SOME RESULTS OF $\eta$-RICCI SOLITONS ON $(LCS)_n$-MANIFOLDS

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Abstract. In this paper, we consider an $\eta$-Ricci soliton on the $(LCS)_n$-manifolds $(M, \phi, \xi, \eta, g)$ satisfying certain curvature conditions like: $R(\xi, X) \cdot S = 0$ and $W_2(\xi, X) \cdot S = 0$. We show that on the $(LCS)_n$-manifolds $(M, \phi, \xi, \eta, g)$, the existence of $\eta$-Ricci soliton implies that $(M, g)$ is a quasi-Einstein. Further, we discuss the existence of Ricci solitons with the potential vector field $\xi$. In the end, we construct the non-trivial examples of $\eta$-Ricci solitons on the $(LCS)_n$-manifolds.

1 Introduction

In 2003, Shaikh [33] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifold) with an example, which generalize the notion of LP-Sasakian manifolds introduced by Matsumoto [27] and also by Mihai and Rosca [28]. The properties of $(LCS)_n$-manifolds have been studied by many geometer, for instance we refer ([7], [8], [22]-[25], [29], [34], [36], [39]-[42]).

The Ricci solitons are natural generalization of Einstein metrics on a Riemannian manifold, being generalized fixed points of Hamilton’s Ricci flow $\frac{\partial}{\partial t} g = -2S$ [20]. The evolution equation defining the Ricci flow is a kind of nonlinear diffusion equation, an analogue of heat equation for metrics. Under Ricci flow, a metric can be improved to evolve into more canonical one by smoothing out its irregularities, depending on the Ricci curvature of the manifold: it will expand in the directions of negative Ricci curvature and shrink in the positive case. The geometrical properties of the Ricci solitons have been studied in ([1]-[5], [7]-[13], [17]-[21], [26], [31], [37], [38], [43]) and by others. In paracontact geometry, the Ricci soliton first appeared in the paper of G. Calvaruso and D. Perrone [6]. C. L. Bejan and M. Crasmareanu studied the properties of Ricci solitons on the 3-dimensional normal paracontact manifolds [3]. A more general notion of a Ricci soliton is that of $\eta$-Ricci soliton introduced by J. T. Cho and M. Kimura [18], which was treated by C. Calin and M. Crasmareanu on Hopf hypersurfaces in complex-space-forms [4]. Metrics satisfying

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Ricci flow equations are interesting and useful in physics and are often referred as quasi-Einstein ([12]-[16]).

2 \((LCS)_n\)-manifolds \((M, \phi, \xi, \eta, g)\)

Let \(M\) be an \(n\)-dimensional smooth connected paracontact Hausdroff manifold equipped with a Lorentzian metric \(g\). Then \((M, g)\) is a Lorentzian manifold, that is, \(M\) admits a smooth symmetric tensor field \(g\) of type \((0, 2)\) such that for each point \(p \in M\), the tensor \(g_p : T_p M \times T_p M \to \mathbb{R}\) is a non degenerate inner product of signature \((-\ldots, +-\ldots, +)\), where \(T_p M\) denotes the tangent space of \(M\) at \(p\) and \(\mathbb{R}\) is the real number. A non-zero vector field \(v \in T_p M\) is said to be timelike (resp., non-spacelike, null, and spacelike) if it satisfies \(g_p(v, v) < 0\) (resp., \(\leq 0\), \(= 0\), \(> 0\)) [30].

Definition 1. A non-vanishing vector field \(\rho\) on a Lorentzian manifold \((M, g)\) defined by \(g(X, \rho) = A(X), \forall X \in \chi(M)\) is said to be a concircular vector field [41] if \[(\nabla_X A)(Y) = \alpha \{g(X, Y) + \omega(X)A(Y)\},\]
where \(\alpha\) is a non-zero scalar and \(\omega\) is a closed 1-form.

If the Lorentzian manifold \(M\) admits a unit timelike concircular vector field \(\xi\), called the generator of the manifold, then we have
\[g(\xi, \xi) = -1,\quad g(X, \xi) = \eta(X),\quad (\nabla_X \eta)(Y) = \alpha \{g(X, Y) + \eta(X)\eta(Y)\},\] (2.1)
where \(\alpha \neq 0\) and \(\eta\) is a non-zero 1-form. It is obvious from (2.1) that
\[\nabla_X \xi = \alpha\{X + \eta(X)\xi\}\] (2.2)
for all vector field \(X\) on \(M\). Here \(\nabla\) denotes the operator of the covariant differentiation with respect to the Lorentzian metric \(g\) and \(\alpha\) satisfies
\[\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X),\] (2.3)
\(\rho\) being a certain scalar function given by \(\rho = -(\xi\alpha)\). If we put
\[\alpha \phi X = \nabla_X \xi,\] (2.4)
then (2.2) and (2.4) give
\[\phi X = X + \eta(X)\xi,\] (2.5)
where \(\phi\) is a \((1,1)\)-tensor, called the structure tensor of \(M\). Thus the Lorentzian manifold \(M\) together with a unit timelike concircular vector field \(\xi\), its associated 1-form \(\eta\) and \((1,1)\)-tensor field \(\phi\) is said to be a Lorentzian concircular structure manifold (briefly \((LCS)_n\)-manifold) [33]. Especially, if we take \(\alpha = 1\), then we can
obtain the \( LP \)-Sasakian structure of Matsumoto \([27]\). For details, we refer \([11]\) and the references therein. In an \((LCS)_n\)-manifold, \( n > 2 \), the following relations
\[
\begin{align*}
\eta(\xi) = -1, & \quad \phi \xi = 0, \quad \phi^2 X = X + \eta(X) \xi, \\
\eta(\phi X) = 0, & \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X) \eta(Y), \quad (2.6)
\end{align*}
\]
hold for any vector fields \( X, Y, Z \) on \( M \).

### 3 \( \eta \)-Ricci solitons on \((LCS)_n\)-manifolds \((M, \phi, \xi, \eta, g)\)

Let \((M, \phi, \xi, \eta, g)\) be an \((LCS)_n\)-manifold, then the quartet \((g, \xi, \lambda, \mu)\) on \( M \) is said to be an \( \eta \)-Ricci soliton \([18]\) if it satisfies
\[
\begin{align*}
L_\xi g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0, \quad (3.1)
\end{align*}
\]
where \( L_\xi \) is the Lie-derivative operator along the vector field \( \xi \), \( \lambda \) and \( \mu \) are real constants. We write \( L_\xi g \) in term of the Levi-Civita connection \( \nabla \) as:
\[
(L_\xi g)(X, Y) = g(\nabla_Y \xi, X) + g(Y, \nabla_X \xi) = 2\alpha [g(X, Y) + \eta(X) \eta(Y)], \quad (3.2)
\]
where equations (2.1) and (2.2) are used. In view of (3.1) and (3.2), we get
\[
\begin{align*}
QX &= -(\alpha + \lambda) X - (\alpha + \mu) \eta(X) \xi, \quad (3.3) \\
r &= -n\lambda - (n - 1)\alpha + \mu, \quad (3.4) \\
S(X, Y) &= -(\alpha + \lambda) g(X, Y) - (\alpha + \mu) \eta(X) \eta(Y), \quad (3.5) \\
S(X, \xi) &= S(\xi, X) = (\mu - \lambda) \eta(X), \quad (3.6) \\
\mu - \lambda &= (n - 1)(\alpha^2 - \rho) \quad (3.7)
\end{align*}
\]
for any \( X, Y \in \chi(M) \). Here \( r \) is the scalar curvature of \((M, g)\) and is defined by
\[
r = S(e_i, e_i)\bigg|_{i=1}^n, \quad \text{where } \{e_1, e_2, ..., e_n\} \text{ is a set of linearly independent vector fields on } M.
\]
In particular, if \( \mu = 0 \) then the triplet \((g, \xi, \lambda)\) is a Ricci soliton \([20]\) and it is called shrinking, steady or expanding according as \( \lambda \) is negative, zero or positive, respectively \([19]\).

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Proposition 2. The following relations hold on an \((LCS)_n\)-manifold \((M, \phi, \xi, \eta, g)\)

\[(i) \; \eta(\nabla_X \xi) = 0, \quad (ii) \; \nabla_\xi = 0, \quad (iii) \; \nabla_\xi \eta = 0, \quad (iv) \; L_\xi \phi = 0,\]

\[(v) \; L_\xi \eta = 0, \quad (vi) \; L_\xi (\eta \otimes \eta) = 0, \quad (vii) \; L_\xi g = 2\alpha (g + \eta \otimes \eta).\]

Also, if \(\eta\) is closed the distribution is involuntary and the Nijenhuis tensor of \(\phi\) vanishes identically, i.e., the structure is normal.

\textit{Proof.} Since \((\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}\)
and therefore
\[
\nabla_X \phi Y - \phi(\nabla_X Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}.
\]

Taking \(Y = \xi\) in the above equation, we have \(\phi(\nabla_X \xi) = \alpha \phi X\). Applying \(\phi\) on either sides, we get
\[
\nabla_X \xi + \eta(\nabla_X \xi)\xi = \alpha\{X + \eta(X)\xi\}.
\]

Since \(X(g(\xi, \xi)) = 2\eta(\nabla_X \xi, \xi)\) and \(\nabla_X \xi = \alpha \phi X\), therefore \(\eta(\nabla_X \xi) = 0\), and hence \(\nabla_\xi \xi = 0\). As we know that \(\eta(X) = g(X, \xi)\) and \(\nabla\) is metric, then we have \(\nabla_\xi \eta = 0\).

The Lie-derivative of \(\phi\) along \(\xi\) gives
\[
(L_\xi \phi)(X) = [\xi, \phi X] - \phi([\xi, X]) = \nabla_\xi \phi X - \phi(\nabla_\xi X) = (\nabla_\xi \phi)(X) = 0, \text{ i.e., } L_\xi \phi = 0.
\]

Again, \((L_\xi \eta)(X) = \xi(\eta(X) - \eta([\xi, X])) = g(X, \nabla_\xi \xi) + g(\nabla_X \xi, \xi) = 0\), i.e., \(L_\xi \eta = 0\).

Also, if \(L_\xi \eta = 0\), then \(L_\xi \eta \otimes \eta = 0\), as \(L_\xi \eta \otimes = (L_\xi \eta) \otimes \eta + \eta \otimes (L_\xi \eta)\). Again \((L_\xi g)(X, Y) = \xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y])\), implies that
\[
(L_\xi g)(X, Y) = \alpha\{g(\phi X, Y) + g(X, \phi Y)\}.
\]

Using \((2.5)\), we get
\[
L_\xi g = 2\alpha(g + \eta \otimes \eta).
\]

It is well known that
\[
(d\eta)(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])
\]
implies that
\[
(d\eta)(X, Y) = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi)
\]
\[
= \alpha\{g(Y, X) + \eta(X)\eta(Y)\} - \alpha\{g(X, Y) + \eta(X)\eta(Y)\} = 0, \text{ i.e., } d\eta = 0.
\]

Finally,
\[
N_\phi(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]
\]
yields that
\[
N_\phi(X, Y) = \phi^2(\nabla_X Y) - \phi^2(\nabla_Y X) - \phi(\nabla_X \phi Y) + \phi(\nabla_Y \phi X)
\]
\[
+ \nabla_\phi X \phi Y - \phi(\nabla_\phi X Y) - \nabla_\phi Y \phi X + \phi(\nabla_\phi Y X) = 0,
\]
i.e., the structure is normal. \(\blacksquare\)
In [7] and [8], Shaikh et al. proved that a second order parallel symmetric tensor on a Lorentzian concircular structure manifold with $\alpha^2 - \rho \neq 0$ is a constant multiple of the Ricci tensor. Thus we apply this concept for $\eta$-Ricci soliton and prove the following results.

**Theorem 3.** Let $(M, \phi, \xi, \eta, g)$ is an $(LCS)_n$-manifold. If the symmetric tensor field $h = L_\xi g + 2S + 2\mu \eta \otimes \eta$ of type $(0,2)$ is parallel with respect to the Levi-Civita connection $\nabla$, then $(g, \xi, \lambda)$ on $M$ yields an $\eta$-Ricci soliton.

**Proof.** In consequence of (3.2), we have

$$h(X, Y) = 2\alpha g(X, Y) + 2S(X, Y) + 2(\alpha + \mu)\eta(X)\eta(Y).$$

Replacing $X$ and $Y$ with $\xi$ in the above equation, we get

$$h(\xi, \xi) = (L_\xi g)(\xi, \xi) + 2S(\xi, \xi) + 2\mu\eta(\xi)\eta(\xi) = 2\lambda,$$

and therefore

$$\lambda = \frac{1}{2} h(\xi, \xi).$$

From [7] and [8], we have

$$h(X, Y) = -h(\xi, \xi)g(X, Y), \forall X, Y \in \chi(M).$$

Thus, $L_\xi g + 2S + 2\mu\eta \otimes \eta = -2\lambda g$. Hence the statement of the theorem.

If $\mu = 0$, it follows that $L_\xi g + 2S + 2(\alpha^2 - \rho)g = 0$. Thus we conclude the following corollary:

**Corollary 4.** On an $(LCS)_n$-manifold $(M, \phi, \xi, \eta, g)$ with the property that a symmetric tensor field $h = L_\xi g + 2S$ of type $(0,2)$ is parallel with respect to the Levi-Civita connection associated to $g$, then the equation (3.1), for $\mu = 0$ and $\lambda = (n-1)(\alpha^2 - \rho)$, define a Ricci soliton.

An $(LCS)_n$ manifold $(M, \phi, \xi, \eta, g)$ is said to be quasi-Einstein if its Ricci tensor $S$ is a linear combination (with real scalars $\lambda$ and $\mu(\neq 0)$) of $g$ and the tensor product of a non-zero 1-form $\eta$ satisfying (2.1) and for an Einstein if $S$ is collinear with $g$ [6]. From (3.5), we state the results in the form of corollary as:

**Corollary 5.** If the equation (3.5) define an $\eta$-Ricci soliton on an $(LCS)_n$-manifold, then $(M, g)$ is quasi-Einstein.

Next, we prove the following theorem as:

**Theorem 6.** Let $(g, \xi, \lambda, \mu)$ is an $\eta$-Ricci soliton on an $(LCS)_n$-manifold $(M, \phi, \xi, \eta, g)$. If the Ricci tensor $S$ of $M$ is

(i) cyclic parallel, then $\mu = -\alpha - \frac{\rho}{2\alpha}$, and $\lambda = -\frac{\rho}{2\alpha}(1 - 2\alpha(n-1)) - \alpha(1 + (n-1)\alpha)$.

(ii) cyclic parallel $\eta$-recurrent, then there does not exist an $\eta$-Ricci soliton or a Ricci soliton with the potential vector field $\xi$ on $M$.
Proof. It is well known that
\[(\nabla_X S)(Y, Z) = X(S(Y, Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).\] (3.8)
In view of (2.2), (2.3) and (3.5), the equation (3.8) reduces to
\[(\nabla_X S)(Y, Z) = -\rho g(\phi Y, \phi Z)\eta(X) - \alpha(\alpha + \mu)\{g(\phi X, \phi Z)\eta(Y) + g(\phi X, \phi Y)\eta(Z)\}.\] (3.9)
If possible, we suppose that the Ricci tensor \(S\) of \(M\) is cyclic parallel, that is,
\[(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(Z, Y) = 0\ \forall\ X, Y, Z \in \chi(M).\] The cyclic sum of (3.9) together with the last argument give
\[-\rho\{g(\phi Y, \phi Z)\eta(X) + g(\phi X, \phi Z)\eta(Y) + g(\phi Y, \phi X)\eta(Z)\} - 2\alpha(\alpha + \mu)\{g(\phi X, \phi Z)\eta(Y) + g(\phi X, \phi Y)\eta(Z) + g(\phi Y, \phi Z)\eta(X)\} = 0.\] (3.10)
Replacing \(Z = \xi\) in (3.10), we have
\[(\rho + 2\alpha(\alpha + \mu))g(\phi X, \phi Y) = 0\]
for any \(X, Y \in \chi(M)\). It follows that \(\rho + 2\alpha(\alpha + \mu) = 0\) and thus (3.7) gives \(\mu = -\alpha - \frac{2\rho}{2n}\) and \(\lambda = -\frac{\rho}{2(n - 1)} - \alpha(1 + (n - 1)\alpha)\). To prove the result \((ii)\), we suppose that \(M\) is \(\eta\)-recurrent, that is, \((\nabla_X S)(Y, Z) = \eta(X)S(Y, Z)\ \forall\ X, Y, Z \in \chi(M)\). If the Ricci tensor \(S\) of the \(\eta\)-recurrent \((LCS)_{n}\)-manifold is cyclic parallel, then
\[\eta(X)S(Y, Z) + \eta(Y)S(Z, X) + \eta(Z)S(X, Y) = -\rho\{g(\phi Y, \phi Z)\eta(X) + g(\phi X, \phi Z)\eta(Y) + g(\phi Y, \phi X)\eta(Z)\} - 2\alpha(\alpha + \mu)\{g(\phi X, \phi Z)\eta(Y) + g(\phi X, \phi Y)\eta(Z) + g(\phi Y, \phi Z)\eta(X)\} = 0\] (3.11)
for any \(X, Y, Z \in \chi(M)\). Taking \(Y = Z = \xi\) in (3.11) and then using (3.5) and (3.6), we get \(3(\mu - \lambda)\eta(X) = 0\) for any \(X \in \chi(M)\). It follows that \(\lambda = \mu\), which is a contradiction. Thus the statements of the theorem are proved. \(\square\)

In view of the Theorem 6, we can state the following corollaries.

**Corollary 7.** In an \((LCS)_{n}\)-manifold \((M, \phi, \xi, \eta, g)\) equipped with a cyclic parallel Ricci tensor, there is no Ricci soliton with the potential vector field \(\xi\).

**Corollary 8.** If an \((LCS)_{n}\)-manifold \((M, \phi, \xi, \eta, g)\) possesses a cyclic parallel \(\eta\)-recurrent Ricci tensor, then \(M\) does not admit \(\eta\)-Ricci soliton or Ricci soliton with the potential vector field \(\xi\).

**Theorem 9.** Let \((g, \xi, \lambda, \mu)\) be an \(\eta\)-Ricci soliton on an \((LCS)_{n}\)-manifold \((M, \phi, \xi, \eta, g)\). If the Ricci tensor \(S\) of \(M\) satisfies
\[(i)\ \nabla S = 0, \text{ then } \mu = -\alpha + \frac{\lambda}{\alpha}, \text{ and } \lambda = \frac{\lambda}{\alpha} - \alpha - (n - 1)(\alpha^2 - \rho),\]
\[(ii)\ \nabla S = \eta \otimes S, \text{ then there does not exist } \eta\text{-Ricci soliton or Ricci soliton with the potential vector field } \xi \text{ on } M.\]
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Proof. Let us suppose that the Ricci tensor $S$ of $M$ satisfies $\nabla S = 0$, that is, $M$ is Ricci symmetric $(LCS)_n$-manifold. Replacing $Z$ by $\xi$ in (3.10), we obtain

$$\{\alpha(\alpha + \mu) + \rho\}g(\phi X, \phi Y) = 0, \quad \forall \ X, Y \in \chi(M).$$

It follows that $\mu = -\alpha + \frac{\xi \alpha}{\alpha}$, and $\lambda = \frac{\xi \alpha}{\alpha} - \alpha - (n - 1)(\alpha^2 - \rho)$, the statement (i).

Let $M$ is $\eta$-recurrent $(LCS)_n$-manifold, that is, $\nabla S = \eta \otimes S$. From (3.5) we obtain $\lambda = \mu$, which is not possible. Thus our theorem is proved. \qed

In consequence of the Theorem 9, we state the following corollaries.

Corollary 10. If an $(LCS)_n$-manifold $(M, \phi, \xi, \eta, g)$ is Ricci symmetric, then there is no Ricci soliton with the potential vector field $\xi$ on $M$.

Corollary 11. If an $(LCS)_n$-manifold $(M, \phi, \xi, \eta, g)$ is admitting an $\eta$-recurrent Ricci tensor, then there does not exist $\eta$-Ricci soliton or Ricci soliton with the potential vector field $\xi$ on $M$.

4 \quad \eta$-Ricci solitons satisfying certain curvature conditions on the $(LCS)_n$-manifolds $(M, \phi, \xi, \eta, g)$

In 1970, Pokhariyal et al. [32], defined and studied the properties of $W_2$-curvature tenor, and is given by

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}\{g(X, Z)QY - g(Y, Z)QX\} \quad (4.1)$$

for $X, Y, Z \in \chi(M)$.

Theorem 12. If an $(LCS)_n$-manifold $(M, \phi, \xi, \eta, g)$ equipped with an $\eta$-Ricci soliton $(g, \xi, \lambda, \mu)$ satisfies $R(\xi, X) \cdot S = 0$, then $\mu = -\alpha$ and $\lambda = -\alpha - (n - 1)(\alpha^2 - \rho)$.

Proof. Suppose $M$ satisfies $R(\xi, X) \cdot S = 0$. Then we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0$$

for any $X, Y, Z \in \chi(M)$. Using (2.9) and (3.5) in the above equation, we yield

$$(\alpha^2 - \rho)(\mu + \alpha)\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)\} = 0.$$ 

For $Z = \xi$, we have

$$(\alpha^2 - \rho)(\mu + \alpha)\{g(X, Y) + \eta(X)\eta(Y)\} = 0.$$ 

It is obvious from the above equation that $\mu = -\alpha$, provided $\alpha^2 - \rho \neq 0$. Equation (3.7) together with the last result give $\lambda = -\alpha - (n - 1)(\alpha^2 - \rho)$. Hence the statement of the theorem is proved. \qed

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With the help of the Theorem 12, we state the following corollaries.

**Corollary 13.** Let an \((LCS)_n\)-manifold \((M, \phi, \xi, \eta, g)\) equipped with the \(\eta\)-Ricci soliton satisfies \(R(\xi, X) \cdot S = 0\). Then there is no Ricci soliton on \(M\) with the potential vector field \(\xi\).

**Corollary 14.** An \((LCS)_n\)-manifold \((M, \phi, \xi, \eta, g)\) together with the \(\eta\)-Ricci soliton \((g, \xi, \lambda, \mu)\) and \(R(\xi, X) \cdot S = 0\) is Einstein.

**Theorem 15.** If an \((LCS)_n\)-manifold \((M, \phi, \xi, \eta, g)\) with an \(\eta\)-Ricci soliton satisfies \(W_2(\xi, X) \cdot S = 0\), then either \(\mu = -\alpha\), \(\lambda = -\alpha + (n - 1)(\alpha^2 - \rho)\) or \(\lambda = -\alpha\), \(\mu = -\alpha + (n - 1)(\alpha^2 - \rho)\).

**Proof.** If possible, we assume that the \((LCS)_n\)-manifolds endowed with the \(\eta\)-Ricci solitons are \(W_2\)-Ricci symmetric, that is, \(W_2(\xi, X) \cdot S = 0\). Thus we have
\[
S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0 \tag{4.2}
\]
for any \(X, Y, Z \in \chi(M)\). Using (3.5) and (4.1) in (4.2), we get
\[
\begin{align*}
&(\alpha^2 - \rho) [g(X, Y)S(\xi, Z) + g(X, Z)S(Y, \xi) - S(X, Z)\eta(Y) - S(X, Y)\eta(Z)] \\
&\quad - \frac{1}{n-1} [(\alpha + \lambda) \{S(X, Z)\eta(Y) + \eta(Z)S(Y, X)\} + (\alpha + \mu) \{\eta(X)\eta(Y)S(\xi, Z) \\
&\quad \quad + \eta(X)\eta(Z)S(\xi, Y)\} + (\mu - \lambda) \{g(X, Y)S(\xi, Z) + g(X, Z)S(Y, \xi)\}] = 0. \tag{4.3}
\end{align*}
\]
In consequence of (3.5)-(3.7), equation (4.3) consider the form
\[
\frac{(\alpha + \mu)(\alpha + \lambda)}{n-1} \{\eta(Y)g(X, Y) + \eta(Z)g(X, Y) + 2\eta(X)\eta(Y)\eta(Z)\} = 0. \tag{4.4}
\]
Taking \(Z = \xi\) in (4.4), we yield
\[
\frac{(\alpha + \mu)(\alpha + \lambda)}{n-1} g(\phi X, \phi Y) = 0 \tag{4.5}
\]
for any \(X, Y \in \chi(M)\). In general, \(g \neq 0\) on \(M\), therefore (4.5) shows that either \(\mu = -\alpha\) or \(\lambda = -\alpha\), for \(n > 1\). These results together with (3.7) reflect that either \(\mu = -\alpha\), \(\lambda = -\alpha + (n - 1)(\alpha^2 - \rho)\) or \(\lambda = -\alpha\), \(\mu = -\alpha + (n - 1)(\alpha^2 - \rho)\) on \(M\). \(\square\)

**Corollary 16.** If an \((LCS)_n\)-manifold \((M, \phi, \xi, \eta, g)\) satisfies \(W_2(\xi, X) \cdot S = 0\), then there is no Ricci soliton with the potential vector field \(\xi\) on \(M\).

5 **Examples of \(\eta\)-Ricci soliton on \((LCS)_n\)-manifolds**

**Example 17.** Let a 3-dimensional manifold \(M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}\), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). Let \(\{E_1, E_2, E_3\}\) be a linearly independent global frame on \(M\) given by
\[
E_1 = e^z \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad E_2 = e^z \frac{\partial}{\partial y}, \quad E_3 = e^{2z} \frac{\partial}{\partial z}.
\]

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Assume that \( g \) be the Lorentzian metric on \( M \), and is defined by
\[
g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, \quad g(E_1, E_1) = g(E_2, E_2) = 1, \quad g(E_3, E_3) = -1.
\]

Let \( \eta \) be the 1-form defined by \( \eta(V) = g(V, E_3) \) for any \( V \in \chi(M) \) and \( \phi \) is a \((1,1)\)-tensor field defined by \( \phi E_1 = E_1, \ \phi E_2 = E_2, \ \phi E_3 = 0 \). Then using the linearity of \( \phi \) and \( g \) we have
\[
\eta(E_3) = -1, \quad \phi^2 V = V + \eta(V) E_3, \quad g(\phi V, \phi W) = g(V, W) + \eta(V) \eta(W)
\]
for any \( V, W \in \chi(M) \). Let \( \nabla \) be the Levi-Civita connection with respect to the Lorentzian metric \( g \) and \( R \) be the curvature tensor of \( g \). Then we obtain
\[
[E_1, E_2] = -e^2 E_2, \quad [E_1, E_3] = -e^2 z E_1, \quad [E_2, E_3] = -e^2 z E_2.
\]

Taking \( E_3 = \xi \) and using the Koszul’s formula for the Lorentzian metric \( g \), we have
\[
\nabla_{E_1} E_3 = -e^{2z} E_1, \quad \nabla_{E_2} E_3 = -e^{2z} E_2, \quad \nabla_{E_3} E_3 = 0,
\]
\[
\nabla_{E_1} E_1 = -e^{2z} E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_1} E_2 = e^2 E_1, \quad \nabla_{E_2} E_1 = e^2 E_2, \quad \nabla_{E_3} E_1 = 0.
\]

From the above equations, it can be easily seen that \( E_3 = \xi \) is a unit timelike concircular vector field and hence the structure \((\phi, \xi, \eta, g)\) is an \((LCS)_3\)-structure on \( M \). Consequently, \( M^3(\phi, \xi, \eta, g) \) is an \((LCS)_3\)-manifold with \( \alpha = -e^{2z} \neq 0 \) such that \((X\alpha) = \rho \eta(X)\), where \( \rho = 2e^{4z} \). Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor \( R \) and the Ricci tensor \( S \) as follows:
\[
R(E_2, E_3) E_3 = -e^{4z} E_2, \quad R(E_1, E_3) E_3 = -e^{4z} E_1, \quad R(E_1, E_2) E_2 = \{e^{4z} - e^{2z}\} E_1,
\]
\[
R(E_2, E_3) E_2 = e^{4z} E_3 - e^{3z} E_1, \quad R(E_1, E_3) E_1 = -e^{4z} E_3, \quad R(E_2, E_1) E_1 = \{e^{4z} - e^{2z}\} E_2,
\]
\[
S(E_1, E_1) = -e^{2z}, \quad S(E_2, E_2) = -e^{2z}, \quad S(E_3, E_3) = -2e^{4z}.
\]

Also from the equation (3.5), we can see that
\[
S(E_1, E_1) = -(\alpha + \lambda), \quad S(E_2, E_2) = -(\alpha + \lambda), \quad S(E_3, E_3) = (\lambda - \mu).
\]

Thus we conclude from the last two expressions that for \( \alpha = -e^{2z}, \lambda = 2e^{2z} \) and \( \mu = 2\{e^{2z} + e^{4z}\} \), the structure \((g, \xi, \lambda, \mu)\) is an \( \eta \)-Ricci soliton on \( M^3(\phi, \xi, \eta, g) \).

**Example 18.** Let a 3-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\} \), where \((x, y, z)\) are the standard coordinates in \( \mathbb{R}^3 \). In \[35\], Shaikh defined the linear independent vector fields \( \{E_1, E_2, E_3\} \) on \( M \) as:
\[
E_1 = e^{-z} \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad E_2 = e^{-z} \frac{\partial}{\partial y}, \quad E_3 = e^{-2z} \frac{\partial}{\partial z}.
\]
Let $g$ be the Lorentzian metric defined by

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, \quad g(E_1, E_1) = g(E_2, E_2) = 1, \quad g(E_3, E_3) = -1.$$ 

Let $\eta$ be the 1-form defined by $\eta(V) = g(V, E_3)$ for any $V \in \chi(M)$. Let $\phi$ be the $(1, 1)$-tensor field defined by $\phi E_1 = E_1, \phi E_2 = E_2, \phi E_3 = 0$. Then using the linearity of $\phi$ and $g$ we have

$$\eta(E_3) = -1, \quad \phi^2 V = V + \eta(V) E_3, \quad g(\phi V, \phi W) = g(V, W) + \eta(V)\eta(W),$$

for any $V, W \in \chi(M)$. Let $\nabla$ be the Levi-Civita connection with respect to the Lorentzian metric $g$ and $R$ be the curvature tensor of $g$. Then we obtain

$$[E_1, E_2] = -e^{-z} E_2, \quad [E_1, E_3] = -e^{-2z} E_1, \quad [E_2, E_3] = -e^{-2z} E_2.$$

Taking $E_3 = \xi$ and using the Koszul’s formula for the Lorentzian metric $g$, we have

$$\nabla_{E_3} E_3 = e^{-2z} E_3, \quad \nabla_{E_1} E_3 = e^{-2z} E_3, \quad \nabla_{E_2} E_3 = 0,$$

$$\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = e^{-2z} E_3 - e^{-z} E_1, \quad \nabla_{E_3} E_1 = 0.$$

From the above equations, it can be easily seen that $E_3 = \xi$ is a unit timelike concircular vector field and hence $(\phi, \xi, \eta, g)$ is an $(\text{LCS})_3$-structure on $M$. Thus $M^3(\phi, \xi, \eta, g)$ is an $(\text{LCS})_3$-manifold with $\alpha = e^{2z} \neq 0$ such that $(X\alpha) = \rho\eta(X)$, where $\rho = 2e^{-4z}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor $R$ and the Ricci tensor $S$ as follows:

$$R(E_2, E_3)E_3 = e^{-4z} E_2, \quad R(E_1, E_3)E_3 = e^{-4z} E_1, \quad R(E_1, E_2)E_2 = \{e^{-4z} - e^{-2z}\} E_1,$$

$$R(E_2, E_3)E_2 = e^{-4z} E_3, \quad R(E_1, E_3)E_1 = e^{-4z} E_3, \quad R(E_1, E_2)E_1 = \{-e^{-4z} - e^{-2z}\} E_2,$$

$$S(E_1, E_1) = 2e^{-4z} - e^{-2z}, \quad S(E_2, E_2) = 2e^{-4z} - e^{-2z}, \quad S(E_3, E_3) = 2e^{-4z}.$$

Also from (3.5), we calculate that

$$S(E_1, E_1) = -(\alpha + \lambda), \quad S(E_2, E_2) = -(\alpha + \lambda), \quad S(E_3, E_3) = (\lambda - \mu).$$

We conclude from (3.5) that for $\alpha = e^{2z}, \lambda = -2e^{-4z}$ and $\mu = -4e^{-4z}$, the data $(g, \xi, \lambda, \mu)$ admits an $\eta$-Ricci soliton on $M^3(\phi, \xi, \eta, g)$.

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