Cat**//**B**), we show that Y is pseudo-exponentiable in $\mathcal{K}_T$, provided that TY is pseudo-exponentiable in $\mathcal{K}$.

Recall that a diagram

$$\begin{array}{ccc} & X \times Y & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & & Y \end{array}$$

is a called a *pseudo-product* in a bicategory $\mathcal{K}$ if the induced functor

$$\pi_Z : \mathcal{K}(Z, X \times Y) \to \mathcal{K}(Z, X) \times \mathcal{K}(Z, Y)$$

is an equivalence of categories, for all objects Z. Note that since π_Z is pseudo-natural in Z, so is its pseudo-inverse.

3.1. LEMMA. *If $\mathcal{K}$ is a bicategory with binary pseudo-products and T, η, μ is a pseudo-monad on $\mathcal{K}$ such that the canonical map*

$$\rho : T(X \times TY) \xrightarrow{<T\pi_1, \mu_Y T\pi_2>} TX \times TY$$

is an equivalence in $\mathcal{K}$, for all X, Y, then $X \times TY$ is the pseudo-product of X and Y in $\mathcal{K}_T$.

PROOF. The functor $\mathcal{K}_T(Z, X \times TY) \to \mathcal{K}_T(Z, X) \times \mathcal{K}_T(Z, Y)$ is an equivalence of categories since it factors as a composite

$$\mathcal{K}(Z, T(X \times TY)) \xrightarrow{\mathcal{K}(Z,\rho)} \mathcal{K}(Z, TX \times TY) \xrightarrow{\pi_Z} \mathcal{K}(Z, TX) \times \mathcal{K}(Z, TY)$$

of equivalences. ■

Returning to our examples, the functor

$$\rho : \mathbf{B^I} \times_{\mathbf{B}} \mathbf{X} \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y} \to (\mathbf{B^I} \times_{\mathbf{B}} \mathbf{X}) \times_{\mathbf{B}} (\mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y})$$

is an isomorphism in **Cat**/**B**, and the map

$$\rho : B^I \times_B X \times_B B^I \times_B Y \to (B^I \times_B X) \times_B (B^I \times_B Y)$$

is an equivalence in **Top**/B. In fact, both are defined by

$$(b \xrightarrow{\alpha} px, x, px \xrightarrow{\beta} qy, y) \;\mapsto\; ((b \xrightarrow{\alpha} px, x), (b \xrightarrow{\alpha} px \xrightarrow{\beta} qy, y))$$

Thus, $\mathbf{X} \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y}$ is the product in **Cat**//**B** and $X \times_B B^I \times_B Y$ is the pseudo-product in **Top**//B.

Recall that an object Y is *pseudo-exponentiable* in a bicategory $\mathcal{K}$ if the pseudo-functor $- \times Y : \mathcal{K} \to \mathcal{K}$ has a right pseudo-adjoint (i.e., a biadjoint in the sense of Street [17]), or equivalently, for every object Z, these is an object Z^Y together with equivalence

$$\theta_{X,Z} : \mathcal{K}(X \times Y, Z) \longrightarrow \mathcal{K}(X, Z^Y)$$

which are pseudo-natural in X and Z.

Note that any object equivalent to a pseudo-exponentiable one is necessarily pseudo-exponentiable, where Y is *equivalent* to Y', written $Y \simeq Y'$, in a bicategory if there exist $f\colon Y \to Y'$ and $g\colon Y' \to Y$ such that $fg \cong id_{Y'}$ and $gf \cong id_Y$. Moreover, if $\mathcal{K}$ is a 2-category, $Y \simeq Y'$, and Y is 2-exponentiable in $\mathcal{K}$, then composing with the natural isomorphisms $\mathcal{K}(X \times Y, Z) \to \mathcal{K}(X, Z^Y)$ with the equivalences $\mathcal{K}(X \times Y', Z) \to \mathcal{K}(X \times Y, Z)$ gives the pseudo-exponentiability of Y' in $\mathcal{K}$.

Returning to the general case, suppose TY is pseudo-exponentiable in $\mathcal{K}$, and consider the following pseudo-natural transformations

$$\mathcal{K}(X \times TY, TZ) \longrightarrow \mathcal{K}(T(X \times TY), T^2Z) \xrightarrow{\mathcal{K}(T(X\times TY),\mu)}$$

$$\mathcal{K}(T(X \times TY), TZ) \xrightarrow{\simeq} \mathcal{K}(TX \times TY, TZ) \xrightarrow{\simeq} \mathcal{K}(TX, TZ^{TY}) \xrightarrow{\mathcal{K}(TX,\eta)}$$

$$\mathcal{K}(TX, T(TZ^{TY})) \xleftarrow{\mathcal{K}(TX,\mu)} \mathcal{K}(TX, T^2(TZ^{TY})) \longleftarrow \mathcal{K}(X, T(TZ^{TY}))$$

where the first and last functors are given by T. If we can show that these are all equivalences of categories, then we will have an equivalence

$$\mathcal{K}_T(X \times TY, Z) \longrightarrow \mathcal{K}_T(X, TZ^{TY}) \tag{4}$$

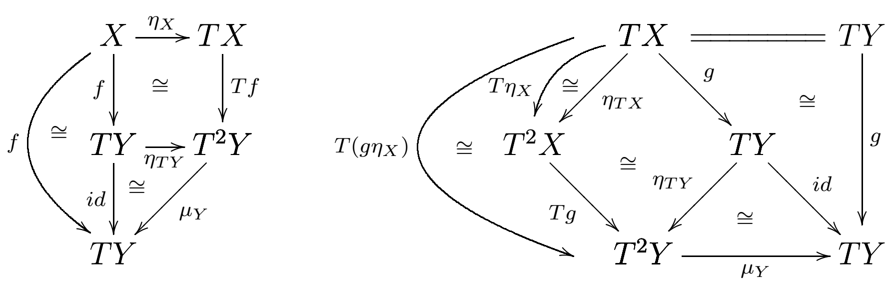
which is pseudo-natural in X, giving the pseudo-exponentiability of Y in $\mathcal{K}_T$.

3.2. Lemma. *If T, η, μ is a pseudo-monad on a bicategory $\mathcal{K}$ and $\eta T \cong T\eta$, then*

$$\tau_{X,Y}\colon \mathcal{K}(X, TY) \longrightarrow \mathcal{K}(TX, T^2Y) \xrightarrow{\mathcal{K}(TX,\mu_Y)} \mathcal{K}(TX, TY)$$

is an equivalence of categories, for all X, Y.

Proof. Consider $\tau'_{X,Y}\colon \mathcal{K}(TX, TY) \xrightarrow{\mathcal{K}(\eta_X,TY)} \mathcal{K}(X, TY)$. To see that $\tau'_{X,Y}$ is a pseudo-inverse of $\tau_{X,Y}$, given $f\colon X \to TY$ and $g\colon TX \to TY$, let $\theta_f\colon f \to \tau'\tau f$ and $\theta'_g\colon \tau\tau' g \to g$ be defined by the invertible 2-cells given by the diagrams



Then naturality of θ and θ' follows from coherence and pseudo-naturality of η and μ, and so $\tau_{X,Y}$ is an equivalence of categories. ∎

Note that since $\tau_{X,Y}: \mathcal{K}_T(X,Y) \to \mathcal{K}(TX,TY)$ is pseudo-natural in X, so is $\tau'_{X,Y}$, and it follows that we have a pseudo-natural transformation

$$\mathcal{K}_T(-,Y) \longrightarrow \mathcal{K}(T(-),TY)$$

considered as pseudo-functors from $\mathcal{K}_T$ to **Cat**.

Returning to the examples, we see that $\eta T \cong T\eta$ in both case. For $\mathbf{Top}/B$, given $p: X \to B$, define $F: B^I \times_B X \times I \to B^I \times_B B^I \times_B X$ by

$$F(\beta, x, t) = (\beta|_{[0,t]}, \beta|_{[t,1]}, x)$$

where $\beta|_{[0,t]}(u) = \beta(ut)$ and $\beta|_{[t,1]}(u) = \beta(u + t - ut)$. Then F is clearly a continuous map over B, $F(\beta, x, 0) = (i_b, \beta, x) = \eta_{TX}(\beta, x)$, and $F(\beta, x, 1) = (\beta, i_{px}, x) = T\eta_X(\beta, x)$, and it follows that $\eta T \cong T\eta$. For $\mathbf{Cat}/\mathbf{B}$, given $p: \mathbf{X} \to \mathbf{B}$, note that

$$\eta_{Tp}, T\eta_p: \mathbf{B^I} \times_{\mathbf{B}} \mathbf{X} \to \mathbf{B^I} \times_{\mathbf{B}} \mathbf{B^I} \times_B \mathbf{X}$$

are given by

$$(\beta, X) \mapsto (id_B, \beta, X) \quad (\beta, X) \mapsto (\beta, id_{pX}, X)$$

respectively. Then it is not difficult to show that

$$m_p(\beta, X) = ((id_B, \beta), (\beta, id_{pX}), id_X)$$

defines an invertible modification $m: \eta T \to T\eta$, as desired.

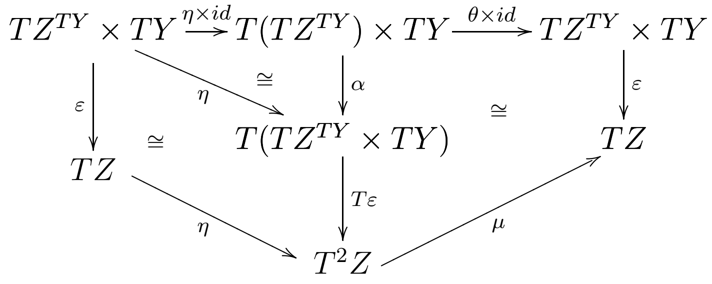
3.3. Lemma. *If T is as in Lemma 3.2 and TY is pseudo-exponentiable in $\mathcal{K}$, then $\eta: TZ^{TY} \to T(TZ^{TY})$ is an equivalence in $\mathcal{K}$, for all Z in $\mathcal{K}$.*

Proof. Let $\alpha_X: TX \times TY \to T(X \times TY)$ denote the pseudo-inverse of the canonical morphism, and let $\varepsilon: TZ^{TY} \times TY \to TZ$ the counit of the pseudo-adjunction. Then

$$T(TZ^{TY}) \times TY \xrightarrow{\alpha_{TZ^{TY}}} T(TZ^{TY} \times TY) \xrightarrow{T\varepsilon} T^2 Z \xrightarrow{\mu} TZ$$

induces a morphism $\theta: T(TZ^{TY}) \longrightarrow TZ^{TY}$.

To see that $\theta\eta \cong id_{TZ^{TY}}$, first note that $\alpha_X(\eta_X \times id) \cong \eta_{X \times TY}$, for all X via the pseudo-adjunction since their composites with $<T\pi_1, \mu_Y T\pi_2>$ are isomorphic. Using the invertible 2-cells



it follows that $\theta\eta \cong id_{TZ^{TY}}$.

Now, since η is pseudo-natural, we know

$$\begin{array}{ccc} T(TZ^{TY}) & \xrightarrow{\eta T} & T^2(TZ^{TY}) \\ {\scriptstyle\theta}\downarrow & \cong & \downarrow{\scriptstyle T\theta} \\ TZ^{TY} & \xrightarrow[\eta]{} & T(TZ^{TY}) \end{array}$$

and $\eta T \cong T\eta$, by assumption. Thus, $\eta\theta \cong (T\theta)(\eta T) \cong (T\theta)(T\eta) \cong T(\theta\eta) \cong T(id_{TZ^{TY}}) \cong id_{T(TZ^{TY})}$. ∎

3.4. Theorem. *Suppose that $\mathcal{K}$ is a bicategory with binary pseudo-products and T, η, μ is a pseudo-monad on $\mathcal{K}$ such that $\eta T \cong T\eta$ and $\rho\colon T(X \times TY) \to TX \times TY$ is an equivalence in $\mathcal{K}$, for all X, Y. If TY is pseudo-exponentiable in $\mathcal{K}$, then Y is pseudo-exponentiable in the Kleisli bicategory $\mathcal{K}_T$.*

Proof. Applying the three lemmas, we get a pseudo-natural transformation

$$\mathcal{K}_T(- \times TY, Z) \longrightarrow \mathcal{K}_T(-, TZ^{TY})$$

defined in (4) above, and so Y is pseudo-exponentiable in $\mathcal{K}_T$. ∎

Since **Top**$/B$ and **Cat**$/$**B** satisfy the hypotheses of this theorem, it will be applied in the following sections to obtain pseudo-exponentiable objects of the homotopy slice **Top**$//B$ and the pseudo-slice **Cat**$//$**B**.

4. Exponentiability in **Top**$//B$

In this section, we apply Theorem 3.4 to obtain pseudo-exponentiable objects of **Top**$//B$, including all fibrations which are exponentiable in **Top**$/B$.

Exponentiable objects of **Top**$/B$ were characterized in [14] as follows. Given a map $q\colon Y \to B$, a functor $(\ )^q\colon \mathbf{Top}/B \to \mathbf{Top}/B$ is defined together with natural transformations

$$\theta_{X,Z}\colon \mathbf{Top}/B(X \times_B Y, Z) \longrightarrow \mathbf{Top}/B(X, Z^Y)$$

where, by abuse of notation, Z^Y denotes the domain of r^q for $r\colon Z \to B$. Then q is exponentiable precisely when this functor is right adjoint to $- \times q$, i.e., when these functions $\theta_{X,Z}$ are bijections, if and only if q satisfies a certain technical condition. This condition yields examples of exponentiable maps including all local homeomorphisms, locally trivial maps with locally compact fibers, locally closed inclusions, and locally compact spaces over a locally Hausdorff space. It is not difficult to show that when q is exponentiable, each $\theta_{X,Z}$ is an isomorphism of categories, so that these are precisely the 2-exponentiable objects of the 2-slice **Top**$/B$. By the remarks following Lemma 3.1, these 2-exponentiable objects provide a source of potential pseudo-exponentiable objects of **Top**$/B$ and hence, by Theorem 3.4, pseudo-exponentiable objects of the homotopy slice **Top**$//B$.

4.1. PROPOSITION. *If $q: Y \to B$ is a (Hurewicz) fibration and q is exponentiable in* **Top**$/B$*, then $ev_0\pi_1: B^I \times_B Y \to B$ is pseudo-exponentiable in* **Top**$/B$*.*

PROOF. It suffices to show that $\eta_Y: Y \to B^I \times_B Y$ is an equivalence in **Top**$/B$, for then $q \simeq ev_0\pi_1$ and q is 2-exponentiable is **Top**$/B$, and so $ev_0\pi_1$ is pseudo-exponentiable in **Top**$/B$. Consider the commutative diagram

$$\begin{array}{ccc} B^I \times_B Y & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \langle id,1\rangle}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle q} \\ (B^I \times_B Y) \times I & \xrightarrow[ev \circ \pi_{13}]{} & B \end{array}$$

where H exists since q is a fibration. Define $\eta'_Y: B^I \times_B Y \to Y$ by

$$\eta'_Y(\beta, y) = H(\beta, y, 0)$$

Then η' is a pseudo-inverse of η and the desired result follows. ■

Thus, we get the following corollary of Theorem 3.4.

4.2. COROLLARY. *If $q: Y \to B$ is a fibration and q is exponentiable in* **Top**$/B$*, then q is pseudo-exponentiable in* **Top**$//B$*.*

Since pseudo-exponentiability is preserved by pseudo-equivalence, the following proposition yields additional examples.

4.3. PROPOSITION. *If $f: X \to Y$ is a homotopy equivalence in* **Top** *and $q: Y \to B$ then $qf \simeq q$ in* **Top**$//B$*.*

PROOF. Suppose $f: X \to Y$ is a homotopy equivalence. Then there exists $g: Y \to X$ such that $gf \cong id_X$ and $fg \cong id_Y$. Moreover, $F: gf \to id_X$ and $G: id_Y \to fg$ can be chosen so that $(fF)(Gf) \sim id_f$ and $(Fg)(gG) \sim id_g$, i.e., f and g are adjoint equivalences. Then the triangles

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \overrightarrow{id_{qf}} & \\ {\scriptstyle qf}\searrow & & \swarrow{\scriptstyle q} \\ & B & \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{g} & X \\ & \overrightarrow{qG} & \\ {\scriptstyle q}\searrow & & \swarrow{\scriptstyle qf} \\ & B & \end{array}$$

give rise to morphisms (f, id_{qf}) and (g, qG) of **Top**$//B$ such that

$$(f, id_{qf})(g, qG) \cong (fg, qG) \quad \text{and} \quad (g, qG)(f, id_{qf}) \cong (gf, qGf) \cong (gf, qfF^{-1})$$

where the latter isomorphism follows from $(fF)(Gf) \sim id_f$. Thus, it suffices to show that $(id_Y, id_q) \cong (fg, qG)$ and $(gf, qfF^{-1}) \cong (id_X, id_{qf})$.

To see that $(id_Y, id_q) \cong (fg, qG)$, consider

$$\begin{array}{ccc} Y \times I & \xrightarrow{G} & Y \\ & \overrightarrow{\Psi} & \\ {\scriptstyle q\pi_1}\searrow & & \swarrow{\scriptstyle q} \\ & B & \end{array}$$

where $\Psi(y,t,u) = qG(y,tu)$. Then $(G_0, \Psi_0) = (id_Y, id_q)$, since

$$(G(y,0), \Psi(y,0,u)) = (y, qG(y,0)) = (y, qy)$$

and $(G_1, \Psi_1) = (fg, qG)$, since $(G(y,1), \Psi(y,1,u)) = (fgy, qG(y,u))$.

To see that $(gf, qfF^{-1}) \cong (id_X, id_{qf})$, consider

$$\begin{array}{ccc} X \times I & \xrightarrow{F} & X \\ & \overrightarrow{\Phi} & \\ {\scriptstyle qf} \searrow & & \swarrow {\scriptstyle qf} \\ & B & \end{array}$$

where $\Phi(x,t,u) = qfF(x, 1-u+tu)$. Then $(F_0, \Phi_0) = (gf, qfF^{-1})$, since

$$(F(x,0), \Phi(x,0,u)) = (gfx, qfF(x,1-u))$$

and $(F_1, \Phi_1) = (id_X, id_{qf})$, since

$$(F(x,1), \Phi(x,1,u)) = (x, qfF(x,1)) = (x, qfx)$$

as desired. ■

4.4. Corollary. *If $f: X \to Y$ is a homotopy equivalence in* **Top** *and $q: Y \to B$ is an exponentiable fibration in* **Top**$/B$*, then qf is pseudo-exponentiable in* **Top**$//B$.

Proof. Since $qf \simeq q$ by Proposition 4.3 and q is pseudo-exponentiable by Corollary 4.2, it follows that qf is pseudo-exponentiable in **Top**$//B$. ■

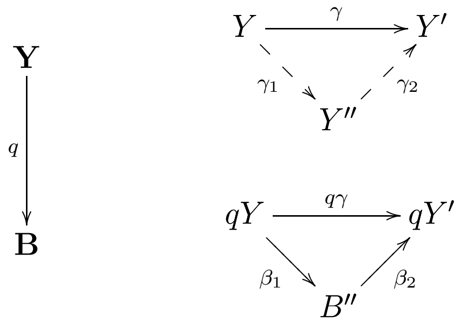
4.5. Corollary. *If $f: X \to Y$ is a homotopy equivalence in* **Top** *and $qf: X \to B$ is an exponentiable fibration in* **Top**$/B$*, then q is pseudo-exponentiable in* **Top**$//B$.

Proof. Since $qf \simeq q$ by Proposition 4.3 and qf is pseudo-exponentiable by Corollary 4.2, it follows that q is pseudo-exponentiable in **Top**$//B$. ■

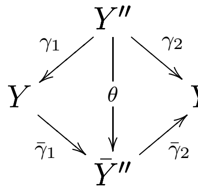
5. Exponentiability in Cat//B

In this section, we prove Johnstone's theorem [6] characterizing pseudo-exponentiable $\mathbf{Y} \to \mathbf{B}$ of the pseudo-slice **Cat**//**B**, and show that this is also equivalent to the exponentiability of $\mathbf{B}^{\mathbf{I}} \times_{\mathbf{B}} \mathbf{Y} \to \mathbf{B}$ in the 2-slice **Cat**/**B**.

Conduché [3] and Giraud [5] independently showed that the exponentiable objects of **Cat**//**B** are those $q: \mathbf{Y} \to \mathbf{B}$ satisfying the following factorization lifting property (FL). Given $\gamma: Y \longrightarrow Y'$ in **Y** and a factorization $q\gamma = \beta_2\beta_1$ in **B**, the following diagram can be completed

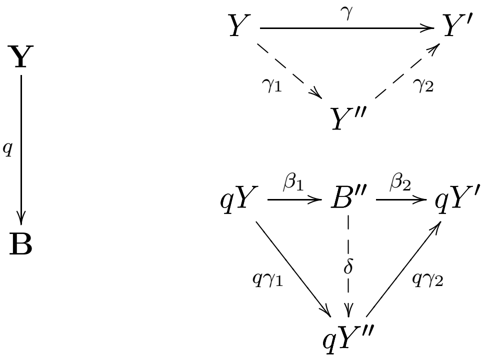


i.e., there exists an object Y'' over B'' and a factorization $\gamma = \gamma_2\gamma_1$ in $\mathbf{Y}$ such that $q\gamma_1 = \beta_1$ and $q\gamma_2 = \beta_2$. Moreover, this factorization is unique in the sense that any two such are equivalent via the equivalence relation generated by the relation $(Y'', \gamma_1, \gamma_2) \sim (\bar{Y}'', \bar{\gamma}_1, \bar{\gamma}_2)$ if there exists a morphism $\theta\colon Y'' \to \bar{Y}''$ over the identity making the following diagram commute



To Conduché and Giraud this was a 1-dimensional problem, but Johnstone [6] pointed out that FL also characterizes 2-exponentiable objects in the 2-slice **Cat**/**B**. As noted in the introduction above, he also defined the following factorization pseudo-lifting property (FPL) in [6] and sketched the proof of its sufficiency for pseudo-exponentiability in **Cat**//**B**. In this section, we prove Johnstone's theorem using Theorem 3.4 for the sufficiency of FPL and a variation of the proof in [13] for its necessity.

A functor $q\colon \mathbf{Y} \to \mathbf{B}$ satisfies the factorization pseudo-lifting property (FPL) if



i.e., given $\gamma\colon Y \to Y'$ in $\mathbf{Y}$ and a factorization $q\gamma = \beta_2\beta_1$ in $\mathbf{B}$, there exists a factorization $\gamma = \gamma_2\gamma_1$ in $\mathbf{Y}$ and an isomorphism $\delta\colon B'' \to qY''$ such that the diagram in $\mathbf{B}$ commutes. Moreover, this factorization is unique in the sense that any two such are equivalent via the equivalence relation generated by

$$(Y'', \gamma_1, \gamma_2, \delta) \sim (\bar{Y}'', \bar{\gamma}_1, \bar{\gamma}_2, \bar{\delta})$$

if there exists a morphism $\theta\colon Y'' \to \bar{Y}''$ such the following diagram commutes

$$\begin{array}{ccc} & B'' & \\ {\scriptstyle\delta}\swarrow & & \searrow{\scriptstyle\bar{\delta}} \\ qY'' & \xrightarrow[q\theta]{} & q\bar{Y}'' \end{array}$$

5.1. Theorem. *The following are equivalent.*

(a) $q\colon \mathbf{Y} \to \mathbf{B}$ *satisfies FPL.*

(b) $ev_0\pi_1\colon \mathbf{B}^{\mathbf{I}} \times_{\mathbf{B}} \mathbf{Y} \to \mathbf{B}$ *satisfies FL.*

(c) $ev_0\pi_1\colon \mathbf{B}^{\mathbf{I}} \times_{\mathbf{B}} \mathbf{Y} \to \mathbf{B}$ *is 2-exponentiable in* $\mathbf{Cat}/\mathbf{B}$.

(d) $q\colon \mathbf{Y} \to \mathbf{B}$ *is pseudo-exponentiable in* $\mathbf{Cat}//\mathbf{B}$.

Proof. It suffices to prove (a)⇒(b) and (d)⇒(a), for (b)⇒(c) is essentially the Conduché/Giraud theorem and (c)⇒(d) follows from Theorem 3.4 since 2-exponentiable objects of $\mathbf{Cat}/\mathbf{B}$ are necessarily pseudo-exponentiable.

For (a)⇒(b), suppose q satisfies FPL. To show $ev_0\pi_1$ satisfies FL, let

$$(\beta, q\gamma)\colon (B \xrightarrow{\alpha} qY, Y) \longrightarrow (B' \xrightarrow{\alpha'} qY', Y')$$

be a morphism of $\mathbf{B}^{\mathbf{I}} \times_{\mathbf{B}} \mathbf{Y}$ and suppose $\beta = \beta_2\beta_1$. Thus, we have a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ qY & \xrightarrow[q\gamma]{} & qY' \end{array}$$

Since

$$qY \xrightarrow{\alpha^{-1}} B \xrightarrow{\beta_1} B'' \xrightarrow{\beta_2} B' \xrightarrow{\alpha'} qY'$$

is a factorization of $q\gamma$, applying FPL for q we get a factorization

$$Y \xrightarrow{\gamma_1} Y'' \xrightarrow{\gamma_2} Y'$$

of γ in $\mathbf{Y}$ and an isomorphism $\delta\colon B'' \to qY''$ such that

$$\begin{array}{ccccc} qY & \xrightarrow{\beta_1\alpha^{-1}} & B'' & \xrightarrow{\alpha'\beta_2} & qY' \\ & {\scriptstyle q\gamma_1}\searrow & \downarrow{\scriptstyle\delta} & \nearrow{\scriptstyle q\gamma_2} & \\ & & qY'' & & \end{array}$$

commutes. Then $(B'' \xrightarrow{\delta} qY'', Y'')$ is an object of $\mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y}$ and the commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{\beta_1} & B'' & \xrightarrow{\beta_2} & B' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\alpha'} \\ qY & \xrightarrow[q\gamma_1]{} & qY'' & \xrightarrow[q\gamma_2]{} & qY' \end{array}$$

gives the desired factorization of $(\beta, q\gamma)$ in $\mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y}$. Moreover, the uniqueness condition of FL follows from that of FPL.

For (d)⇒(a), suppose $q: \mathbf{Y} \to \mathbf{B}$ is pseudo-exponentiable in $\mathbf{Cat}//\mathbf{B}$. Then $- \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y}$ preserves pseudo-pushouts (i.e., cocomma objects) in $\mathbf{Cat}//\mathbf{B}$. To show that q satisfies FPL, suppose $q\gamma = \beta_2\beta_1$ where $\gamma: Y \to Y'$. Then $\beta_2\beta_1$ induces a functor $p: \mathbf{X} \to B$, where $\mathbf{X}$ is defined by the pseudo-pushout

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{0} & \mathbf{2} \\ {\scriptstyle 1}\downarrow & \rightarrow & \downarrow \\ \mathbf{2} & \longrightarrow & \mathbf{X} \end{array} \tag{5}$$

and $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$ is the category with one morphism $0 \to 1$. Thus, (5) becomes a pseudo-pushout in $\mathbf{Cat}//\mathbf{B}$ via $p: \mathbf{X} \to \mathbf{B}$. Note that $\mathbf{X}$ can be constructed as the colimit of the diagram

$$\begin{array}{ccccc} & & \mathbf{I} & & \\ & {\scriptstyle 1}\swarrow & {\scriptstyle 0}\downarrow\;\downarrow{\scriptstyle 1} & \searrow{\scriptstyle 0} & \\ \mathbf{2} & & \mathbf{I} & & \mathbf{2} \end{array}$$

it follows that $\mathbf{X}$ is the category

$$\cdot \xrightarrow{\alpha_1} \cdot \cong \cdot \xrightarrow{\alpha_2} \cdot$$

with $p(\alpha_1) = \beta_1$ and $p(\alpha_2) = \beta_2$. Since $- \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y}$ preserves pseudo-pushouts, it follows that the diagram obtained by applying $- \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y}$ to (5) is a pseudo-pushout in $\mathbf{Cat}$. Thus, $\mathbf{X} \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y}$ is given by the colimit of

$$\begin{array}{ccccc} & & \mathbf{I} \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y} & & \\ & {\scriptstyle 1\times id}\swarrow & {\scriptstyle 0\times id}\downarrow\;\downarrow{\scriptstyle 1\times id} & \searrow{\scriptstyle 0\times id} & \\ \mathbf{2} \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y} & & \mathbf{I} \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y} & & \mathbf{2} \times_{\mathbf{B}} \mathbf{B^I} \times_{\mathbf{B}} \mathbf{Y} \end{array}$$

and so FPL follows from the construction of colimits in $\mathbf{Cat}$. ∎

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