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CHARACTERIZATION OF POINTED VARIETIES OF UNIVERSAL ALGEBRAS WITH NORMAL PROJECTIONS

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ABSTRACT. We characterize pointed varieties of universal algebras in which $(A \times B)/A \approx B$, i.e. all product projections are normal epimorphisms.

1. DEFINITION. *We will say that a pointed category* $\mathbf{C}$ *has normal projections if every product projection* $A \times B \to B$ *in* $\mathbf{C}$ *is a normal epimorphism.*

Equivalently, for any two objects A and B in such a category $\mathbf{C}$, forming the product $A \times B$ and then factoring it by $A \approx A \times 0$ results in B. In particular, every Jónsson-Tarski variety of universal algebras [3] (considered as a category) has this property; the same is true for the pointed subtractive varieties in the sense of Ursini [4].

The purpose of this paper is to characterize pointed varieties with normal projections (Theorem 3 below).

Before stating the theorem, we make a simple reformulation of Definition 1.

2. PROPOSITION. *Let* $\mathbf{C}$ *be a pointed variety. The following conditions are equivalent:*

(a) $\mathbf{C}$ *has normal projections;*

(b) there exists a natural number n*, such that for all* A *and* B *in* $\mathbf{C}$*, and for all* $a \in A$ *and* $b \in B$*,* $((a, b), (0, b)) \in R^n$*, where* R *is the reflexive homomorphic relation on* $A \times B$ *generated by the set* $\{((a', 0), (a'', 0))|\ a' = 0 \text{ or } a'' = 0\}$*;*

(c) let $F[x]$ *be the free algebra in* $\mathbf{C}$ *generated by* x*; there exists a natural number* n *such that* $((x, x), (0, x)) \in Q^n$*, where* Q *is the reflexive homomorphic relation on* $F[x] \times F[x]$ *generated by the set* $\{((x, 0), (0, 0)), ((0, 0), (x, 0))\}$*.*

Moreover, the number n *in (b) and in (c) can be supposed to be the same.*

3. THEOREM. *A pointed variety* $\mathbf{C}$ *has normal projections if and only if the corresponding theory contains*

- *unary terms* $t_1, ..., t_m$ *and* $u_1, ..., u_m$*;*
- $(m + 2)$*-ary terms* $v_1, ..., v_n$*;*

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and the following identities hold in $\mathbf{C}$*:*

- $x = v_1(t_1(x), ..., t_m(x), x, 0)$*;*
- $v_{i+1}(t_1(x), ..., t_m(x), x, 0) = v_i(t_1(x), ..., t_m(x), 0, x)$ *for each* $i \in \{1, ..., n-1\}$*;*
- $0 = v_n(t_1(x), ..., t_m(x), 0, x)$*;*
- $x = v_i(u_1(x), ..., u_m(x), 0, 0)$ *for each* $i \in \{1, ..., n\}$*.*

Moreover, for each n *this characterizes the pointed varieties of universal algebras which satisfy 2(b) (for the same* n*).*

PROOF. According to Proposition 2, we have to characterize pointed varieties for which $((x, x), (0, x)) \in Q^n$, i.e. for which there exist $s_1, ..., s_{n+1} \in F[x]$ such that $s_1 = x, s_{n+1} = 0$, and for each $i \in \{1, ..., n\}$ the pair $((s_i, x), (s_{i+1}, x))$ is in Q. On the other hand, $((s_i, x), (s_{i+1}, x)) \in Q$ if and only if for some $(m+2)$-ary term v_i and unary terms $t_1, ..., t_m$, $u_1, ..., u_m \in F[x]$, one has the equalities $(s_i, x) = v_i((t_1, u_1), ..., (t_m, u_m), (x, 0), (0, 0))$ and $(s_{i+1}, x) = v_i((t_1, u_1), ..., (t_m, u_m), (0, 0), (x, 0))$. Moreover, we can assume that the t's, u's and the number m are the same for each $i \in \{1, ..., n\}$. Writing the equalities above separately for the components of pairs, we obtain $s_i = v_i(t_1, ..., t_m, x, 0)$, $s_{i+1} = v_i(t_1, ..., t_m, 0, x)$, $x = v_i(u_1, ..., u_m, 0, 0)$. Since s's are expressed by v's, we may omit them; after this the identities become exactly as in the formulation of the theorem. ■

4. EXAMPLE. Let $\mathbf{C}$ be a pointed variety with normal projections, for which we could take $n = 1$ in Theorem 3. Then, the theory corresponding to $\mathbf{C}$ has unary terms $t_1, ..., t_m, u_1, ..., u_m$ and an $(m+2)$-ary term v, which satisfy the identities

$$x = v(t_1(x), ..., t_m(x), x, 0), \;\; x = v(u_1(x), ..., u_m(x), 0, 0), \;\; 0 = v(t_1(x), ..., t_m(x), 0, x).$$

When the unary terms are either x or 0, an easy argument shows that we could rewrite these identities as

$$x = w(x, 0, x, x, 0), \;\; x = w(x, x, 0, 0, 0), \;\; 0 = w(x, 0, x, 0, x).$$

If the term $w = w(x_1, x_2, x_3, x_4, x_5)$ depends only on the first variable x_1 and the last variable x_5, then we can write $w(x_1, x_2, x_3, x_4, x_5) = x_1 + x_5$ and our identities become

$$x = x + 0, \;\; 0 = x + x;$$

in this case the variety $\mathbf{C}$ becomes nothing but a pointed subtractive variety in the sense of Ursini [4]. On the other hand, $w(x_1, x_2, x_3, x_4, x_5) = x_2 + x_4$ would give

$$x = 0 + x, \;\; x = x + 0$$

which defines a Jónsson-Tarski variety [3].

5. Remark. It is known that every semi-abelian category has normal projections (see Condition SA*3a in [2]). More generally, every pointed category in which every pair of canonical morphisms $(1_A, 0) : A \to A \times B$, $(0, 1_B) : B \to A \times B$ is jointly epimorphic has this property. In particular, this is the case for the unital categories in the sense of Bourn [1].

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