

ZERO-DIVISOR GRAPHS OF FINITE COMMUTATIVE RINGS: A SURVEY

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Abstract. This article gives a comprehensive survey on zero-divisor graphs of finite commutative rings. We investigate the results on structural properties of these graphs.

1 Introduction

The aim of this article is to survey the most recent developments in describing the structural properties of zero-divisor graphs of finite commutative rings and their applications. The *zero-divisor graph* of a commutative ring R , denoted by $\Gamma(R)$, is the undirected graph with vertex set $Z(R)^*$ and two distinct vertices x and y are adjacent if $xy = 0$. The connection between graph theory and ring theory was established by Beck [15] in 1988. He defined a graph denoted by $\Gamma_0(R)$, with all the elements of a ring as vertices of the graph and two distinct vertices x and y are adjacent provided $xy = 0$. After modifying the definition of Beck [15], Anderson and Livingston [8] were the first who introduced the definition of zero-divisor graph.

Throughout the paper, $\Gamma(\mathbb{Z}_n)$ denotes the zero-divisor graph of ring of integers modulo n . Also $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ unless otherwise stated. *Complement of graph G* is the graph \overline{G} with the vertex set $V(G)$, such that two vertices are adjacent in \overline{G} if and only if they are not adjacent in G . The *line graph* $L(\Gamma(R))$ of a zero-divisor graph $\Gamma(R)$ is defined as a graph with vertex set $V(L(\Gamma(R))) = E(\Gamma(R))$ that is vertices of line graph $L(\Gamma(R))$ represents an edge of $\Gamma(R)$, and distinct vertices in $L(\Gamma(R))$ are connected if and only if their corresponding edges in $\Gamma(R)$ share a common vertex.

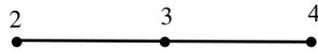
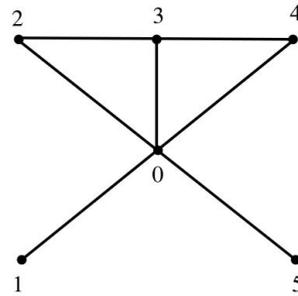
$$V(L(\Gamma(R))) = \{(a, b) \mid a.b = 0, a \text{ and } b \text{ are non-zero zero-divisors of } R\}$$
$$E(L(\Gamma(R))) = \{\{e_1, e_2\} \mid e_1 \text{ and } e_2 \text{ are incident to common vertex in } R\}$$

An element u is called *unit* if there exist element y such that $uy = 1$. If there exist a unit u such that $x = uy$, then x and y are associates. We can partition the

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Figure 1: $\Gamma(\mathbb{Z}_8)$ Figure 2: $\Gamma_0(\mathbb{Z}_8)$

elements of $V(\Gamma(\mathbb{Z}_n))$ in such a way that two vertices are in the same class if and only if they are associates. This partition divides the elements of $V(\Gamma(\mathbb{Z}_n))$ into associate classes. We denote by A_v the associate class represented by vertex v . The set of *Gaussian integers*, denoted by $\mathbb{Z}[i]$, is defined as the set of complex numbers $a + bi$, where $a, b \in \mathbb{Z}$. If x is a prime Gaussian integer, then x is either

1. $(1 + i)$ or $(1 - i)$, or
2. q where q is a prime integer and $q \equiv 3 \pmod{4}$, or
3. $a + bi$, $a - bi$ where $a^2 + b^2 = p$, p is a prime integer and $p \equiv 1 \pmod{4}$.

For ring of Gaussian integers, we denote p and p_i as prime integers which are congruent to 1 modulo 4, while q and q_i are prime integers which are congruent to 3 modulo 4. For better understanding of the notation and terminology related to graph theory, one can refer to [20].

During last 20 years, many research articles have been published on associating graphs with algebraic structures. Some of them to mention are cayley graphs, conjugacy class graphs, unital graphs, conjugacy graphs, zero-divisor graphs etc. Moreover, zero-divisor graphs were also defined and studied for non-commutative rings, near rings, semi-groups, modules, lattices and posets. In 2002, DeMeyer et al. [26] extended the study of zero-divisor graph of rings to semigroups. A detail survey of zero-divisor graph of a commutative semigroup is given in [11]. In the same year, Redmond [59] extended the definition of zero-divisor graph to non-commutative rings. He showed that for non-commutative ring R , $\Gamma(R)$ is connected if the set of left and right zero-divisors coincide. Various other authors ([5], [61], [80]) investigated the properties for zero-divisor graphs of non-commutative rings. For recent survey on graphs related to algebraic structures, one can refer ([10], [11], [22]), [25], [36],

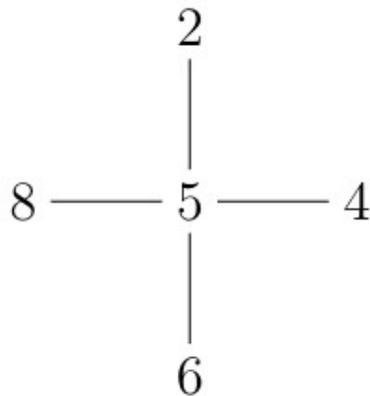


Figure 3: $\Gamma(\mathbb{Z}_{10})$

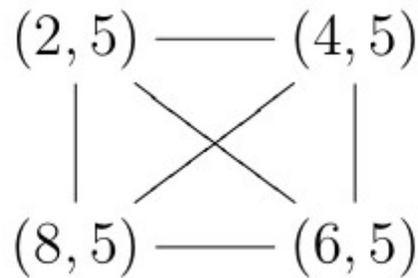


Figure 4: Line graph of $\Gamma(\mathbb{Z}_{10})$

[44].

In 2003, Redmond [60] took a new approach to define zero-divisor graph with the help of an ideal I of a ring R . The graph $\Gamma_I(R)$ has vertex from $R \setminus I$, where distinct vertices x and y are adjacent if and only if $xy \in I$. He investigated the relationship between properties of $\Gamma_I(R)$ and $\Gamma(R/I)$. Zero-divisor graphs of power series rings and polynomial rings were first studied by Axtell et al. [13]. He examined the diameter and girth of these rings.

In 2004, Philips et al. [55] studied the central sets in $\Gamma(\mathbb{Z}_n)$ and $\overline{\Gamma(\mathbb{Z}_n)}$. He also investigated properties like planarity, connectivity, vertex degree etc. of $\Gamma(\mathbb{Z}_n)$. In 2005, Cordova et al. [23] extended the work of [55]. He worked on the clique and coloring of $\Gamma(\mathbb{Z}_n)$ and $\overline{\Gamma(\mathbb{Z}_n)}$. In 2005, Aponte [12] introduced the line graph of zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ and studied properties like planarity, diameter, girth etc. Origin and early history of zero-divisor graph is discussed in detail in [10].

The contribution of the present work is a survey and classification of the literature on the structural properties of zero divisor graph of finite commutative ring. Our aim is to give the flavor of the subject but not be exhaustive.

The remainder of this paper is organised as follows: the structural properties like diameter, radius, girth, vertex degree, connectivity, planarity, Eulerian, Hamiltonian, and many more of zero-divisor graph for finite commutative rings is discussed in section 2. Several results on adjacency matrix, energy and eigen values are provided

in section 3. In section 4, we give some additional results related to these rings. The paper is concluded in section 5.

2 Structure in Zero-Divisor Graph of Finite Commutative Rings

In this section, we investigate the structural properties of zero-divisor graph of finite commutative rings that include-

- diameter, girth and radius
- vertex degree and connectivity
- planarity, Eulerian and Hamiltonian
- domination number and independence number
- chromatic number

We start with few definitions and terminology. A graph G is *complete* if any two distinct vertices are adjacent and is *connected* if there is a path between any two distinct vertices. For distinct vertices x and y in G , the distance between x and y , denoted by $d(x, y)$, is the length of the shortest path connecting x and y .

A *complete bipartite graph* is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. A graph G is *complete r -partite* if G is the disjoint union of r nonempty vertex sets and two distinct vertices are adjacent if and only if they are in distinct vertex sets.

In 2011, Osba [47] determined that for $n = p, 2q$ and $q_1 q_2$, $\Gamma(\mathbb{Z}_n[i])$ is bi-partite and triangle free. He also showed that for any $n > 2$, $\Gamma(\mathbb{Z}_n[i])$ is not a tree.

The *cardinality* of the set of nonzero zero-divisors of \mathbb{Z}_n that is the number of vertices in $\Gamma(\mathbb{Z}_n)$ and $\overline{\Gamma(\mathbb{Z}_n)}$ is equal to $n - \phi(n) - 1$, where $\phi(n)$ is the number of units in \mathbb{Z}_n (Lemma 6.5, [55]). For a ring of gaussian integers modulo n , $|V(\Gamma(\mathbb{Z}_n[i]))| = n - (2^{2k-1} \times \prod_{j=1}^m (q_j^{2\alpha_j} - q_j^{2\alpha_j-2}) \times \prod_{s=1}^l (p_s^{\beta_s} - p_s^{\beta_s-1})^2) - 1$ (lemma 12, [46]).

In 2014, Birch [18] gave a method to find the number of edges in \mathbb{Z}_n [18]. Result for cardinality of the set of non-zero zero-divisors and edges for $\mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{q^2}$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ are discussed in [33]. In 2006, Redmond [63] gave an algorithm to find all commutative, reduced rings with unity which gives rise to a zero-divisor graph on n vertices for any $n \geq 1$ and a list of all commutative rings (up to isomorphism) which produce zero-divisor graph for $n = 6, 7, \dots, 14$ vertices is also given.

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2.1 Diameter, Girth and Radius

This subsection deals with the results of diameter, girth and radius of zero-divisor graph of finite commutative rings. The *diameter* of G is the greatest distance between any pair of vertices. The minimum among all the maximum distances between a vertex to all other vertices is considered as *radius* of graph G . We define *girth* of G denoted by $g(G)$, as the length of the shortest cycle in G , provided G contains a cycle; otherwise $g(G) = \infty$.

Now, we discuss the diameter and girth in zero-divisor graph. We begin with the following result given by Anderson and Livingston [8], which shows that diameter of zero-divisor graph of \mathbb{Z}_n is less than 4.

Theorem 1 (Theorem 2.3, [8]). *Let R be a commutative ring. Then $\Gamma(R)$ is connected and $\text{diam}\Gamma(R) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 7$.*

They noticed that all of the examples they considered had girths of 3, 4 or ∞ . Based on this, they conjectured that if a zero-divisor graph has a cycle, then its girth is 3 or 4. They were able to prove this for Artinian ring (Theorem 2.4, [8]). The conjecture was proved independently by Mulay [42] and DeMeyer and Schneider [26]. Additionally, short proofs have been given by Axtell et al. [13] and Wright [79].

Theorem 2 (Theorem 2.4, [8]). *Let R be a commutative ring. If $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 4$.*

In 2006, Redmond [62] gave several results on radius and centre of zero divisor graph of commutative ring. Following theorem discusses the case of commutative Artinian ring with identity.

Theorem 3. (Theorem 3.10, [62]) *Let R be a commutative Artinian ring with identity that is not a domain*

1. *The radius of $\Gamma(R)$ is zero if and only if the diameter of $\Gamma(R)$ is zero if and only if $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/X^2$*
2. *If the radius of $\Gamma(R)$ is 1, then the diameter of $\Gamma(R)$ is 1 if and only if $\Gamma(R)$ is complete. Otherwise, the diameter is 2.*
3. *If the radius of $\Gamma(R)$ is 2, then the diameter of $\Gamma(R)$ is 2 if and only if $R \cong F_1 \times F_2$, where F_1 and F_2 are both fields and both not isomorphic to \mathbb{Z}_2 . Otherwise the diameter of $\Gamma(R)$ is 3.*

Line graphs of zero-divisor graphs of \mathbb{Z}_n were first studied in detail by Aponte [12] in 2005. The next result gives the relation for diameter and girth of the line graph of $\Gamma(\mathbb{Z}_n)$.

$$\text{Diam}(L(\Gamma(\mathbb{Z}_n))) = \begin{cases} 1, & \text{if } n = 8 \\ 2, & \text{if } n \geq 10, r \leq 2 \\ 3, & \text{otherwise.} \end{cases}$$

$$\text{Girth}(L(\Gamma(\mathbb{Z}_n))) = \begin{cases} 3, & \text{if } n \geq 10 \\ \infty, & \text{if } n = 6, 8, \text{ or } 9. \end{cases}$$

From this result, we can observe that $L(\Gamma(\mathbb{Z}_n))$ is a tree if and only if $n = 6, 8$ or 9 (Corollary 4.2, [12]). Next we discuss properties for ring of Gaussian integers.

Theorem 4 (Theorem 12, [47]). *For any integer $n > 1, m > 1$ with $n \neq q$ for any q ,*

$$\text{rad}(\Gamma(\mathbb{Z}_n[i])) = \begin{cases} 0, & \text{if } n = 2 \\ 1, & \text{if } n = 2^m \text{ or } q^m \\ 2, & \text{otherwise.} \end{cases}$$

The following result shows the diameter and girth of $\Gamma(\mathbb{Z}_n[i])$. It was shown that for prime t and $n > 1$, $g(\Gamma(\mathbb{Z}_{t^n}[i])) = 3$ and $g(\Gamma(\mathbb{Z}_p)) = 4$

Theorem 5 (Theorem 13, [46]). *Let n be a positive integer greater than 1. Then*

1. $\text{diam}(\Gamma(\mathbb{Z}_n[i])) = 1$ if and only if $n = q^2$;
2. $\text{diam}(\Gamma(\mathbb{Z}_n[i])) = 2$ if and only if $n = p$ or $n = 2^m$ with $m \geq 2$ or $n = q^m$ with $m \geq 3$.
3. $\text{diam}(\Gamma(\mathbb{Z}_n[i])) = 3$ if and only if $m = p^m$ with $m \geq 2$ or n is divisible at least by two distinct primes.

Theorem 6 (Theorem 14, [46]). *Let $n = \prod_{j=1}^m t_j^{n_j}$ be the prime factorization of n . Then:*

1. If $n_k > 1$ for some k , then $g(\Gamma(\mathbb{Z}_n[i])) = 3$;
2. If $n_k = 1$ for all k and $m \geq 3$, then $g(\Gamma(\mathbb{Z}_n[i])) = 3$;
3. If $n = p_1 \times p_2$ or $n = p_1 \times q$ or $n = p_1 \times 2$, then $g(\Gamma(\mathbb{Z}_n[i])) = 3$;
4. If $n = q_1 \times q_2$, then $g(\Gamma(\mathbb{Z}_n[i])) = 4$;
5. If $n = 2 \times q$, then $g(\Gamma(\mathbb{Z}_n[i])) = 4$.

Next we discuss the case of complement graph of gaussian integers. In 2012, Osba [48] proved that $\text{diam}(\overline{\Gamma(\mathbb{Z}_{2^n}[i])}) = \text{diam}(\overline{\Gamma(\mathbb{Z}_{q^n}[i])}) = 2$, $n > 2$ and $\text{diam}(\overline{\Gamma(\mathbb{Z}_{p^n}[i])}) = 3$, $n \geq 2$. Following theorem gives the general case of diameter of $\overline{\Gamma(\mathbb{Z}_{2^n}[i])}$.

Theorem 7 (Theorem 3, [48]). *Let $n > 1$. if $m \neq p$, $m \neq 2^n$, $m \neq q^n$, and m is not a product of two distinct primes, then $(\Gamma(\mathbb{Z}_m[i]))$ is connected with $\text{diam}(\Gamma(\mathbb{Z}_m[i])) \leq 3$.*

Theorem 8 (Theorem 5, [48]). *Let n be an integer such that $n \neq 2$, $n \neq q$, and $n \neq q^2$. Then $g(\Gamma(\mathbb{Z}_n[i])) = 3$.*

Next we present results on diameter, girth and radius of line graph of $\Gamma(\mathbb{Z}_n[i])$ given by Nazzal and Ghanem [43].

1. $\text{diam}(L(\Gamma(\mathbb{Z}_n[i]))) = 2$ if and only if $n = p, 2q, q_1q_2, q_1q_2q_3, 2q_1q_2, 4q, 2q^2, 2p, qp$ or $n = 2^m, q^m$ with $m \geq 2$.
2. $\text{diam}(L(\Gamma(\mathbb{Z}_n[i]))) = 3$, otherwise.
3. $\text{girth}(L(\Gamma(\mathbb{Z}_n[i]))) = 3$.
4. $\text{radius}(L(\Gamma(\mathbb{Z}_n[i]))) = 2$.

Theorem 9 (Corollary 9.5, [43]). *The centre of the graph $\Gamma(\mathbb{Z}_n[i])$ is*

$$\begin{cases} 2^{m-1}(1+i), & \text{if } n = 2^m, m \geq 2 \\ A_{(m-1)(m-1)}, & \text{if } n = q^m, m \geq 2 \\ \{(u, v) : u, v \in Z(\mathbb{Z}_{p^m})\} \setminus \{(0, 0)\}, & \text{if } n = p^m, m \geq 2. \end{cases}$$

Ghanem and Nazzal proved that $L(\overline{(\Gamma(\mathbb{Z}_n[i]))})$ is connected if and only if $n \neq 2, p, q^2, q_1q_2$, (see Theorem 3.1, [30]). He gave the following result for diameter and girth of $L(\overline{(\Gamma(\mathbb{Z}_n[i]))})$:

1. $\text{diam}(L(\overline{(\Gamma(\mathbb{Z}_n[i]))})) = 2$ if $n = 2^m, m \geq 2$ or $n = q^m, m \geq 3$.
2. $\text{diam}(L(\overline{(\Gamma(\mathbb{Z}_n[i]))})) = 3$ if $n = p^m, m \geq 2$ or n is a composite such that $n \neq q_1q_2$.
3. $\text{girth}(L(\overline{(\Gamma(\mathbb{Z}_n[i]))})) = 3$ if $n \neq 2, q, q^2$.
4. $\text{radius}(L(\overline{(\Gamma(\mathbb{Z}_n[i]))})) = 2$ if and only if $n \neq 2, p, q, q^2$ or q_1q_2 .

Diameter and girth of direct product of rings is discussed in [14]. In 2019, Akgunes and Nacaroglu [6] calculated the diameter and girth for $Z_p \times Z_q \times Z_r$. For more results on diameter and girth of zero-divisor graphs of commutative rings, readers may refer to [22].

Let $B = \{x \in \mathbb{Z}_n : p_i \mid x \text{ for all } i\}$. The centre of $\Gamma(\mathbb{Z}_n)$ is discussed in proposition 3.1 – 3.3 of [55].

$$centre(\Gamma(\mathbb{Z}_n)) = \begin{cases} p, & \text{if } n = 2p \\ A_{p^{e-1}}, & \text{if } n = p^e \\ (\bigcup_{i=1}^r A_{v_i}) \cup B, & \text{otherwise.} \end{cases}$$

Next we discuss the centre of line graph of $\Gamma(\mathbb{Z}_n)$.

Theorem 10 (Theorem 6.2, [12]). *For $L(\Gamma(\mathbb{Z}_n))$:*

1. *when $n = 27$ the centre is the vertex [9, 18].*
2. *when $n = 8$ the centre is $L(\Gamma(\mathbb{Z}_8))$ and the $\varepsilon(v) = 1$ for every $v \in L(\Gamma(\mathbb{Z}_8))$.*
3. *when $n = 2p$ the center is $L(\Gamma(\mathbb{Z}_{2p}))$.*
4. *when $n = 16$ the center is [4, 8], [8, 12].*
5. *otherwise the center is the graph induced by the vertices with eccentricity equal to 2.*

Author	Graph	Property
Anderson and Livingston (1999)	Commutative ring	Diameter and Girth
Philips et al. (2004)	$\Gamma(\mathbb{Z}_n), \overline{\Gamma(\mathbb{Z}_n)}$	$ V(\Gamma(\mathbb{Z}_n)) , \overline{\Gamma(\mathbb{Z}_n)} ,$ Centre
Aponte et al. (2005)	$L(\Gamma(\mathbb{Z}_n))$	Diameter, Girth, Centre
Osba et al. (2008)	$\Gamma(\mathbb{Z}_n[i])$	$ V(\Gamma(\mathbb{Z}_n)[i]) ,$ Diameter, Girth
Osba et al. (2011)	$\overline{\Gamma(\mathbb{Z}_n[i])}$	Radius, Bi-partite, Tree
Osba et al. (2012)	$\overline{\Gamma(\mathbb{Z}_n[i])}$	Diameter, Girth
Nazzal and Ghanem (2012)	$L(\Gamma(\mathbb{Z}_n[i])),$ $\Gamma(\mathbb{Z}_n[i])$	Diameter, Girth, Radius of $L(\Gamma(\mathbb{Z}_n[i]))$ and centre of $\Gamma(\mathbb{Z}_n[i])$
Ghanem and Nazzal (2012)	$\overline{L(\Gamma(\mathbb{Z}_n[i]))}$	Diameter, Girth and Radius
Birch et al. (2014)	$\Gamma(\mathbb{Z}_n)$	Method to find number of edges in $\Gamma(\mathbb{Z}_n)$
Koam et al. (2019)	$\Gamma(\mathbb{Z}_{p_1 p_2}) \times \Gamma(\mathbb{Z}_{q^2}),$ $\Gamma(\mathbb{Z}_{q^2}) \times \Gamma(\mathbb{Z}_{q^2})$	$ V , E $
Akgunes and Nacaroglu (2019)	$\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$	Diameter, Girth and Radius

2.2 Vertex degree and connectivity

In this subsection, vertex connectivity, edge connectivity and minimum degree of zero-divisor graph of some finite commutative rings and their relationship with each other are discussed. A graph G is *connected* if there is a path joining every pair of vertices. *Vertex connectivity* of a graph is the minimum number of vertices whose removal results in disconnection of the graph.

Degree of a vertex is the number of edges associated with it. The *minimum degree* of G is the minimum degree of its vertices. A set $A \subset V(G)$, is said to be a *cut-set* if there exist distinct vertices a and b in $V(G) \setminus A$ such that every path in G from a to b involves atleast one element from A , and no proper subset of A satisfies the same condition.

The application of connectivity can be seen in communication network, for example, if we think of a graph as representing a communication network, the connectivity (or edge connectivity) becomes the smallest number of communication stations (or communication links) whose breakdown would jeopardise communication in the system. The higher the connectivity and edge connectivity, the more reliable the network.

Now we discuss the degree of vertices in zero-divisor graphs of finite commutative rings. In 2004, Philips [55] gave the relation for vertex degree in $\Gamma(\mathbb{Z}_n)$, with the help of which Aponte [12] calculated vertex degree for line graph of $\Gamma(\mathbb{Z}_n)$.

For any vertex $u \in V(\Gamma(\mathbb{Z}_n))$:

$$\deg(u) = \begin{cases} \gcd(u, n) - 1, & \text{if } u^2 \neq 0 \\ \gcd(u, n) - 2, & \text{if } u^2 = 0. \end{cases}$$

Also for vertex $v \in V(\overline{\Gamma(\mathbb{Z}_n)})$:

$$\deg(v) = \begin{cases} n - \phi(n) - \gcd(v, n) - 1, & \text{if } v^2 \neq 0 \\ n - \phi(n) - \gcd(v, n), & \text{if } v^2 = 0. \end{cases}$$

Philips [55] showed that the vertex $v_1 = p_1^{e_1-1} p_2^{e_2} \dots p_r^{e_r}$ in $G = \overline{\Gamma(\mathbb{Z}_n)}$ has minimum degree and minimum degree is $\delta(G) = p_1^{e_1-1} (p_1 - 1) p_2^{e_2} \dots p_r^{e_r} - p_1^{e_1-1} (p_1 - 1) p_2^{e_2-1} (p_2 - 1) \dots p_r^{e_r-1} (p_r - 1)$ (see, lemma 7.1). He gave a relation for vertex connectivity of $\overline{\Gamma(\mathbb{Z}_n)}$ in terms of minimum degree (see, Theorem 7.1- 7.2).

$$\kappa(\overline{\Gamma(\mathbb{Z}_n)}) = \begin{cases} \delta(\overline{\Gamma(\mathbb{Z}_n)}), & \text{when } n = p_1^{e_1} p_2^{e_2}, e_1 > 1 \\ \delta(\overline{\Gamma(\mathbb{Z}_n)}) - |A_{v_1}| + 1, & \text{when } n = p_1 p_2^{e_2}, e_2 > 1. \end{cases}$$

where v_1 is the vertex of minimum degree.

Cote et al. [24], showed that a set A is a cut set of $\Gamma(\mathbb{Z}_n)$ if and only if $A = \text{ann}(p) \setminus \{0\}$ for some prime p which divides n , where $n \neq p, 2p, p^2$. This result was extended to $\Gamma(\prod_{i=1}^m \mathbb{Z}_{n_i})$ (Theorem 2.9, [24]).

Conjecture 11 (Conjecture 8.1, [55]). *Let $G = \overline{\Gamma(\mathbb{Z}_n)}$ where $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ for primes $p_1 < p_2 < \dots < p_r$. Then $\kappa(G) = \delta(G) - |A_{v_1}| + 1$ when $e_1 = 1$ and $\kappa(G) = \delta(G)$ when $e_1 > 1$.*

The maximal degree in $G(\Gamma(\mathbb{Z}_n))$ has the vertex $\frac{n}{p_1}$ and the maximum degree is equal to $\frac{n}{p_1} - 1$ (cf. [70], [32]).

In 2020, Chattopadhyay and Patra [21] proved that algebraic connectivity and vertex connectivity of $\Gamma(\mathbb{Z}_n)$ coincide if and only if n is a product of two distinct primes or $n = p^t$ for some prime p and integer $t \geq 3$. Aponte et al. [12] proved that if a graph G is connected then its line graph $L(G)$ is also connected. In Theorem 2.1, he gave a relation for degree of vertex $[u, v] \in V(L(\Gamma(\mathbb{Z}_n)))$: $\text{deg}_{L(\Gamma(\mathbb{Z}_n))}[u, v] =$

$$\begin{cases} \gcd(u, n) + \gcd(v, n) - 4, & \text{if } u^2 \neq 0 \text{ and } v^2 \neq 0; \\ \gcd(u, n) + \gcd(v, n) - 5, & \text{if either } u^2 = 0 \text{ or } v^2 = 0; \\ \gcd(u, n) + \gcd(v, n) - 6, & \text{if } u^2 = 0 \text{ and } v^2 = 0; \end{cases}$$

In [32], Ju and Wu proved that $\kappa(\Gamma(\mathbb{Z}_n)) = \lambda(\Gamma(\mathbb{Z}_n)) = \delta(\Gamma(\mathbb{Z}_n))$ that is, vertex connectivity, edge connectivity and minimum degree of the zero-divisor graph of ring \mathbb{Z}_n always coincide (Theorem 3.2, [32]). Also minimum degree of $\Gamma(\mathbb{Z}_n)$ is discussed. Let $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ for primes $p_1 < p_2 < \dots < p_r$. Then

$$\delta(\Gamma(\mathbb{Z}_n)) = \begin{cases} p_1 - 1, & \text{if } r > 1 \text{ or } r = 1, e_r = 2 \\ p_1 - 2, & \text{if } r = 1, e_r > 2 \end{cases}$$

In [58], same result for connectivity and minimum degree is proved by using a different approach. [30] gives the relation for vertex degrees of $\Gamma(\mathbb{Z}_n[i])$, and $L(\Gamma(\mathbb{Z}_n[i]))$:

Theorem 12 (Theorem 10.2, [30]). *Let $v \in V(\Gamma(\mathbb{Z}_n[i]))$ and $\text{g.c.d}(v, n) = c + di$. Then*

$$\text{deg}(v) = \begin{cases} c^2 + d^2 - 1, & \text{if } v^2 \neq 0 \\ c^2 + d^2 - 2, & \text{if } v^2 = 0. \end{cases}$$

Corollary 13 (Corollary 10.3, [30]). *Let $[u, v] \in V(L(\Gamma(\mathbb{Z}_n[i])))$, $\text{g.c.d}(u, n) = a + bi$ and $\text{g.c.d}(v, n) = c + di$. Then*

$$\text{deg}([u, v]) = \begin{cases} a^2 + b^2 + c^2 + d^2 - 4, & \text{if } u^2 \neq 0 \text{ and } v^2 \neq 0 \\ a^2 + b^2 + c^2 + d^2 - 5, & \text{if } u^2 = 0 \text{ and } v^2 \neq 0 \\ a^2 + b^2 + c^2 + d^2 - 6, & \text{if } u^2 = 0 \text{ and } v^2 = 0. \end{cases}$$

From these, one can easily compute the degrees of complement of these graphs. It is also shown that $L(\Gamma(\mathbb{Z}_n[i]))$ is regular if and only if $n = p, q^2, q_1q_2$ and $L(\overline{\Gamma(\mathbb{Z}_n[i])})$ is regular if and only if $n = p$ or q^3 (see, Theorem 11.4 and Theorem 11.6).

The graph $\Gamma(\mathbb{Z}_n[i])$ is regular if and only if $n = 2$ or $n = p$ or $n = q^2$ (Theorem 25, [46]). Complement of graph $\overline{\Gamma(R)}$ is regular if and only if $\Gamma(R)$ is regular. Therefore $\overline{\Gamma(\mathbb{Z}_n[i])}$ is regular if and only if $n = 2$ or $n = p$ or $n = q^2$, [48].

Also, to determine the degree of every vertex in graph $\Gamma(\mathbb{Z}_n[i])$ when $n = 2^m, m \geq 2, n = q^m, m \geq 3$ and $n = p^m, m \geq 1$, (see, Theorem 4-Theorem 6, [30]).

Author	Graph	Property
Philips et al. (2004)	$\Gamma(\mathbb{Z}_n), \overline{\Gamma(\mathbb{Z}_n)}$	Vertex degree, Vertex connectivity of $\overline{\Gamma(\mathbb{Z}_n)}$ for some cases, Vertex of minimum degree in $\Gamma(\mathbb{Z}_n)$
Aponte et al. (2005)	$L(\Gamma(\mathbb{Z}_n))$	Vertex degree
Skoweonek-Kaziow (2008)	$\Gamma(\mathbb{Z}_n)$	Vertex of maximum degree
Osba et al. (2008)	$\Gamma(\mathbb{Z}_n[i])$	Regularity
Osba et al. (2012)	$\overline{\Gamma(\mathbb{Z}_n[i])}$	Regularity
Ghanem and Nazzal (2012)	$\Gamma(\mathbb{Z}_n[i]), L(\Gamma(\mathbb{Z}_n[i])), L(\overline{\Gamma(\mathbb{Z}_n[i])})$	Vertex degree of all, Regularity of $L(\Gamma(\mathbb{Z}_n[i]))$ and $L(\overline{\Gamma(\mathbb{Z}_n[i])})$
Ju and Wu (2014)	$\Gamma(\mathbb{Z}_n)$	$\kappa(\Gamma(\mathbb{Z}_n)) = \lambda(\Gamma(\mathbb{Z}_n)) = \delta(\Gamma(\mathbb{Z}_n))$, Maximum degree
Reddy et al. (2018)	$\Gamma(\mathbb{Z}_n)$	Vertex and Edge connectivity, Minimum degree

2.3 When a graph is Planar, Eulerian and Hamiltonian

A *planar graph* is a graph that can be drawn in a plane in such a way that its edges intersect only at their end points. The first work in this direction was given by Anderson et al. [9] in 2001, where they asked a question- Which finite commutative rings R have $\Gamma(R)$ planar? Akbari et al. [4], Smith [71], and Belshoff and Chapman [16] worked on the question posed by them on planarity of zero-divisor graphs. They give a complete list of finite commutative rings R for which $\Gamma(R)$ is planar. For more detail, one can see section 6 of [10].

In the following theorems, we discuss the planarity of complement graph of \mathbb{Z}_n :

Theorem 14 (Theorem 4.1, [55]). $\overline{\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r})}$ is planar if and only if $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r} \cong \mathbb{Z}_{p^2}$ for some prime p or $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, \mathbb{Z}_8 , $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_4 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_5$.

Next, we discuss the planarity of zero-divisor graphs of Gaussian integers.

Theorem 15. Let $\Gamma(\mathbb{Z}_n[i])$ be a zero-divisor graph of Gaussian integers modulo n . Then

1. (Theorem 22, [46]) The graph $\Gamma(\mathbb{Z}_n[i])$ is planar if and only if $n = 2, 4$.
2. (Theorem 9, [48]) $\overline{\Gamma(\mathbb{Z}_n[i])}$ is planar if and only if $n = 2, 5$ or q^2 .
3. (Theorem 2.3, [30]) The graph $L(\overline{\Gamma(\mathbb{Z}_n[i])})$ is planar if and only if $n = 5$.
4. (Theorem 2.3, [43]) $L(\Gamma(\mathbb{Z}_n[i]))$ is never planar.

The *genus* of a graph is the minimal integer n such that the graph can be drawn without crossing itself on a sphere with n handles. More generally, G has genus g if it can be embedded in a surface of genus g , but not in one of genus $g - 1$. A graph G is *planar* if it can be embedded in the plane and is *toroidal* if it is not planar, but can be embedded in a torus. Let $\gamma(G)$ denote the genus of G ; so G is planar (resp. toroidal) when $\gamma(G) = 0$ (resp. $\gamma(G) = 1$). Genus of zero-divisor graphs of commutative rings is given in [77], [78], [76], [19].

Now we discuss the results of Eulerian and Hamiltonian graphs. A graph G is *Eulerian* if the graph is both connected and has a closed trail containing all edges of the graph. A graph is *hamiltonian* if it is connected and there exists a closed walk that visits every vertex exactly once except starting vertex without repeating the edges.

Following two theorems discuss when $\Gamma(R)$ is Eulerian and Hamiltonian:

Theorem 16. Let $n = p_1^{e_1} \dots p_r^{e_r}$. Then

1. (Theorem 3.1, [23]) $\Gamma(\mathbb{Z}_n)$ is Eulerian if and only if n is odd and square-free or $n = 4$.
2. (Theorem 8.1, [12]) $L(\Gamma(\mathbb{Z}_n))$ is Eulerian if and only if n is odd and square-free.
3. (Theorem 8.2, [12]) $\overline{\Gamma(\mathbb{Z}_n)}$ is Eulerian if and only if $n = p^2$ for some prime p .
4. (Theorem 29, [46]) The graph $\Gamma(\mathbb{Z}_n[i])$ is Eulerian if and only if $n = 2$ or n is a prime congruent to 1 modulo 4 or n is a composite integer which is a product of distinct odd primes.

5. (Theorem 12, [48]) If $\overline{\Gamma(\mathbb{Z}_n[i])}$ is connected, then it is Eulerian if and only if $n = 2$ or n is odd.
6. (Theorem 2.1, [30]) $L(\overline{\Gamma(\mathbb{Z}_n[i])})$ is Eulerian if and only if n is a product of distinct odd primes.
7. (Theorem 3.2, [43]) $L(\Gamma(\mathbb{Z}_n[i]))$ is Eulerian graph if and only if $n = 2$, p , q^2 , or n is a composite integer which is a product of distinct odd primes.

Theorem 17. Let $n = p_1^{e_1} \dots p_r^{e_r}$. Then

1. (Theorem 7.1, 7.2, [23]) $\Gamma(\mathbb{Z}_n)$ is Hamiltonian for $n = p_1^{e_1} p_2^{e_2}$ and $p_1^2 p_2 \dots p_r$.
2. (Lemma 9.1, 9.2, [12]) $L(\Gamma(\mathbb{Z}_n))$ is Hamiltonian if $n = 2p$ and $n = p_1 p_2 \dots p_r$.
3. (Corollary 11, [47]) $\Gamma(\mathbb{Z}_n[i])$ is Hamiltonian if $n = p$ or $n = q^2$.
4. (Theorem 6.1, [30]) If $n = 2^m$, $m \geq 2$ or $n = q^m$, $m \geq 3$, then $L(\overline{\Gamma(\mathbb{Z}_n[i])})$ is Hamiltonian.
5. (Theorem 6.2, [30]) If $n = p^m$, $m \geq 2$ or n is a composite integer such that $n \neq q_1 q_2$, then $L(\overline{\Gamma(\mathbb{Z}_n[i])})$ is hamiltonian.
6. (Corollary 4.4, [43]) If $n = p$, 2^m , q^m , where $m \geq 2$, or n is a composite integer which is a product of distinct odd primes then $L(\Gamma(\mathbb{Z}_n[i]))$ is Hamiltonian.

The graph $\Gamma(\mathbb{Z}_n[i])$ and $\overline{\Gamma(\mathbb{Z}_n[i])}$ are locally H if and only if $n = 2$, p , or q^2 (Theorem 1, [47], Theorem 10, [48]). If $n = p^m$, $m \geq 1$ or n is a composite integer such that $n \neq q_1 q_2$, then both $\Gamma(\mathbb{Z}_n[i])$ and $L(\overline{\Gamma(\mathbb{Z}_n[i])})$ are locally connected (Theorem 5.3, [30]). From the following theorem, $L(\Gamma(\mathbb{Z}_n[i]))$ is locally H if and only if $n = p$, q^2 , $q_1 q_2$ and $L(\overline{\Gamma(\mathbb{Z}_n[i])})$ is locally H if and only if $n = p$ or q^3 .

Theorem 18 (Theorem 12.2, [30]). The following statements are equivalent

1. The graph $L(\Gamma(\mathbb{Z}_n[i]))/L(\overline{\Gamma(\mathbb{Z}_n[i])})$ is regular.
2. The graph $L(\Gamma(\mathbb{Z}_n[i]))/L(\overline{\Gamma(\mathbb{Z}_n[i])})$ is locally H .

Author	Graph	Property
Philips et al. (2004)	$\Gamma(\mathbb{Z}_n)$	Planarity
Aponte et al. (2005)	$L(\Gamma(\mathbb{Z}_n)), \overline{\Gamma(\mathbb{Z}_n)}$	Eulerian, Hamiltonian
Cordova et al. (2005)	$\Gamma(\mathbb{Z}_n)$	Eulerian, Hamiltonian
Osba et al. (2008)	$\Gamma(\mathbb{Z}_n[i])$	Planarity, Eulerian
Osba et al. (2011)	$\Gamma(\mathbb{Z}_n[i])$	Hamiltonian, Locally H
Osba et al. (2012)	$\Gamma(\mathbb{Z}_n[i])$	Planarity, Eulerian, Locally H

Ghanem and Nazzal (2012)	$L(\overline{\Gamma(\mathbb{Z}_n[i])})$	Planarity, Eulerian, Hamiltonian, Locally H
Nazzal and Ghanem (2012)	$L(\Gamma(\mathbb{Z}_n[i]))$	Planarity, Eulerian, Hamiltonian

2.4 Domination number and Independence number

A dominating set in a graph G is a subset of the vertex set of G with the property that every vertex in G is either in the dominating set or adjacent to a vertex that is in the dominating set. The *domination number* of G , denoted by $Domn(G)$, is defined as the cardinality of a minimum dominating set of G .

An independent set in a graph G is a subset I of the vertex set of G such that no two vertices of I are adjacent i.e. I is a subset of null vertices. The *independence number* of G , denoted by $Indep(G)$, is defined as the cardinality of a maximum independent set of G .

Let $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ where, p_1, p_2, \dots, p_k are distinct primes and that e_i 's are positive integers. Also $n \neq 2p$. Then domination number of $\Gamma(\mathbb{Z}_n)$ is k (Theorem 1, [1]). When $n = 2p$, domination number is 1.

Proposition 19 (Proposition 5.1, [55]). $A_{p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}}$ is an independent set if and only if $a_i \geq \frac{e_i}{2}$ for all i .

In 2008, AbdAlJawad [1] gave the independence number of $\Gamma(\mathbb{Z}_n)$ for $n = p^2, p^k q, pq^k, p^2 q^2$ where $p < q$, (see, Theorem 2-5). In 2012, Ghanem and Nazzal examined the domination number of $\Gamma(\mathbb{Z}_n[i])$ and its line graph. For $L(\Gamma(\mathbb{Z}_n[i]))$, domination number is only calculated for $n = 2^m, q^m, p^m, m \geq 2$. For domination number of $\Gamma(\mathbb{Z}_n[i])$, they gave the following theorem:

Theorem 20 (Theorem 8.2, [43]). If $n = \prod_{j=1}^k \pi_j^{m_j}$, where $k \geq 1$ and π_j 's are distinct gaussian prime and m_j 's are positive integers and $n \neq 2$ or q . Then $\gamma(\Gamma(\mathbb{Z}_n[i])) = k$, if n is odd, and $\gamma(\Gamma(\mathbb{Z}_n[i])) = k - 1$, if n is even.

Theorem 21 (Theorem 8.1, [30]). The domination number of complement graph of $\Gamma(\mathbb{Z}_n[i])$ is given as

1. If $n \neq 2, q^m$, then $\gamma(\overline{\Gamma(\mathbb{Z}_n[i])}) = 2$.
2. $\gamma(\overline{\Gamma(\mathbb{Z}_{q^2}[i])}) = q^2 - 1$ and $\gamma(\overline{\Gamma(\mathbb{Z}_{q^m}[i])}) = q^2, m \geq 3$.

The study of the domination number of the line graph of G leads to the study of edge or line domination number of G , that is, $\gamma(L(G)) = \gamma'(G)$. Also for any graph

G , $\gamma'_i(G) = \gamma'(G)$. In 2012, Ghanem and Nazzal [30] gave relation for domination number of $\gamma(L(\overline{\Gamma(\mathbb{Z}_n[i])}))$. $\gamma(L(\overline{\Gamma(\mathbb{Z}_n[i])})) =$

$$\begin{cases} 2^{m-1}(2^{m-1} - 1), & \text{if } n = 2^m, m \geq 2 \\ \frac{1}{2}q^2 \lfloor \frac{m}{2} \rfloor \left(q^{\lceil \frac{m}{2} \rceil} - 1 \right)^2, & \text{if } n = q^m, m \geq 3 \\ \frac{1}{2} \left(2p^{2m-1} - p^{2m-2} - p^{2m-\lceil \frac{m}{2} \rceil} - 2 \right), & \text{if } n = p^m. \end{cases}$$

The *connected domination number* of a connected graph G denoted by $\gamma_c(G)$, is the size of a minimum connected dominating set of a graph G . Following theorems gave the results for connected domination number:

Theorem 22 (Theorem 4.3, [65]). *Let $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, where p_1, p_2, \dots, p_k are distinct primes and the e_i 's are positive integers, then $\gamma_c(\Gamma(\mathbb{Z}_n)) = k$.*

Theorem 23. *For any graph $\Gamma(\mathbb{Z}_n)$, $\gamma(\Gamma(\mathbb{Z}_n)) = \gamma_c(\Gamma(\mathbb{Z}_n))$ if and only if $\Gamma(\mathbb{Z}_n)$ has a spanning tree T with maximum number of pendent vertices such that for every set A of pendent vertices with $\langle A \rangle$ independent of G , there exists a non-pendent vertex v in T such that $A \subseteq N(v)$, where $\langle A \rangle$, means subgraph induced by the set A .*

The *weak domination number* of a graph G denoted by $\gamma_\omega(G)$, is the minimum cardinality of a weak dominating set. In 2013, Sankar and Meena [66] calculated weak domination number of $\Gamma(\mathbb{Z}_n)$ for $n = p^2, pq, 2^n, 2^np, 3^n$.

1. $\gamma_\omega(\Gamma(\mathbb{Z}_{p^2})) = 1$
2. $\gamma_\omega(\Gamma(\mathbb{Z}_{pq})) = q - 1, p < q$
3. $\gamma_\omega(\Gamma(\mathbb{Z}_{2^n})) = \begin{cases} 2^{\frac{n}{2}} \sum_{i=0}^{\frac{n-4}{2}} 2^i + 1, & \text{if } n \text{ is even} \\ 2^{\frac{n-1}{2}} \sum_{i=0}^{\frac{n-3}{2}} 2^i, & \text{otherwise.} \end{cases}, n > 2$
4. $\gamma_\omega(\Gamma(\mathbb{Z}_{2^np})) = 2^{n-1}(p - 1), p > 2^n$
5. $\gamma_\omega(\Gamma(\mathbb{Z}_{3^n})) = 3^{n-1} - 8, p > 3$.

In [73], point covering number, edge covering number and independence number of $L(\Gamma(\mathbb{Z}_n))$, for $n = pq, p^2, p^3$ are discussed. It is proved that edge covering number is less than the point covering number and independence number in $L(\Gamma(\mathbb{Z}_{pq}))$. In 2019, Abu Hijleh et al. [2] determined graph invariants matching number, vertex covering number and independence number for the zero-divisor graph over the rings \mathbb{Z}_{p^k} and $\mathbb{Z}_{p^k q^r}$.

Remark 24 (Remark 1, [2]). For $\Gamma(\mathbb{Z}_n)$, $\beta(\Gamma(\mathbb{Z}_n)) + \alpha(\Gamma(\mathbb{Z}_n)) = |V(\Gamma(\mathbb{Z}_n))|$

Author	Graph	Property
Philips et al. (2004)	$\overline{\Gamma(\mathbb{Z}_n)}$	Independent set
AbdAlJawad and Al-Ezeh (2008)	$\Gamma(\mathbb{Z}_n)$	Domination number
Ghanem and Nazzal (2012)	$\overline{\Gamma(\mathbb{Z}_n[i])},$ $L(\overline{\Gamma(\mathbb{Z}_n[i])})$	Domination number
Nazzal and Ghanem (2012)	$\Gamma(\mathbb{Z}_n[i]),$ $L(\Gamma(\mathbb{Z}_n[i]))$	Domination number
Sankar and Meena (2012)	$\Gamma(\mathbb{Z}_n)$	Connected domination number
Sankar and Meena (2012)	$\Gamma(\mathbb{Z}_n)$	Weak domination number for $n = p^2, pq, 2^n, 2^n p, 3^n$
Suthar and Prakash (2015)	$L(\Gamma(\mathbb{Z}_n))$	Point covering number, Edge covering number, Independence number for $n = pq, p^2, p^3$
AbuHijleh et al. (2019)	$\Gamma(\mathbb{Z}_n)$	Matching number, Vertex covering number and Independence number for $n = p^k$ and $p^k q^r$

2.5 Chromatic Number

Beck [15] introduced the notion of coloring in a commutative ring R in 1988. The *chromatic number* $\chi(\Gamma_0(R))$ of a graph is the minimal number of colors needed to color the elements of $\Gamma_0(R)$ so that no two adjacent elements have the same color. A subset $\{x_1, x_2, \dots, x_n\}$ of $\Gamma_0(R)$ is called a clique if $x_i x_j = 0$.

Beck [15] showed that for a ring R , the following conditions are equivalent: 1) $\chi(\Gamma_0(R))$ is finite. 2) $cl(\Gamma_0(R))$ is finite. 3) the nilradical of R , $nil(R)$, is finite and $nil(R)$ is a finite intersection of prime ideals. He called a ring satisfying any of these three equivalent conditions a coloring. He gave a conjecture that $\chi(\Gamma_0(R)) = cl(\Gamma_0(R))$ for any coloring R . Anderson and Naseer [10] gave a counter example to Beck's [15] conjecture (Theorem 2.1, [7]). They also determined the finite commutative rings R with $\chi(\Gamma_0(R)) = 4$ (Theorem 7.4(c)). All finite commutative rings with clique equals to 1, 2 or 3 are listed in Theorem 7.4 of [10].

Hedetniemi [31] conjectured that for all graphs G and G' , $\chi(G \times G') = \min(\chi(G), \chi(G'))$. In 2005, Cordova et al. [23] proved this conjecture for all $\Gamma(\mathbb{Z}_n)$, $\chi(\Gamma(\mathbb{Z}_n)) \times$

$\Gamma(\mathbb{Z}_m) = \min(\chi(\Gamma(\mathbb{Z}_n)), \chi(\Gamma(\mathbb{Z}_m)))$. The following lemma gives the clique size in $\Gamma(\mathbb{Z}_n)$.

Lemma 25 (Lemma 4.2, [23]). *Let $n = p_1^{e_1} \dots p_r^{e_r} q_1^{f_1} \dots q_s^{f_s}$, where p_i and q_j are distinct primes, such that for every $1 \leq i \leq r$, e_i is even, and for every $1 \leq j \leq s$, f_j is odd. In $\Gamma(\mathbb{Z}_n)$ there exists a clique of size $p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1-1}{2}} \dots q_s^{\frac{f_s-1}{2}} + s - 1$.*

He also proved that for $\Gamma(\mathbb{Z}_n)$, chromatic number is equal to clique number, $\chi(\Gamma(\mathbb{Z}_n)) = cl(\Gamma(\mathbb{Z}_n))$ (see Corollary 4.3). A graph G is a core if any homomorphism from G to itself is an automorphism. Also, a subgraph H of G is called a core of G if H is a core itself, and there is a homomorphism from G to H . Next theorem gives the core of $\Gamma(\mathbb{Z}_n)$.

Theorem 26 (Theorem 5.1, [23]). *The core of the graph $\Gamma(\mathbb{Z}_n)$ is the maximal clique in $\Gamma(\mathbb{Z}_n)$.*

In 2006, Duane [27] determined the chromatic number of $\Gamma(\mathbb{Z}_n)$. She gave the following results in this context:

Theorem 27 (Theorem 4.9, [27]). *Suppose $m = p_1 p_2 \dots p_n$, where each p_i is a distinct prime. Then $\Gamma(\mathbb{Z}_m)$ is n -colorable.*

Theorem 28 (Theorem 4.10, [27]). *Suppose $m = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$, where each p_i is a distinct prime, $n \leq 2$, $a_i > 0$ for all i , and $a_i > 1$ for some i . Let $s = p_1^{\lfloor a_1/2 \rfloor} p_2^{\lfloor a_2/2 \rfloor} \dots p_n^{\lfloor a_n/2 \rfloor}$. Then $\Gamma(\mathbb{Z}_m)$ is s -colorable.*

Corollary 29 (Corollary 4.11, [27]). *If m and s are defined as in the previous theorem, then $\chi(\Gamma(\mathbb{Z}_m)) = s$ if at least one a_i is odd. Otherwise, $\chi(\Gamma(\mathbb{Z}_m)) = s - 1$.*

Chromatic number of $\Gamma(\mathbb{Z}_n)$ is also investigated in [70], [81]. In the following theorem, Aponte et al. [12] gave a relation between vertex coloring of $\Gamma(\mathbb{Z}_n)$ and chromatic number of $L(\Gamma(\mathbb{Z}_n))$.

Theorem 30 (Theorem 7.1, [12]). *In $\Gamma(\mathbb{Z}_n)$, the edge leads to the vertex coloring in $L(\Gamma(\mathbb{Z}_n))$. Let $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$. Moreover, $\chi(L(\Gamma(\mathbb{Z}_n))) = \omega(L(\Gamma(\mathbb{Z}_n))) = \Delta(\Gamma(\mathbb{Z}_n))$.*

Following theorem gives the general formula for the chromatic number of $\Gamma(\mathbb{Z}_n[i])$:

Theorem 31 (Theorem 26, [47]). *Let $n = 2^l \times (\prod_{k=1}^r q_k^{m_k} \times \prod_{k=r+1}^t q_k^{m_k}) \times (\prod_{k=1}^z p_k^{n_k} \times \prod_{k=z+1}^c p_k^{n_k})$, m_k is odd for all $k \leq r$ while m_k is even integer otherwise, n_k is odd for all $k \leq z$ while n_k is even integer otherwise and let $s = 2^l \times \prod_{k=1}^t q_k^{2 \lfloor \frac{m_k}{2} \rfloor} \times \prod_{k=1}^c p_k^{2 \lfloor \frac{n_k}{2} \rfloor}$. Then*

$$\chi(\Gamma(\mathbb{Z}_n[i])) = s + r + 2z - 1$$

Next we give results on chromatic number of line graph of zero-divisor graph of gaussian integers and their complement.

Theorem 32 (Theorem 5.1, [43]). $\omega(L(\Gamma(\mathbb{Z}_n[i]))) = \chi(L(\Gamma(\mathbb{Z}_n[i]))) =$

$$\begin{cases} 2^{2m-1} - 2, & \text{if } n = 2^m, m \geq 2, \\ q^{2m-2} - 2, & \text{if } n = q^m, m \geq 2, \\ p^{2m-1} - 1, & \text{if } n = p^m, m \geq 1. \end{cases}$$

Theorem 33 (Theorem 5.2, [43]). *If $n = 2^m \prod_{j=1}^r p_j^{r_j} \prod_{j=1}^t q_j \prod_{j=1}^s q_j^{s_j}$, where $m, r_j \geq 1$ and $s_j \geq 2$, then*

$$\omega(L(\Gamma(\mathbb{Z}_n[i]))) = \chi(L(\Gamma(\mathbb{Z}_n[i]))) = (2^{2m-1} - 1) \prod_{j=1}^r (p_j^{2r_j-1}) \prod_{j=1}^s (q_j^{2s_j-2} - 1) - 1$$

Corollary 34 (Corollary 7.8, [30]). *Let $\chi(L(\overline{\Gamma(\mathbb{Z}_n[i])}))$ be chromatic number of line graph of zero-divisor graph of gaussian integers then following results holds*

1. *If $n = 2^m, m \geq 2$, then $\chi(L(\overline{\Gamma(\mathbb{Z}_n[i])})) = 2^{2m-1} - 3$.*
2. *If $n = q^m, m \geq 3$, then $\chi(L(\overline{\Gamma(\mathbb{Z}_n[i])})) = q^{2m-2} - q^2 - 1$.*
3. *If $n = p^m, m \geq 2$, then $\chi(L(\overline{\Gamma(\mathbb{Z}_n[i])})) = 2p^{2m-1} - p^{2m-2} - p - 1$.*

Author	Graph	Property
Cordova et al. (2005)	$\Gamma(\mathbb{Z}_n)$	Chromatic number, Core
Aponte et al. (2005)	$\Gamma(\mathbb{Z}_n)$	Clique, Chromatic number
Duane (2006)	$\Gamma(\mathbb{Z}_n)$	Chromatic number and Coloring
Osba et al. (2011)	$\Gamma(\mathbb{Z}_n[i])$	Chromatic number
Ghanem and K. Nazzal (2012)	$L(\Gamma(\mathbb{Z}_n))$	Chromatic number
Nazzal and M. Ghanem (2012)	$L(\Gamma(\mathbb{Z}_n))$	Chromatic number
Ju and M. Wu (2014)	$\Gamma(\mathbb{Z}_n)$	Clique number
Yanjhao and Qijiao (2015)	$\Gamma(\mathbb{Z}_n)$	Clique number

3 Adjacency matrix, energy and eigen values

In this section, we discuss the results related to adjacency matrix, energy and eigen values. For any vertices x_i, x_j and for all $i, j \in \mathbb{N}$, the adjacency matrix of the zero divisor graph $\Gamma(R)$ of R is defined as $\mathbf{A} = [A_{ij}]$, where

$$A_{ij} = \begin{cases} 1, & x_i, x_j = 0, \\ 0, & \text{otherwise} \end{cases}$$

Adjacency matrix for zero-divisor graphs over finite ring of gaussian integer is studied by Sharma et al. [67] in 2010. Here properties like order of matrix, trace, singularity etc are discussed. In 2011, Sharma et al. [68] studied adjacency matrix of $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$, where p is a prime.

In 2013, Patra and Baruah [50] discussed adjacency matrix of zero-divisor graph of rings $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, $\mathbb{Z}_p \times \mathbb{Z}_{2p}$ and $\mathbb{Z}_p \times \mathbb{Z}_{p^2-2}$. Properties like neighborhood of vertex, neighborhood number, order of matrix etc are also discussed. Laplacian eigen values of the $\Gamma(\mathbb{Z}_n)$ are discussed in [21].

In [3], wiener index and energy of graph of $\Gamma(\mathbb{Z}_n)$ are studied for $n = pq$ or $n = p^2$. A MATLAB code is also presented for same.

$$\text{Energy}(\Gamma(\mathbb{Z}_n)) = \begin{cases} 2p - 4, & \text{if } n = p^2 \\ 2\sqrt{(p-1)(q-1)}, & \text{if } n = pq. \end{cases}$$

$$\text{Wiener index}(\Gamma(\mathbb{Z}_n)) = \begin{cases} \frac{(p-1)(P-2)}{2}, & \text{if } n = p^2 \\ p^2 + q^2 + pq - 4p - 4q + 5, & \text{if } n = pq. \end{cases}$$

function $Gz = \text{graph_zero_divisor_zn2}(p)$

$n = p;$

$M = [];$

for $i = 1 : n - 1$

for $j = 1 : n - 1$

if $\text{mod}(i * j, n) == 0$

$M = [M, i];$

break;

end

end

end

$n = \text{length}(M);$

for $i = 0 : n - 1$

$\text{axes}(i + 1, :) = [\cos(2 * pi * i/n), \sin(2 * pi * i/n)];$

end

```

Gz = zeros(n);
holdon
for i = 1 : n
plot(axes(i, 1), axes(i, 2), ' *')
if mod(M(i)^2, p) == 0
Gz(i, i) = 1;
plot(axes(i, 1), axes(i, 2), ' rO')
end
end
for i = 1 : n - 1
for j = i + 1 : n
if mod(M(i) * M(j), p) == 0
Gz(i, j) = 1; Gz(j, i) = 1;
plot(axes([i, j], 1), axes([i, j], 2));
end
end
end
functions = Wiener(B)
B(B == 0) = inf;
A = triu(B, 1) + tril(B, -1);
m = length(A);
B = zeros(m);
j = 1;
while j <= m
for i = 1 : m
fork = 1 : m
B(i, k) = min(A(i, k), A(i, j) + A(j, k));
end
end
A = B;
j = j + 1;
end
s = sum(sum(A))/2;
functione = energy(a)
eg = eig(a);
e = sum(abs(eg));

```

Eigen values and wiener index of $\Gamma(\mathbb{Z}_{p^3})$ and $\Gamma(\mathbb{Z}_{p^2q})$ are discussed in [57].

Theorem 35 (Theorem 4.1, [57]). *Let $n = p^3$. Then the only non-zero eigenvalues of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ are given by $\frac{(p-1)(1 \pm \sqrt{1+4p})}{2}$.*

Theorem 36 (Theorem 4.3, [57]). *Let $n = p^2q$ with p and q primes. If $\lambda \neq 0$ is a*

nonzero eigenvalue of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$, then

$$\lambda^4 - (p-1)\lambda^3 - 2p(q-1)^2(p-1)\lambda^2 + p(p-1)^2(q-1)\lambda + p(p-1)^3(q-1)^2 = 0$$

$$\begin{aligned} & \text{Wiener index of } \Gamma(\mathbb{Z}_n) \\ = & \begin{cases} \binom{p-1}{2}(2p^3 - 3p - 2), & \text{if } n = p^3 \\ \frac{1}{2}[p^2(2p^2 - 4p - 1) + p(4p^2q + 2pq^2 - 8pq - 2q + 5) + 2], & \text{if } n = p^2q. \end{cases} \end{aligned}$$

In [75], energy and wiener index of line graph of zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ for $n = pq$ and p^2 are discussed.

$$\begin{aligned} \text{Energy}(L(\Gamma(\mathbb{Z}_n))) &= \begin{cases} 2p^2 - 10p + 8, & \text{if } n = p^2, p \geq 5 \\ 4pq - 8p - 8q + 16, & \text{if } n = pq. \end{cases} \\ \text{W}(L(\Gamma(\mathbb{Z}_n))) &= \begin{cases} \frac{1}{4}(p-1)(p-2)^2(p-3), & \text{if } n = p^2, p \geq 5 \\ \frac{1}{2}(p-1)(q-1)(2pq - 3p - 3q + 4), & \text{if } n = pq, q > p > 2. \end{cases} \end{aligned}$$

Neighbourhood number of line graph of zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ is also discussed. A MATLAB program for the same is given.

Now we give relations for Hosoya polynomial and wiener index of $\Gamma(\mathbb{Z}_{p^m})$ and $\Gamma(\mathbb{Z}_{p^m q})$.

Theorem 37 (Theorem 2.10, [40]). $H(\Gamma(\mathbb{Z}_{p^m}); x) = a_0 + a_1x + a_2x^2$, where $a_0 = p^{m-1} - 1$, $a_1 = \frac{1}{2}[(m-1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2]$, and $a_2 = \frac{1}{2}[p^{2(m-1)} - (m-1)p^m + (m-3)p^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}]$

Theorem 38 (Theorem 3.6, [40]). $H(\Gamma(\mathbb{Z}_{p^m q}); x) = a_0 + a_1x + a_2x^2 + a_3x^3$, where $a_0 = (p+q-1)p^{m-1} - 1$, $a_1 = \frac{1}{2}[2mq(p-1) - (m+1)p + m]p^{m-1} - \frac{1}{2}p^{\lfloor \frac{m}{2} \rfloor} + 1$, $a_2 = \frac{1}{2}(p^2 + q^2 - 1)p^{2m-2} + \frac{1}{2}[(m-4)p - 2(m-1)pq + (2m-5)q - m + 5]p^{m-1} + \frac{1}{2}p^{\lfloor \frac{m}{2} \rfloor}$, and $a_3 = (q-1)(p-1)(p^{2m-2} - pm - 1)$.

$$\text{W}(\Gamma(\mathbb{Z}_n)) =$$

$$\begin{cases} \frac{1}{2}[2p^{2(m-1)} - (m-1)p^m + (m-6)p^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} + 2], & \text{if } n = p^m \\ [p^2 + q^2 + 3(pq - p - q) + 2]p^{2m-2} + \frac{1}{2}[(m-3)p - 2(m+1)pq + 2(m-2)q]p^{m-1} + \frac{1}{2}p^{\lfloor \frac{m}{2} \rfloor} + 1, & \text{if } n = p^m q. \end{cases}$$

Hosoya polynomial and wiener index of $\Gamma(\mathbb{Z}_{p^m q^2})$ is discussed in Corollary 3.6 and theorem 3.2 of [69]. In 2019, Monius [41] investigated the spectra of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ for some prime p .

Author	Graph	Property
Sharma et al. (2010)	$\Gamma(\mathbb{Z}_n[i])$	Adjacency matrix for some values of n
Sharma et al. (2010)	$\Gamma(\mathbb{Z}_n[i]), \Gamma(\mathbb{Z}_n)$	Adjacency matrix of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ and $\Gamma(\mathbb{Z}_p[i] \times \mathbb{Z}_p[i])$
Ahmadi and Jahani-Nezhad (2011)	$\Gamma(\mathbb{Z}_n)$	Wiener index and energy for $n = p^2$ and pq
Patra and Baruah (2013)	Direct product of $\Gamma(\mathbb{Z}_n)$	Adjacency matrix of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}, \mathbb{Z}_p \times \mathbb{Z}_{2p}$ and $\mathbb{Z}_p \times \mathbb{Z}_{p^2-2}$
Reddy et al. (2017)	$\Gamma(\mathbb{Z}_n)$	Eigen values and Wiener index for $n = p^3$ and p^2q
Suthar and Prakash (2018)	$L(\Gamma(\mathbb{Z}_n))$	Energy and Wiener index for $n = pq$ and p^2
Mohammad and Authman (2018)	$\Gamma(\mathbb{Z}_n)$	Hosoya polynomial and Wiener index for $n = p^m$ and $p^m q$
Shuker et al. (2019)	$\Gamma(\mathbb{Z}_n)$	Hosoya polynomial and Wiener index for $n = p^m q^2$
Monius (2019)	$\Gamma(\mathbb{Z}_n)$	Spectra of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$

4 Other results

By referencing a couple of other results on zero-divisor graphs, we conclude this survey article. As far as references is concerned, we restrict ourselves in the content for the most part to key references that give the reader further access to the literature; we apologize for any unintentional exclusions.

In [34], Lu and Wu determined that $\Gamma(\mathbb{Z}_n)$ is end-regular if and only if $n = pq$ or p^3 . Sankar et al. [64] evaluated the crossing number of $\Gamma(\mathbb{Z}_n)$ for some cases of $n = p^2, p^3$. In [37], authors studied rectilinear crossing number for $n = pq, pqr$ and $4p$ and compare that with crossing number. Nikmehar et al. [45] computed wiener, hyper-wiener, reverse wiener index and many more. Suthar and Parkash [73] studied the point covering number and independence number of $L(\mathbb{Z}_n)$, for some positive integer $n = p^2, p^3, pq$. Malathi et al. ([37], [38], [39]) found the consistency of rectilinear crossing number of zero-divisor graphs of $n = p^2, p^3, pq$ and edge non-edge crossing number of $\Gamma(\mathbb{Z}_n)$ for $n = p^3, pqr$ respectively. The properties like sum cordial labelling, geometric mean labelling and pair sum labelling of zero-divisor graph of \mathbb{Z}_n for some n is investigated in [51], [53] and [72]. Suthar and Parkash [74] studied energy, Laplacian energy, distance energy, and Wiener index of the total graph of \mathbb{Z}_n and also find the relation among these energies. In [56], Rajakumaran and Selvi determined the decomposition of $\Gamma(\mathbb{Z}_n)$ into a finite number of paths and cycles for $n = p^2, p^3$. Periaswamy and Selvi in [52], [54] find the edge sum index and integral sum of zero divisor graph and studied their properties for some values of n . In [17], strong metric dimension of the zero-divisor graphs of \mathbb{Z}_n and $\mathbb{Z}_n[i]$ is given. Compressed zero-divisor graph of commutative ring is studied in [35]. In [2] and [49], odd & even number and matching number & covering vertex number are studied.

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