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# EXPONENTIABILITY OF PERFECT MAPS: FOUR APPROACHES

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ABSTRACT. Two proofs of the exponentiability of perfect maps are presented and compared to two other recent approaches. One of the proofs is an elementary approach including a direct construction of the exponentials. The other, implicit in the literature, uses internal locales in the topos of set-valued sheaves on a topological space.

## 1. Introduction

This is a paper about the exponentiability of perfect maps in the category **Top** of topological spaces and continuous maps. Perfect maps can be thought of as generalized compact Hausdorff spaces, and it has been known since at least the mid 1940's that *all* locally compact Hausdorff spaces are exponentiable in **Top** [Fo]. However, although generalized locally compact spaces and exponentiable morphisms in **Top** have been extensively considered since the late 1970's, it was not until just last year that a proof appeared establishing the exponentiability of perfect maps. In fact, there have been two recent proofs, but neither is constructive and both depend on the axiom of choice. One [CHT], by Clementino, Hofmann, and Tholen, involves an ultrafilter-interpolation characterization of exponentiable maps. The exponentials are constructed via partial products (as in [DT]) in the category of grizzly spaces and are shown to be in **Top** using the Extension Lemma of [P] which is an application of the Prime Filter Theorem [J2]. The other [RT], by Richter and Tholen, is a general categorical proof that pullback along a perfect morphism preserves quotient maps. The setting there is a finitely complete category with a proper pullback stable factorization system. Though constructive itself, one then uses Freyd's Adjoint Functor Theorem [Fr] to obtain exponentiability of perfect maps in **Top**.

Here we present two additional proofs. One is implicit in the literature but uses internal locales in the category of sheaves on a topological space and a theorem from [N3] for which it is necessary to assume that the codomain of the perfect map is a sober space. The other consists of an elementary proof using the characterization [N1,N2] of exponentiable morphism in **Top** together with an explicit description of the exponentials so that the approach is self-contained relative to its reliance on [N1,N2]. Moreover, it is entirely constructive and makes no assumption on the spaces involved.

We begin with a review of exponentiability (Section 2) and perfect maps (Section 3) in **Top**. In the next two sections, we present the elementary and internal locale proofs,

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respectively, concluding in Section 6 with a few remarks about their relationship to the two earlier approaches.

## 2. Exponentiability in **Top**

Let $\mathbf{A}$ be a category with finite limits. Recall that an object $Y$ is said to be *exponentiable* in $\mathbf{A}$ if the functor $- \times Y\colon \mathbf{A} \to \mathbf{A}$ has a right adjoint, usually denoted by $(\ )^Y$. A morphism $p\colon Y \to T$ is said to be *exponentiable* in $\mathbf{A}$ if it is exponentiable in the category $\mathbf{A}/T$, whose objects are morphisms of $\mathbf{A}$ and morphisms are commutative triangles over $T$. Note that we will follow the customary abuse of notation and write $- \times_T Y$ and $(\ )^Y$ for $- \times p$ and $(\ )^p$, respectively.

When $\mathbf{A} = \mathbf{Top}$, taking $X$ to be a one-point space in the natural bijection

$$\mathbf{Top}(X \times Y, Z) \cong \mathbf{Top}(X, Z^Y)$$

one easily sees that the space $Z^Y$ can be identified with the set $\mathbf{Top}(Y, Z)$ of continuous maps from $Y$ to $Z$, and so the question of exponentiability becomes one of finding suitable topologies on the function spaces $Z^Y$.

Perhaps the first exponentiability result in print appeared in the 1945 paper [Fo] by R. H. Fox where the function space problem was clearly stated, and it was shown that a separable metric space is exponentiable if and only if it is locally compact. Fox attributed his interest in this problem to a question posed to him in a letter from Hurewicz.

A complete characterization of exponentiable spaces appeared in the 1970 paper [DK] of Day and Kelly where they proved that the functor $- \times Y$ preserves quotient maps if and only if the lattice $\mathcal{O}(Y)$ of open subsets of $Y$ is a continuous lattice in the sense of Scott [S], and that a Hausdorff space satisfies this property if and only if it is locally compact. Since $- \times Y$ preserves coproducts in any case, the preservation of quotients is necessary and sufficient for the exponentiability of $Y$. Note that sufficiency follows from Freyd's Adjoint Functor Theorem but one can also directly construct the exponentials and use the given condition to establish the adjunction.

In 1978, Hoffmann and Lawson [HL] extended the connection with local compactness to all sober spaces, and Hyland [H] proved that the exponentiable objects in the category **Loc** of locales are precisely the locally compact ones. That same year, Niefield [N1,N2] also extended the Day/Kelly characterization to $\mathbf{Top}/T$, and used it to show that every locally compact space over a Hausdorff space $T$ is exponentiable, and that the inclusion of a subspace of $T$ is exponentiable if and only if it is locally closed, i.e., the intersection of an open and a closed subspace.

More recently, Lowen-Colebunders and Richter [LR] showed that the exponentials $Z^Y$ in **Top** can be described using the way-below relation on the continuous lattice $\mathcal{O}(Y)$, and then Richter [R] generalized this result to the fiberwise case.

For a further discussion of the influence of [DK], the reader is referred to Isbell's article "General function spaces, products, and continuous lattices" [I]. For more on exponentiability, see also [N4].

## 3. Perfect Maps

Recall that a continuous function $p: Y \to T$ is called *proper* if it closed and has compact fibers, *separated* if the diagonal $Y \to Y \times_T Y$ is closed, and *perfect* if it is both proper and separated. Note that $p$ is proper if and only if $1 \times p: X \times Y \to X \times T$ is closed for every topological space $X$ [B], and separated if and only if all $y \neq y'$ with $py = py'$ can be separated by disjoint open neighborhoods in $Y$.

Perfect is clearly a generalization of compact Hausdorff since $Y \to 1$ is proper (respectively, separated) if and only if $Y$ is compact (respectively, Hausdorff). In fact, Johnstone showed in [J3] that this relationship goes much further. The setting in [J3] is the category $Sh(T)$ of set-valued sheaves on $T$. As in any topos, the subobject classifier $\Omega_T$ (given by $U \mapsto \mathcal{O}(U)$) is an internal locale in $Sh(T)$. The map $p: Y \to T$ induces a geometric morphism with direct image $p_*: Sh(Y) \to Sh(T)$ which takes $F$ to the sheaf $U \mapsto F(p^{-1}U)$. Since the direct image of any geometric morphism preserves internal locales, one can consider properties of $p_*(\Omega_Y)$ as a locale in the topos $Sh(T)$. In this context, Johnstone shows that if $p$ is proper, then $p_*(\Omega_Y)$ is compact and if $p$ is perfect, then $p_*(\Omega_Y)$ is compact and regular. He also establishes the converse to each of these statements in the case where $T$ is a $T_D$-space in the sense of [A], i.e., points of $T$ are locally closed.

For more on internal locales, the reader is referred to [J1], [J2], and [JT].

## 4. Exponentiability of Perfect Maps

In this section, we recall the characterization of exponentiable morphisms in **Top** presented in [N1,N2], and apply it to obtain the exponentiability of perfect maps. We conclude with a description of the relevant exponentials so that the interested reader can verify that perfect maps are exponentiable without explicit reference to the general characterization in [N1,N2].

Recall that if $Y$ is a topological space, then $H \subseteq \mathcal{O}(Y)$ is called *Scott-open* if it is upward closed, i.e.,

$$U \in H,\ U \subseteq V \Rightarrow V \in H$$

and it satisfies the finite union property, i.e.,

$$\bigcup_{\alpha \in A} U_\alpha \in H \Rightarrow \bigcup_{\alpha \in F} U_\alpha \in H$$

for some finite $F \subseteq A$.

For a continuous map $p: Y \to T$ and $t \in T$, let $Y_t$ denote the *fiber of* $Y$ *over* $t$, i.e., the set $p^{-1}t$ with the subspace topology. Given $H \subseteq \bigcup_{t \in T} \mathcal{O}(Y_t)$, we write $\bigcap H$ for the subset of $Y$ whose fiber over $t$ is given by

$$\left(\bigcap H\right)_t = \bigcap H_t = \bigcap \{V_t | V_t \in H_t\}$$

In [N1,N3], it was shown that:

4.1. Theorem. *A map $p: Y \to T$ is exponentiable in* **Top** *if and only if whenever $U$ is open neighborhood of $y$ in $Y$, there exists $H \subseteq \bigcup_{t \in T} \mathcal{O}(Y_t)$ such that*
*(i) $U_{py} \in H_{py}$*
*(ii) $H_t$ is Scott-open, for all $t \in T$*
*(iii) $\{t \in T | V_t \in H_t\}$ is open in $T$, for all $V \in \mathcal{O}(Y)$*
*(iv) $\bigcap H$ is a neighborhood of $y$ in $Y$*

To show that perfect maps satisfy this condition, we first prove the following lemma.

4.2. Lemma. *If $p: Y \to T$ is a separated map with compact fibers and $U$ is open neighborhood of $y$ in $Y$, then $F_{py} \subseteq U_{py}$, for some closed neighborhood $F$ of $y$ in $Y$.*

*Proof.* For each $x \in Y_{py} \setminus U_{py}$, since $p$ is separated, there are disjoint open neighborhoods $V$ and $W$ of $x$ and $y$ in $Y$, respectively. Since $Y_{py} \setminus U_{py}$ is a compact set which is covered by the $V$'s, there is a finite covering family $V_1, \ldots, V_n$. Let $F = Y \setminus (V_1 \cup \ldots \cup V_n)$. Then $F_{py} \subseteq U_{py}$, since $Y_{py} \setminus U_{py} \subseteq (V_1 \cup \ldots \cup V_n)_{py}$. To see that $F$ is a neighborhood of $y$, one easily shows that $W \subseteq F$, where $W = W_1 \cap \ldots \cap W_n$ is clearly a neighborhood of $y$ in $Y$.

4.3. Theorem. *Every perfect map is exponentiable in* **Top**.

*Proof.* Suppose $p: Y \to T$ is a perfect map. To apply Theorem 4.1, let $U$ be open neighborhood of $y$ in $Y$ and let $F$ be as in Lemma 4.2. Consider

$$H_t = \{V_t \in \mathcal{O}(Y_t) | F_t \subseteq V_t\}$$

Then $U_{py} \in H_{py}$ since $F_{py} \subseteq U_{py}$, $H_t$ is Scott-open since $F_t$ is a closed subset of a compact set and hence compact, and $\bigcap H$ is a neighborhood of $y$ in $Y$ since $F \subseteq \bigcap H$. It remains to show that the set

$$G = \{t \in T | V_t \in H_t\}$$

is open in $T$, for all $V$ open in $Y$. Since $p$ is a closed map, it suffices to show that $p(F \setminus V) = T \setminus G$. But, $t \notin G \iff V_t \notin H_t \iff F_t \not\subseteq V_t \iff \exists y \in F_t$ such that $y \notin V_t \iff t \in p(F \setminus V)$, to complete the proof.

We conclude this section with a description of the exponentials $Z^Y$ when $p: Y \to T$ is a perfect map. This is the same construction given in [N1,N2] for an arbitrary exponentiable map. Note that here, as in the general case, all but the continuity of the counit $\epsilon$ holds when no assumption is made about the map $p: Y \to T$.

Given $q: Z \to T$, let $Z^Y$ denote the set of pairs $(\sigma, t)$, where $t \in T$ and $\sigma: Y_t \to Z_t$ is continuous. Suppose $Z^Y$ has the topology generated by the sets

$$\langle H, W \rangle = \{(\sigma, t) | \sigma^{-1} W_t \in H_t\}$$

where $H \subseteq \bigcup_{t \in T} \mathcal{O}(Y_t)$ satisfies (ii) and (iii) of Theorem 4.1 and $W$ is open in $Z$. Then the projection $q^p: Z^Y \to T$ is clearly continuous, for if $G$ is open in $T$, then $(q^p)^{-1} G = \langle H, Z \rangle$, where $H = \bigcup_{t \in G} \mathcal{O}(Y_t)$.

Consider $\eta\colon Z \to (Z \times_T Y)^Y$ given by $\eta z = ((z,-),t)$, where $qz = t$. To see that $\eta$ is continuous, suppose $\eta z \in \langle H, W\rangle$, where $H$ satisfies (ii) and (iii), and $W$ is open in $(Z \times_T Y)^Y$. Then

$$(z,-)^{-1}W_t = \{y \in Y_t | (z,y) \in W_t\} \in H_t$$

For each $y$ in this set, let $U \times V$ be an open neighborhood of $(z,y)$ in $W$. Since $H_t$ is Scott-open, we know $(\cup V)_t \in H_t$, and so $V'_t \in H_t$, for some finite union $V' = V_1 \cup \ldots \cup V_n$. Let

$$U' = U_1 \cap \ldots \cap U_n \cap q^{-1}G$$

where $G = \{t' \in T | V'_{t'} \in H_{t'}\}$ which is open in $T$ by (iii). Then it is not difficult to show that $z \in U'$, and $\eta U' \subseteq \langle H, W\rangle$ since

$$V'_{t'} \subseteq \{y' \in Y_{t'} | (z',y') \in W_{t'}\}$$

for all $z' \in U'$ where $qz' = t'$, and it follows that $\eta$ is continuous.

Consider $\epsilon\colon Z^Y \times_T Y \to Z$ given by $\epsilon((\sigma,t),y) = \sigma y$. To see that $\epsilon$ is continuous, suppose $\epsilon((\sigma,t),y) \in W$, where $W$ is open in $Z$, and let $U_t = \sigma^{-1}W_t$. Then $y \in U_t$, and taking $F$ and $H$ as in the proof of Theorem 4.3, it follows that $((\sigma,t),y) \in \langle H, W\rangle \times_T F^\circ$ and $\epsilon(\langle H, W\rangle \times_T F^\circ) \subseteq W$, where $F^\circ$ denotes the interior of $F$ in $Y$, establishing the continuity of $\epsilon$.

## 5. Exponentiability and Internal Locales

As noted in Section 3, Johnstone [J3] showed that if $p\colon Y \to T$ is perfect, then $p_*(\Omega_Y)$ is a compact regular locale in $Sh(T)$. He also proved that compact regular locales in $Sh(T)$ are (stably) locally compact. Now, Niefield [N3] showed that if $T$ is a sober $T_D$-space, then the map $p\colon Y \to T$ is exponentiable in **Top** if and only if $p_*(\Omega_Y)$ is locally compact as an internal locale in $Sh(T)$. Thus, with these conditions on $T$, it follows that perfect maps are exponentiable in **Top**.

Note that since Hyland gave a constructive proof [H] of the exponentiability of locally compact locales, it establishes the result for internal locales, as well. Consequently, $p_*(\Omega_Y)$ is exponentiable as a locale in $Sh(T)$, whenever $p$ is perfect.

## 6. Concluding Remarks

We conclude by comparing the two "new" proofs with the others. As noted in the introduction, the proofs presented here are constructive while the others depend on the axiom of choice. The internal locale approach (Section 5) has the advantage of being short and yielding an explicit construction of the exponentials (since the theorem from [N3] does) but it is not the most general result possible (there is the assumption that $T$ is a sober $T_D$-space). Moreover, it is highly non-elementary due to its reliance on internal locales in a topos. The approach suggested by [N1,N2] (Section 4) is direct and elementary especially if one bypasses the reliance on Theorem 4.1 in establishing the adjunction.

## References

[AT] C. E. Aull and W. J. Thron, Separation axioms between $T_0$ and $T_1$, *Indag. Math.* **24** (1963), 26–37.

[B] N. Bourbaki, Elements of General Topology, Part 1, Hermann, Paris, 1966.

[CHT] M. M. Clementino, D. Hofmann, and W. Tholen, The convergence approach to exponentiable maps, *Port. Math.*, (to appear).

[DK] B. J. Day and G. M. Kelly, On topological quotients preserved by pullback or products, *Proc. Camb. Phil. Soc.* **67** (1970) 553–558.

[DT] R. Dyckhoff and W. Tholen, Exponentiable morphisms, partial products and pullback complements, *J. Pure Appl. Alg.*, **49** (1987) 103–116.

[Fo] R. H. Fox, On topologies for function spaces, *Bull. Amer. Math. Soc.* **51** (1945) 429–432.

[Fr] P. Freyd, Abelain Categories, Harper & Row, New York, 1964.

[HL] K. H. Hofmann and J. D. Lawson, The spectral theory of distributive continuous lattices, *Trans. Amer. Math. Soc.* **246** (1978) 285–310.

[H] J. M. E. Hyland, Function spaces in the category of locales, *Springer Lecture Notes in Math.* **871** (1981) 264–281.

[I] J. R. Isbell, General function spaces, products, and continuous lattices, *Proc. Camb. Phil. Soc.* **100** (1986) 193–205.

[J1] P. T. Johnstone, Topos Theory, Academic Press, 1977.

[J2] P. T. Johnstone, Stone Spaces, Cambridge University Press, 1982.

[J3] P. T. Johnstone, The Gleason cover of a topos, II, *J. Pure Appl. Alg.*, **22** (1981) 229–247.

[JT] A. Joyal and M. Tierney, An Extension of the Galois Theory of Grothendieck, Amer. Math. Soc. Memoirs 309 (1984).

[LR] E. Lowen-Colebunders and G. Richter, An Elementary approach to exponentiable spaces, *Appl. Categorical Structures* **9** (2001) 303–310.

[N1] S. B. Niefield, Cartesianness, Ph.D. Thesis, Rutgers University, 1978.

[N2] S. B. Niefield, Cartesianness: topological spaces, uniform spaces, and affine schemes, *J. Pure Appl. Alg.* **23** (1982) 147–167.

[N3] S. B. Niefield, Cartesian spaces over $T$ and locales over $\Omega(T)$, *Cahiers de Top. et Géom. Diff. Catégoriques* **23-3** (1982) 257–267.

[N4] S. B. Niefield, Exponentiable morphisms: posets, spaces, locales, and Grothendieck toposes, *Theory and Applications of Categories* **8** (2001) 16–32.

[P] C. Pisani, Convergence in exponentiable spaces, *Theory and Applications of Categories* **5** (1999) 148–162.

[R] G. Richter, Exponentiability for maps means fibrewise core-compactness, preprint.

[RT] G. Richter and W. Tholen, Perfect maps are exponentiable – categorically, *Theory and Applications of Categories* **8** (2001) 457–464.

[S] D. S. Scott, Continuous lattices, *Springer Lecture Notes in Math.* **274** (1972) 97–137.

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