

## CHARACTERIZATION OF POINTED VARIETIES OF UNIVERSAL ALGEBRAS WITH NORMAL PROJECTIONS

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ABSTRACT. We characterize pointed varieties of universal algebras in which  $(A \times B)/A \approx B$ , i.e. all product projections are normal epimorphisms.

1. DEFINITION. We will say that a pointed category  $\mathbf{C}$  has normal projections if every product projection  $A \times B \rightarrow B$  in  $\mathbf{C}$  is a normal epimorphism.

Equivalently, for any two objects  $A$  and  $B$  in such a category  $\mathbf{C}$ , forming the product  $A \times B$  and then factoring it by  $A \approx A \times 0$  results in  $B$ . In particular, every Jónsson-Tarski variety of universal algebras [3] (considered as a category) has this property; the same is true for the pointed subtractive varieties in the sense of Ursini [4].

The purpose of this paper is to characterize pointed varieties with normal projections (Theorem 3 below).

Before stating the theorem, we make a simple reformulation of Definition 1.

2. PROPOSITION. Let  $\mathbf{C}$  be a pointed variety. The following conditions are equivalent:

- (a)  $\mathbf{C}$  has normal projections;
- (b) there exists a natural number  $n$ , such that for all  $A$  and  $B$  in  $\mathbf{C}$ , and for all  $a \in A$  and  $b \in B$ ,  $((a, b), (0, b)) \in R^n$ , where  $R$  is the reflexive homomorphic relation on  $A \times B$  generated by the set  $\{((a', 0), (a'', 0)) \mid a' = 0 \text{ or } a'' = 0\}$ ;
- (c) let  $F[x]$  be the free algebra in  $\mathbf{C}$  generated by  $x$ ; there exists a natural number  $n$  such that  $((x, x), (0, x)) \in Q^n$ , where  $Q$  is the reflexive homomorphic relation on  $F[x] \times F[x]$  generated by the set  $\{((x, 0), (0, 0)), ((0, 0), (x, 0))\}$ .

Moreover, the number  $n$  in (b) and in (c) can be supposed to be the same.

3. THEOREM. A pointed variety  $\mathbf{C}$  has normal projections if and only if the corresponding theory contains

- unary terms  $t_1, \dots, t_m$  and  $u_1, \dots, u_m$ ;
- $(m + 2)$ -ary terms  $v_1, \dots, v_n$ ;

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Received by the editors 2003-01-08 and, in revised form, 2003-05-14.

Transmitted by Walter Tholen. Published on 2003-05-20.

2000 Mathematics Subject Classification: 18A20, 18A30, 08B05, 08B25.

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and the following identities hold in  $\mathbf{C}$ :

- $x = v_1(t_1(x), \dots, t_m(x), x, 0)$ ;
- $v_{i+1}(t_1(x), \dots, t_m(x), x, 0) = v_i(t_1(x), \dots, t_m(x), 0, x)$  for each  $i \in \{1, \dots, n - 1\}$ ;
- $0 = v_n(t_1(x), \dots, t_m(x), 0, x)$ ;
- $x = v_i(u_1(x), \dots, u_m(x), 0, 0)$  for each  $i \in \{1, \dots, n\}$ .

Moreover, for each  $n$  this characterizes the pointed varieties of universal algebras which satisfy 2(b) (for the same  $n$ ).

PROOF. According to Proposition 2, we have to characterize pointed varieties for which  $((x, x), (0, x)) \in Q^n$ , i.e. for which there exist  $s_1, \dots, s_{n+1} \in F[x]$  such that  $s_1 = x, s_{n+1} = 0$ , and for each  $i \in \{1, \dots, n\}$  the pair  $((s_i, x), (s_{i+1}, x))$  is in  $Q$ . On the other hand,  $((s_i, x), (s_{i+1}, x)) \in Q$  if and only if for some  $(m+2)$ -ary term  $v_i$  and unary terms  $t_1, \dots, t_m, u_1, \dots, u_m \in F[x]$ , one has the equalities  $(s_i, x) = v_i((t_1, u_1), \dots, (t_m, u_m), (x, 0), (0, 0))$  and  $(s_{i+1}, x) = v_i((t_1, u_1), \dots, (t_m, u_m), (0, 0), (x, 0))$ . Moreover, we can assume that the  $t$ 's,  $u$ 's and the number  $m$  are the same for each  $i \in \{1, \dots, n\}$ . Writing the equalities above separately for the components of pairs, we obtain  $s_i = v_i(t_1, \dots, t_m, x, 0)$ ,  $s_{i+1} = v_i(t_1, \dots, t_m, 0, x)$ ,  $x = v_i(u_1, \dots, u_m, 0, 0)$ . Since  $s$ 's are expressed by  $v$ 's, we may omit them; after this the identities become exactly as in the formulation of the theorem. ■

4. EXAMPLE. Let  $\mathbf{C}$  be a pointed variety with normal projections, for which we could take  $n = 1$  in Theorem 3. Then, the theory corresponding to  $\mathbf{C}$  has unary terms  $t_1, \dots, t_m, u_1, \dots, u_m$  and an  $(m + 2)$ -ary term  $v$ , which satisfy the identities

$$x = v(t_1(x), \dots, t_m(x), x, 0), \quad x = v(u_1(x), \dots, u_m(x), 0, 0), \quad 0 = v(t_1(x), \dots, t_m(x), 0, x).$$

When the unary terms are either  $x$  or  $0$ , an easy argument shows that we could rewrite these identities as

$$x = w(x, 0, x, x, 0), \quad x = w(x, x, 0, 0, 0), \quad 0 = w(x, 0, x, 0, x).$$

If the term  $w = w(x_1, x_2, x_3, x_4, x_5)$  depends only on the first variable  $x_1$  and the last variable  $x_5$ , then we can write  $w(x_1, x_2, x_3, x_4, x_5) = x_1 + x_5$  and our identities become

$$x = x + 0, \quad 0 = x + x;$$

in this case the variety  $\mathbf{C}$  becomes nothing but a pointed subtractive variety in the sense of Ursini [4]. On the other hand,  $w(x_1, x_2, x_3, x_4, x_5) = x_2 + x_4$  would give

$$x = 0 + x, \quad x = x + 0$$

which defines a Jónsson-Tarski variety [3].

5. REMARK. It is known that every semi-abelian category has normal projections (see Condition SA\*3a in [2]). More generally, every pointed category in which every pair of canonical morphisms  $(1_A, 0) : A \rightarrow A \times B$ ,  $(0, 1_B) : B \rightarrow A \times B$  is jointly epimorphic has this property. In particular, this is the case for the unital categories in the sense of Bourn [1].

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