THE LOCAL-GLOBAL PRINCIPLE IN LEAVITT PATH ALGEBRAS

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Abstract. This is a short note on how a particular graph construction on a subset of edges that lead to a subalgebra construction, provided a tool in proving some ring theoretical properties of Leavitt path algebras.

1 Introduction

This paper is an expository note publicizing how a particular subalgebra construction which first appeared in the paper [5] by G.Abrams and K.M.Rangaswamy was used in proving many theorems on Leavitt path algebras. The power of the subalgebra construction relies on extending a particular property on a Leavitt path algebra over a “smaller” graph to the Leavitt path algebra of the whole graph. This can be visualised as from a local view to a global setting, “local-to-global jump”.

We start by recalling the definitions of a path algebra and a Leavitt path algebra, (see [2] for a more extended study on Leavitt path algebras). A directed graph $E = (E^0, E^1, r, s)$ consists of two countable sets $E^0, E^1$ and functions $r, s : E^1 \to E^0$. The elements $E^0$ and $E^1$ are called vertices and edges, respectively. For each $e \in E^0$, $s(e)$ is the source of $e$ and $r(e)$ is the range of $e$. If $s(e) = v$ and $r(e) = w$, then we say that $v$ emits $e$ and that $w$ receives $e$. A vertex which does not receive any edges is called a source, and a vertex which emits no edges is called a sink. A graph is called row-finite if $s^{-1}(v)$ is a finite set for each vertex $v$. For a row-finite graph the edge set $E^1$ of $E$ is finite if its set of vertices $E^0$ is finite. Thus, a row-finite graph is finite if $E^0$ is a finite set.

A path in a graph $E$ is a sequence of edges $\mu = e_1 \ldots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$. In such a case, $s(\mu) := s(e_1)$ is the source of $\mu$ and $r(\mu) := r(e_n)$ is the range of $\mu$, and $n$ is the length of $\mu$, i.e., $l(\mu) = n$.

If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then $\mu$ is called a cycle. If $E$ does not contain any cycles, $E$ is called acyclic. For $n \geq 2$, define $E^n$ to be the set...
of paths of length \( n \), and \( E^* = \bigcup_{n \geq 0} E^n \) the set of all finite paths. Denote by \( E^\infty \) the set of all infinite paths of \( E \), and by \( E^{\leq \infty} \) the set \( E^\infty \) together with the set of finite paths in \( E \) whose range vertex is a sink. We say that a vertex \( v \in E^0 \) is **cofinal** if for every \( \gamma \in E^{\leq \infty} \) there is a vertex \( w \) in the path \( \gamma \) such that \( v \geq w \). We say that a graph \( E \) is cofinal if every vertex in \( E \) is cofinal.

The path \( K \)-algebra over \( E \) is defined as the free \( K \)-algebra \( K[E^0 \cup E^1] \) with the relations:

1. \( v_i v_j = \delta_{ij} v_i \) for every \( v_i, v_j \in E^0 \).
2. \( e_i = e_i r(e_i) = s(e_i) e_i \) for every \( e_i \in E^1 \).

This algebra is denoted by \( KE \). Given a graph \( E \), define the extended graph of \( E \) as the new graph \( \hat{E} = (E^0, E^1 \cup (E^1)^*, r', s') \) where \( (E^1)^* = \{ e_i^* \mid e_i \in E^1 \} \) and the functions \( r' \) and \( s' \) are defined as

\[
r'(i) = r, \quad s'(i) = s, \quad r'(e_i^*) = s(e_i) \quad \text{and} \quad s'(e_i^*) = r(e_i).
\]

The Leavitt path algebra of \( E \) with coefficients in \( K \) is defined as the path algebra over the extended graph \( \hat{E} \), with relations:

1. \( e_i^* e_j = \delta_{ij} r(e_j) \) for every \( e_j \in E^1 \) and \( e_i^* \in (E^1)^* \).
2. \( v_i = \sum_{\{ e_j \in E^1 \mid s(e_j) = v_i \}} e_j e_j^* \) for every \( v_i \in E^0 \) which is not a sink.

This algebra is denoted by \( L_K(E) \). The conditions (CK1) and (CK2) are called the Cuntz-Krieger relations. In particular condition (CK2) is the Cuntz-Krieger relation at \( v_i \). If \( v_i \) is a sink, we do not have a (CK2) relation at \( v_i \). Note that the condition of row-finiteness is needed in order to define the equation (CK2).

Given a graph, we define a new graph built upon the given one that will be necessary for the subalgebra construction. The construction is based on an idea presented by Raeburn and Szymański in [12, Definition 1.1]. Then, we construct several examples.

**Definition 1.** [5, Definition 2] Let \( E \) be a graph, and \( F \) be a finite set of edges in \( E \). We define \( s(F) \) (resp. \( r(F) \)) to be the sets of those vertices in \( E \) which appear as the source (resp. range) vertex of at least one element of \( F \). We define a graph \( E_F \) as follows:

\[
E_F^0 = F \cup (r(F) \cap s(F)) \cup (E^1 \setminus F) \cup (r(F) \setminus s(F)),
\]

\[
E_F^1 = \{(e, f) \in F \times E_F^0 \mid r(e) = s(f)\},
\]

and where \( s((x, y)) = x \), \( r((x, y)) = y \) for any \( (x, y) \in E_F^1 \).

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Example 2. [5, Example 1] Let $E$ be the rose with $n$-petals graph

Let $F = \{y_1\}$. Then $E_F^0 = \{y_1\} \cup \{v\}$, and $E_F^1 = \{(y_1, y_1), (y_1, v)\}$. Pictorially, $E_F$ is given by

This example indicates that various properties of the graph $E$ need not pass to the graph $E_F$. For instance, $E$ is cofinal, while $E_F$ is not. In particular, $L_K(E)$ is a simple algebra, while $L_K(E_F)$ is not.

Example 3. Let $E$ be the graph

and $F = \{f_1, g_1\}$. Then, $E_F$ is given by

In this example $E$ is not cofinal but $E_F$ is cofinal. Also, $L_K(E)$ is not purely infinite simple while $L_K(E_F)$ is.

Example 4. Consider the infinite clock graph $E$ with one source which emits countably many edges as follows:

Let $F = \{f\}$ and then $E_F$ is

This is an example which shows that both $E$ and $E_F$ are acyclic graphs where $F$ is any subset of vertices. Actually, if $E$ is any acyclic graph and $F$ any subset of vertices then $E_F$ is acyclic is proved in [5, Lemma 1].
2 The Subalgebra Construction

Although in general $E_F$ need not be a subgraph of $E$, the Leavitt path algebras $L_K(E_F)$ and $L_K(E)$ are related via a homomorphism which leads to a subalgebra construction of $L_K(E)$.

In [5, Proposition 1], for a finite set of edges $F$ in a graph $E$, the algebra homomorphism $\theta : L_K(E_F) \to L_K(E)$ having the properties

1. $F \cup F^* \subseteq \text{Im}(\theta)$,
2. If $w \in r(F)$, then $w \in \text{Im}(\theta)$,
3. If $w \in E^0$ has $s^{-1}_E(w) \subseteq F$, then $w \in \text{Im}(\theta)$,

is defined by using the following subsets $G^0$ and $G^1$ of $L_K(E)$

\[
G^0 = \{ee^* \mid e \in F\} \cup \{v - \sum_{f \in F, s(f) = v} ff^* \mid v \in r(F) \cap s(F) \cap s(E^1 \setminus F)\} \\
\cup \{v \mid v \in r(F) \setminus s(F)\}
\]

and

\[
G^1 = \{eff^* \mid e, f \in F, s(f) = r(e)\} \cup \{e - \sum_{f \in F, s(f) = r(e)} eff^* \mid r(e) \in r(F) \cap s(F) \cap s(E^1 \setminus F)\} \\
\cup \{e \in F \mid r(E) \in r(F) \setminus s(F)\}
\]

In particular, $\theta(w) \in G^0$ for all vertices in $E_F$ and $\theta(w) \in G^1$ for all edges in $E_F$.

Let $E$ be any graph, $K$ any field, and \{a_1, a_2, \ldots, a_l\} any finite subset of nonzero elements of $L_K(E)$.

For each $1 \leq r \leq l$ write

\[
a_r = k_{c_1}v_{c_1} + k_{c_2}v_{c_2} + \ldots + k_{c_{t(r)}}v_{c_{t(r)}} + \sum_{i=1}^{t(r)} k_{r_i}p_{r_i}q_{r_i}^*
\]

where each $k_j$ is a nonzero element of $K$, and, for each $1 \leq i \leq t(r)$, at least one of $p_{r_i}$ or $q_{r_i}$ has length at least 1. Let $F$ be denote the (necessarily finite) set of those edges in $E$ which appear in the representation of some $p_{r_i}$ or $q_{r_i}$, $1 \leq r_i \leq t(r)$, $1 \leq r \leq l$.

Now consider the set

\[
S = \{v_{c_1}, v_{c_2}, \ldots, v_{c_{t(r)}} \mid 1 \leq r \leq l\}
\]
of vertices which appear in the displayed description of \(a_r\) for some \(1 \leq r \leq l\). We partition \(S\) into subsets as follows:

\[ S_1 = S \cap r(F), \]
and, for remaining vertices \(T = S \setminus S_1\), we define

\[
S_2 = \{v \in T \mid s_E^{-1}(v) \subseteq F \text{ and } s_E^{-1}(v) \neq \emptyset\} \\
S_3 = \{v \in T \mid s_E^{-1}(v) \cap F = \emptyset\} \\
S_4 = \{v \in T \mid s_E^{-1}(v) \not\subseteq F \text{ and } s_E^{-1}(v) \cap (E^1 \setminus F) \neq \emptyset\}. 
\]

**Definition 5.** [5, Definition 3] Let \(B\) of mutually orthogonal idempotents and \(\oplus\) subalgebra of \(L\) acyclic, so is each graph \(F\).

**Theorem 6.** [5, Proposition 1] Let \(E\) be any graph, \(K\) any field, and \(\{a_1, a_2, \ldots, a_l\}\) any finite subset of nonzero elements of \(L_K(E)\). Consider the notation presented in The Subalgebra Construction. We define \(B(a_1, a_2, \ldots, a_l)\) to be the \(K\)-subalgebra of \(L_K(E)\) generated by the set \(\text{Im}(\theta) \cup S_3 \cup S_4\). That is,

\[ B(a_1, a_2, \ldots, a_l) = \langle \text{Im}(\theta). S_3, S_4 \rangle. \]

**Proposition 6.** [5, Proposition 1] Let \(E\) be any graph, \(K\) any field, and \(\{a_1, a_2, \ldots, a_l\}\) any finite subset of nonzero elements of \(L_K(E)\). Let \(F\) denote the subset of \(E^1\) presented in The Subalgebra Construction. For \(w \in S_4\) let \(w_w\) denote the element \(w = \sum_{f \in F, s(f) = w} f f^*\) of \(L_K(E)\). Then

1. \(\{a_1, a_2, \ldots, a_l\} \subseteq B(a_1, a_2, \ldots, a_l)\).
2. \(B(a_1, a_2, \ldots, a_l) = \text{Im}(\theta) \oplus (\oplus_{v_1 \in S_3} K v_1) \oplus (\oplus_{w_j \in S_4} K w_j)\).
3. The collection \(\{B(S) \mid S \subseteq L_K(E), S \text{ finite}\}\) is an upward directed set of subalgebras of \(L_K(E)\).
4. \(L_K(E) = \liminf_{S \subseteq L_K(E), S \text{ finite}} B(S)\).

Proposition 6, can be modified to include some more properties of the subalgebra construction in [5]. For instance, the morphism \(\theta\) in the construction is actually a graded morphism whose image is a graded submodule of \(L_K(E)\) and it also reveals some properties of cycles.

The stronger version of Proposition 6 is given in [10] as Theorem 4.1

**Theorem 7.** [10, Theorem 4.1] For an arbitrary graph \(E\), the Leavitt path algebra \(L_K(E)\) is a directed union of graded subalgebras \(B = A \oplus K\epsilon_1 \oplus \cdots \oplus K\epsilon_n\) where \(A\) is the image of a graded homomorphism \(\theta\) from a Leavitt path algebra \(L_K(F_B)\) to \(L_K(E)\) where \(F_B\) a finite graph which depends on \(B\), the elements \(\epsilon_i\) are homogeneous mutually orthogonal idempotents and \(\oplus\) is a ring direct sum. Moreover, if \(E\) is acyclic, so is each graph \(F_B\) and in this case \(\theta\) is a graded monomorphism.
Moreover, any cycle $c$ in the graph $F_B$ gives rise to a cycle $c'$ in $E$ such that if $c$ has an exit in $F_B$ then $c'$ has an exit in $E$. In particular, a cycle in $F_B$ is of the form $(f_1, f_2)(f_2, f_3)\ldots (f_n, f_1)$ and this case $f_1f_2\ldots f_n$ is a cycle in $E$.

Throughout recent literature this subalgebra construction has been a powerful tool. The first theorem that appears in the literature is the following:

**Theorem 8.** [5, Theorem 1] $L_K(E)$ is von Neumann regular if and only if $E$ is acyclic. If $E$ is acyclic, then $L_K(E)$ is locally $K$-matricial; that is, $L_K(E)$ is the direct union of subrings, each of which is isomorphic to a finite matrix rings over $K$.

Now, we give one implication of the statement to demonstrate how the subalgebra construction is used in the proof:

**Proof.** We assume $E$ is acyclic. Let $\{B(S) \mid S \subseteq L_K(E), S \text{ finite}\}$ be the collection of subalgebras of $L_K(E)$ indicated in Proposition 6(3). By Proposition 6(4), it suffices to show that each such $B(S)$ is of the indicated form. But by Proposition 6(2), $B(S) = B(a_1, a_2, \ldots, a_l) = \text{Im}(\theta) \oplus (\oplus_{v_i \in S_3} K v_i) \oplus (\oplus_{w_i \in S_4} K u w_i)$. Since terms appearing in the second and third summands are clearly isomorphic as algebras to $K \cong M_1(K)$, it suffices to show that $\text{Im}(\theta)$ is isomorphic to a finite direct sum of finite matrix rings over $K$. Since $E$ is acyclic, by Lemma 1 in [5] we have that $E_F$ is acyclic. But $E_F$ is always finite by definition, so we have by [3, Proposition 3.5], that $L_K(E_F) \cong \oplus_{i=1}^t M_{m_i}(K)$ for some $m_1, \ldots, m_t \in \mathbb{N}$. Since each $M_{m_i}(K)$ is a simple ring, we have that any homomorphic image of $L_K(E_F)$ must have this same form. So we get that $\text{Im}(\theta) \cong \oplus_{i=1}^t M_{m_i}(K)$ for some $m_1, \ldots, m_t \in \mathbb{N}$, and we are done. (As remarked previously, since $\theta$ is in fact an isomorphism we have $t = l$.)

We list the following theorems which are using the same Subalgebra Construction in their proofs. In particular, we only quote the parts that uses the Subalgebra Construction.

**Theorem 9.** [10, Theorem 5.1] Let $E$ be an arbitrary graph. Then for the Leavitt path algebra $L_K(E)$ the following are equivalent:

1. Every left/right ideal of $L_K(E)$ is graded;
2. The class of all simple left/right $L_K(E)$-modules coincides with the class of all graded-simple left/right $L_K(E)$-modules;
3. The graph $E$ is acyclic.

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Proof. (3) ⇒ (1) For the sake of simplicity of the notation, let $L := L_K(E)$. Suppose $E$ is acyclic. Now, by Theorem 7, $L$ is a direct union of graded subalgebras $B_{\lambda}$ where $\lambda \in I$, an index set and where each $B_{\lambda}$ is a finite direct sum of copies of $K$ and a graded homomorphic image of a Leavitt path algebra of a finite acyclic graph. By [8, Theorem 4.14], Leavitt path algebras of finite acyclic graphs are semisimple algebras which have elementary gradings, that is, all the matrix units are homogeneous. Consequently, every ideal of each $B_{\lambda}$ is graded. Let $L = \bigoplus_{n \in \mathbb{Z}} L_n$ be the $\mathbb{Z}$-graded decomposition of $L$. Since the $B_{\lambda}$ are graded subalgebras, each $B_{\lambda} = \bigoplus_{n \in \mathbb{Z}} (B_{\lambda} \cap L_n)$. Let $M$ be a left ideal of $L$. To show that $M$ is graded, we need only to show that $M = \bigoplus_{n \in \mathbb{Z}} (M \cap L_n)$. Let $a \in M$. Then, for some $\lambda$, $a \in M \cap B_{\lambda}$. Note that $M \cap B_{\lambda} = B_{\lambda}$ or a left ideal of $B_{\lambda}$. Since every left ideal of $B_{\lambda}$ and in particular $M \cap B_{\lambda}$ is graded, we can write $a = a_{n_1} + \cdots + a_{n_k}$ where $a_{n_i} \subset (M \cap B_{\lambda}) \cap (B_{\lambda} \cap L_{n_i}) \subset M \cap L_{n_i}$ for $i = 1, \ldots, k$. This show that $M = \bigoplus_{n \in \mathbb{Z}} (M \cap L_n)$ and hence $M$ is a graded left ideal of $L$. 

The next result is about graded von Neumann regular Leavitt path algebras. A ring $R$ is von Neumann regular if for every $x \in R$ there exists $y \in R$ such that $x = yxy$. Moreover, a graded ring $R$ is graded von Neumann regular if each homogeneous element is von Neumann regular.

**Theorem 10.** [10, Theorem 4.2]; [9, Theorem 10] Every Leavitt path algebra $L_K(E)$ of an arbitrary graph $E$ is a graded von Neumann regular ring.

**Proof.** [10, Proof of Theorem 4.2] Suppose $E$ is an arbitrary graph. By [10, Theorem 4.1], $L_K(E)$ is a directed union of graded subalgebras $B = A \oplus K\epsilon_1 \oplus \cdots \oplus K\epsilon_m$ where $A$ is the image of a graded homomorphism $\theta$ from a Leavitt path algebra $L_K(F_B)$ to $L_K(E)$ with $F_B$ a finite graph (depending on $B$), the elements $\epsilon_i$ are homogeneous mutually orthogonal idempotents and $\oplus$ is a ring direct sum. Since $F_B$ is a finite graph, $L_K(F_B)$ and hence $B$ is graded von Neumann regular by [9]. It is then clear from the definition that the direct union $L_K(E)$ is also graded von Neumann regular.

Recall that a ring $R$ is called left Bézout in case every finitely generated left ideal of $R$ is principal. If the graph $E$ is finite, then $L_K(E)$ is Bézout [4, Theorem 15]. The proof of this statement is given via a nice induction argument which we do not quote here. The generalization of this result to arbitrary graphs, which again appears in [4], uses the subalgebra construction.
Theorem 11. [4, Corollary 16] Let $E$ be an arbitrary graph and $K$ any field. Then $L_K(E)$ is Bézout.

Proof. By Theorem 7, $L_K(E)$ is the direct limit of unital subalgebras, each of which is isomorphic to the Leavitt path $K$-algebra of a finite graph. By [4, Theorem 15], each of these unital subalgebras is a Bézout subring of $L_K(E)$.

Now, we are going to prove that for any ring $R$, if every finite subset of $R$ is contained in a unital Bézout subring of $R$, then $R$ is Bézout. Let us consider a finitely generated left ideal of $R$ with generators $x_1, x_2, \ldots, x_n \in R$. Then there is a unital Bézout subring $S$ of $R$ that contains $\{x_1, x_2, \ldots, x_n\}$. Hence, there exists $x \in S$ such that the left $S$-ideal $Sx_1 + Sx_2 + \cdots + Sx_n = Sx$.

Since $1_S x_i = x_i$ for all $1 \leq i \leq n$, and each $x_i$ is in $Sx_1 + Sx_2 + \cdots + Sx_n = Sx$ which implies that for each $i$ there exists $s_i \in S$ with $x_i = s_i x$.

Hence $Rx_1 + Rx_2 + \cdots + Rx_n = Rs_1 x + Rs_2 x + \cdots + Rs_n x \subseteq Rx$. Also, $x = 1_S x \in Sx$ implies $x \in Sx_1 + Sx_2 + \cdots + Sx_n \subseteq Rx_1 + Rx_2 + \cdots + Rx_n$. Therefore, $Rx_1 + Rx_2 + \cdots + Rx_n = Rx$ and $R$ is a Bézout ring.

Hence, if $R$ is taken to be $L_K(E)$, the result follows.

Recall that a ring with local units $R$ is said to be directly finite if for every $x, y \in R$ and an idempotent element $u \in R$ such that $xu = ux = x$ and $yu = uy = y$, we have that $xy = u$ implies $yx = u$.

Theorem 12. [13, Proposition 4.3] $L_K(E)$ is directly finite if and only if no cycle in $E$ has an exit.

The converse of Theorem 12 for Leavitt path algebras of finite graphs has been proven in [7, Theorem 3.3]. To get the infinite graphs, Lia Vas proved the theorem by using Cohn-Leavitt approach. In particular, the localization of the graph is used by considering a finite subgraph generated by the vertices and edges of just those paths that appear in representations of $x, y$ and $u$ in $L_K(E)$ where $xy = u$ for some local unit $u$. However, the subgraph $F$ defined in this way may not produce a subalgebra $L_K(F)$ of $L_K(E)$. This problem is avoided by considering an appropriate finite subgraph $F$ such that the Cohn-Leavitt algebra of $F$ is a subalgebra of $L_K(E)$ and then adapts [7, Theorem 3.3] to Cohn-Leavitt algebras of finite graphs.

An alternative proof using the subalgebra construction is pointed out in [11, Theorem 3.7] using the grading on matrices. We outline the proof below (without considering the grading to refer to Theorem 12).

Theorem 13. ([11, Theorem 3.7] rephrased) For an arbitrary graph $E$, the following properties are equivalent for $L_K(E)$:

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(a) No cycle in $E$ has an exit;

(b) $L_K(E)$ is a directed union of graded semisimple Leavitt path algebras; specifically, $L_K(E)$ is a directed union of direct sums of matrices of finite order over $K$ or $K[x, x^{-1}]$.

(c) $L_K(E)$ is directly-finite.

Proof. (a) implies (b) Assume (a). By Theorem 7, $L_K(E)$ is a directed union of graded subalgebras $B = A \oplus K\epsilon_1 \oplus \cdots \oplus K\epsilon_n$, where $A$ is the image of a graded homomorphism $\theta$ from a Leavitt path algebra $L_K(F_B)$ to $L_K(E)$ with $F_B$ a finite graph depending on $B$. Moreover, any cycle with an exit in $F_B$ gives rise to a cycle with an exit in $E$. Since no cycle in $E$ has an exit, no cycle in the finite graph $F_B$ has an exit. So by using [2, Theorem 2.7.3],

$$L_K(F_B) \cong \bigoplus_{i \in I} M_{n_i}(K) \oplus \bigoplus_{j \in J} M_{m_j}(K[x, x^{-1}]),$$

where $n_i$ and $m_j$ are positive integers $I$, $J$ are index sets. Since the matrix rings $M_{n_i}(K)$ and $M_{m_j}(K[x, x^{-1}])$ are simple rings, $A$ and hence $B$ is a direct sum of finitely many matrix rings of finite order over $K$ and/or $K[x, x^{-1}]$. This proves (b).

(b) implies (c) follows from the known fact that matrix rings $M_{n_i}(K)$ and $M_{m_j}(K[x, x^{-1}])$ are directly-finite and finite ring direct sums of such matrix rings are directly-finite. Hence, by condition (b), $L_K(E)$ is directly-finite.

We want to finish the survey with another application of the Subalgebra Construction. In [6], the authors do not use the exact results, however they carry the same techniques and proofs to another subgraph (dual graph) construction.

The authors present the notion of a dual of a subgraph in a graph, which is the generalization of the usual notion of dual graph found in the literature that we quote here:

**Usual dual:** Let $E$ be an arbitrary graph. The *usual dual of $E$, $D(E)$*, is the graph formed from $E$ by taking

\[
\begin{align*}
D(E)^0 &= \{ e \mid e \in E^1 \} \\
D(E)^1 &= \{ ef \mid ef \in E^2 \} \\
s_{D(E)}(ef) &= e, \quad r_{D(E)}(ef) = f \text{ for all } ef \in E^2.
\end{align*}
\]

The interest on the usual dual graph notion in the context of Leavitt path algebras lies on the fact that, if $E$ is a row-finite graph without sinks, then there is an algebra isomorphism $L_K(E) \cong L_K(D(E))$ ([1, Proposition 2.11]). These statement is untrue.
for usual dual of a graph with sinks. The authors propose a new definition of dual graph which generalizes this important property to row-finite graphs with sinks.

Dual of \( F \) in \( E \): Let \( E \) be a graph and let \( F \) be a subgraph of \( E \). Denote \( F^0_1 = \{ v \in F^0 \mid s_{E}^{-1}(v) = \emptyset \} \), \( F^1_1 = r_{E}^{-1}(F^0) \) and \( F^0_2 = s(F^1) \cap s(E^1 \setminus F^1) \), \( F^1_2 = r_{F}^{-1}(F^0_2) \). The graph \( D_E(F) \), the dual of \( F \) in \( E \) is defined by

\[
D_E(F)^0 = D(F)^0 \cup F^0_1 \cup F^0_2 \\
D_E(F)^1 = D(F)^1 \cup F^1_1 \cup F^1_2 \\
s_{D_E(F)}|D(F) = s_{D(F)}, \quad r_{D_E(F)}|D(F) = r_{D(F)}
\]

For all \( e \in F^1_i \) with \( i \in \{1, 2\} \), \( s_{D_E(F)} = e \in D(F)^0 \), \( r_{D_E(F)}(e) = r_F(e) \in F^0_i \).

**Dual graph:** Given a graph \( E \), they define \( d(E) = D_E(E) \) and call it the dual graph of \( E \).

Then they prove the graded algebra isomorphism \( L_K(d(E)) \cong L_K(E) \) when \( E \) is a row-finite graph ([6, Proposition 3.6]). In this paper the authors also prove that for a graph \( E \) and a row-finite subgraph of \( E \) there is a graded monomorphism \( \theta : L_K(D_E(F)) \rightarrow L_K(E) \). In addition, \( F^0 \cup F^1 \subseteq \theta(L_K(D_E(E))) \). This result is stated as [6, Proposition 3.8] and the proof is basically rephrasing [5, Proposition 1.2].

**Acknowledgement.** The author would like to thank the following: Kulumani M. Rangaswamy for his suggestion on writing this article, Müge Kanuni for all her support and the referee for a careful and detailed review of the article and his well-suited corrections.

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The local-global principle in Leavitt path algebras


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