RC-CLASS AND LC-CLASS ON FIXED POINT THEOREMS FOR $\alpha$-CARISTI TYPE CONTRACTION MAPPINGS

Arslan Hojat Ansari and Muhammad Usman Ali

Abstract. In this paper, we introduce the notion of $(\alpha, H_{\text{RC}}, f_{\text{RC}})$-Caristi type contraction mappings and prove fixed point theorem by using this notion on complete metric space. To illustrate our result, we construct an example.

1 Introduction

Caristi [9] proved that if a self mapping $T$ on a complete metric space $(X, d)$ satisfies the condition:

$$d(x, Tx) \leq \phi(x) - \phi(Tx) \quad \forall \ x \in X$$

where $\phi : X \to [0, \infty)$ is a lower semicontinuous function, then $T$ has a fixed point. The mapping $T$ satisfying the condition (1.1) is known as Caristi mapping. Kirk [15] showed that if Caristi mapping for $(X, d)$ has a fixed point, then $(X, d)$ is complete and vice versa. Semat et al. [19] introduced the notion of $\alpha$-admissible and $\alpha$-$\psi$-contractive type mappings. These notions were extended by several authors, see for example, [1, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 20]. Recently, Ali [1] introduced the notion of $\alpha$-Caristi type contraction mapping and proved a fixed point theorem on complete metric space. On the other hand Ansari [2] introduced the family of functions known as $\text{RC}$-class and $\text{LC}$-class to generalize some existing fixed point theorems. In this paper we introduce a new Caristi type contraction condition by combining the above ideas. Note that, we denote by $\text{CL}(X)$ the space of all nonempty closed subsets of $X$. For $x \in X$ and $A \in \text{CL}(X)$, $d(x, A) = \inf\{d(x, a) : a \in A\}$. A function $H : \text{CL}(X) \times \text{CL}(X) \to [0, \infty]$ defined by

$$H(A, B) = \begin{cases} \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} \quad \text{if exists} \\ \infty \quad \text{otherwise} \end{cases}$$

is a generalized Hausdorff metric space induced by metric $d$.

2010 Mathematics Subject Classification: 47H10; 54H25.
Keywords: $\alpha$-admissible; $\alpha$-admissible; Caristi mappings; $\text{RC}$-class and $\text{LC}$-class.

******************************************************************************

http://www.utgjiu.ro/math/sma
Mohammadi et al. [18] and Asl et al. [8] extended the notion of $\alpha$-admissible mapping from singlevalued to multivalued mapping in the following way:

**Definition 1.** [18] Let $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CL(X)$ is $\alpha$-admissible if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for each $z \in Ty$.

**Definition 2.** [8] Let $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CL(X)$ is $\alpha^*$-admissible mapping if for each $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha^*(T x, Ty) \geq 1$, where $\alpha^*(T x, Ty) = \inf\{\alpha(u, v) : u \in T x \text{ and } v \in Ty\}$.

Minak and Altun [17] showed that every $\alpha^*$-admissible mapping is $\alpha$-admissible, but converse is not true in general, and gave the following example.

**Example 3.** Let $X = [-1, 1]$. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} 
\{0, 1\} & \text{if } x = -1 \\
\{1\} & \text{if } x = 0 \\
\{-x\} & \text{if } x \notin \{-1, 0\}
\end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 
0 & \text{if } x = y \\
1 & \text{if } x \neq y.
\end{cases}$$

This $T$ is $\alpha$-admissible but not $\alpha^*$-admissible.

Kutbi and Sintunavarat [16] introduced the notion of $\alpha$-continuous multivalued mapping which is more general than continuous multivalued mappings.

**Definition 4.** Let $(X, d)$ be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping $T : X \to CL(X)$ is said to be an $\alpha$-continuous, if for each sequence $\{x_n\}$ in $X$ such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, we have $Tx_n \to Tx$, that is, $\lim_{n \to \infty} H(Tx_n, Tx) = 0$.

**Definition 5.** [2] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. The function $f$ is said to be a $RC$-class if $f$ is continuous and satisfies

$$f(s, t) > 0 \implies s > t;$$

$$f(t, t) = 0;$$

$$s \leq t \implies f(e, s) \geq f(e, t) \text{ for each } e \in \mathbb{R};$$

$$t \leq e \leq s \implies f(s, e) + f(e, t) \leq f(s, t);$$

$$\exists g : \mathbb{R} \to \mathbb{R}, \quad f(g(s), g(t)) \geq 0 \implies s \leq t,$$

where $s, t, e \in \mathbb{R}$. 
In the following, we can see some examples for \( RC \)-class functions.

**Example 6.** For \( n \in \mathbb{N} \) and \( a > 1 \),
\[
\begin{align*}
f(s, t) &= s - t, & g(t) &= -t \\
f(s, t) &= \frac{s - t}{1 + t}, & g(t) &= \frac{1}{t} - 1 \\
f(s, t) &= s^{2n+1} - t^{2n+1}, & g(t) &= -t \\
f(s, t) &= a^s - a^t, & g(t) &= -t \\
f(s, t) &= e^{s^{2n+1} - t^{2n+1}} - 1, & g(t) &= -t \\
f(s, t) &= e^{s - t} - 1, & g(t) &= -t. 
\end{align*}
\]

**Definition 7.** \([2]\) We say that \( H : \mathbb{R}^+ \to \mathbb{R}^+ \) is a \( LC \)-class if \( h \) is continuous and satisfies the following conditions
\[
\begin{align*}
H(t) &> 0 \text{ if and only if } t > 0; \\
H(0) &= 0; \\
H(s + t) &\leq H(s) + H(t); \\
\end{align*}
\]
and
\[
x \leq y \implies H(x) \leq H(y).
\]

**Example 8.** For \( a > 1, m > 0 \) and \( n \in \mathbb{N} \)
\[
\begin{align*}
H(t) &= 1 - a^{-t} \\
H(t) &= mt \\
H(t) &= m \sqrt[2n+1]{t} \\
H(t) &= \log_a(1 + t) \\
H(t) &= \log_a(1 + \sqrt[n]{t}),
\end{align*}
\]
are some examples for \( LC \)-class.

## 2 Main Results

We begin this section with the following definition.

**Definition 9.** Let \( (X, d) \) be a metric space, \( \alpha : X \times X \to [0, \infty) \) and \( \phi : X \to [0, \infty) \) be two mappings, further, \( f \) is a \( RC \)-class and \( H \) is a \( LC \)-class function. A mapping \( T : X \to CL(X) \) is said to be an \((\alpha, H_{LC}, f_{RC})\)-Caristi type contraction if for each \( x \in X \) and \( u \in Tx \) with \( \alpha(x, u) \geq 1 \) there exists \( v \in Tu \) such that
\[
H(d(u, v)) \leq f(\phi(x), \phi(u)). \tag{2.1}
\]
Remark 10. If we take $\mathcal{H}(t) = t$ and $f(s, t) = s - t$, then above definition reduces to the Definition 2.1 [1].

Theorem 11. Let $(X, d)$ be a complete metric space and let $T : X \to CL(X)$ be an $(\alpha, \mathcal{H}_{LC}, f_{RC})$-Caristi type contraction mapping. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(ii) $T$ is $\alpha$-admissible;

(iii) $T$ is $\alpha$-continuous.

Then $T$ has a fixed point.

Proof. By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. By Definition 9, for $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$ there exists $x_2 \in Tx_1$ such that

$$\mathcal{H}(d(x_1, x_2)) \leq f(\phi(x_0), \phi(x_1)).$$

As $T$ is $\alpha$-admissible, then $\alpha(x_0, x_1) \geq 1$ implies $\alpha(x_1, x_2) \geq 1$. Again, by Definition 9, for $x_1 \in X$ and $x_2 \in Tx_1$ with $\alpha(x_1, x_2) \geq 1$ there exists $x_3 \in Tx_2$ such that

$$\mathcal{H}(d(x_2, x_3)) \leq f(\phi(x_1), \phi(x_2)).$$

Continuing in the same way, we get a sequence $\{x_n\}$ in $X$ such that $x_n \in Tx_{n-1}$, $\alpha(x_{n-1}, x_n) \geq 1$ and

$$0 \leq \mathcal{H}(d(x_n, x_{n+1})) \leq f(\phi(x_{n-1}), \phi(x_n)) \text{ for each } n \in \mathbb{N}. \quad (2.2)$$

By using the properties of $\mathcal{H}$, $f$ and above inequality, we conclude that the sequence $\{\phi(x_{n-1})\}$ is a nonincreasing sequence, there exists $r \geq 0$ such that $\phi(x_n) \to r$. Now consider $n, p \in \mathbb{N}$, by using the triangular inequality and subadditivity of $\mathcal{H}$, we have

$$\begin{align*}
\mathcal{H}(d(x_n, x_{n+p})) & \leq \mathcal{H}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\
& \quad + \cdots + d(x_{n+p-1}, x_{n+p})) \\
& \leq \mathcal{H}(d(x_n, x_{n+1})) + \mathcal{H}(d(x_{n+1}, x_{n+2})) + \mathcal{H}(d(x_{n+2}, x_{n+3})) \\
& \quad + \cdots + \mathcal{H}(d(x_{n+p-1}, x_{n+p})) \\
& \leq f(\phi(x_{n-1}), \phi(x_n)) + f(\phi(x_n), \phi(x_{n+1})) + f(\phi(x_{n+1}), \phi(x_{n+2})) \\
& \quad + \cdots + f(\phi(x_{n+p-2}), \phi(x_{n+p-1})) \\
& = f(\phi(x_{n-1}), \phi(x_{n+p-1})). \quad (2.3)
\end{align*}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $X$, since $\phi \to r$. By completeness of $X$, we have $x^* \in X$ such that $x_n \to x^*$. By hypothesis (iii), we have $\lim_{n \to \infty} H(Tx_n, Tx^*) = 0$. By using the triangular inequality, we have

$$\begin{align*}
d(x^*, Tx^*) & \leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\
& \leq d(x^*, x_{n+1}) + H(Tx_n, Tx^*).
\end{align*}$$
Letting \( n \to \infty \) in the above inequality, we have \( d(x^*, Tx^*) = 0 \). This implies that \( x^* \in Tx^* \).

**Example 12.** Let \( X = \mathbb{R} \) be endowed with the usual metric \( d(x, y) = |x - y| \). Define \( T : X \to CL(X) \) by

\[
Tx = \begin{cases}
[0, x] & \text{if } x \geq 0 \\
\{-e^x\} & \text{if } x < 0,
\end{cases}
\]

and \( \alpha : X \times X \to [0, \infty) \) by

\[
\alpha(x, y) = \begin{cases}
1 & \text{if } x, y \geq 0 \\
0 & \text{otherwise}.
\end{cases}
\]

Define \( \phi : X \to [0, \infty) \) by

\[
\phi(x) = \begin{cases}
x & \text{if } x > 0 \\
0 & \text{otherwise}.
\end{cases}
\]

Take \( H(x) = \frac{x}{2} \) and \( f(x, y) = x - y \) for each \( x, y \in X \). Then, for each \( x \in X \) and \( u \in Tx \) with \( \alpha(x, u) = 1 \), there exists \( v \in Tu \) such that

\[
H(d(u, v)) \leq f(\phi(x), \phi(u)).
\]

Therefore, \( T \) is \((\alpha, H_{LC}, f_RC)\)-Caristi type contraction mapping. For \( x_0 = 3 \) we have \( x_1 = 3/2 \in Tx_0 \) such that \( \alpha(x_0, x_1) = 1 \). Clearly, \( T \) is \( \alpha \)-admissible. Let \( \{x_n\} \) is any sequence in \( X \) such that \( x_n \to x^* \) and \( \alpha(x_n, x_{n+1}) = 1 \) for each \( n \in \mathbb{N} \), then by definition of \( \alpha \), it clear that \( x_n \geq 0 \) for each \( n \in \mathbb{N} \). Since \( x_n \to x^* \), then \( x^* \geq 0 \). Thus, \( Tx_n = [0, x_n] \) and \( Tx^* = [0, x^*] \). Therefore, \( \lim_{n \to \infty} H(Tx_n, Tx^*) = 0 \). This shows that \( T \) is \( \alpha \)-continuous. Thus, by Theorem 11, \( T \) has a fixed point.

**Example 13.** Let \( X = \mathbb{R} \) be endowed with the usual metric \( d(x, y) = |x - y| \). Define \( T : X \to CL(X) \) by

\[
Tx = \begin{cases}
[0, \frac{x}{x+1}] & \text{if } x \geq 0 \\
\{-x^2\} & \text{if } x < 0,
\end{cases}
\]

and \( \alpha : X \times X \to [0, \infty) \) by

\[
\alpha(x, y) = \begin{cases}
1 & \text{if } x, y \geq 0 \\
0 & \text{otherwise}.
\end{cases}
\]

Define \( \phi : X \to [0, \infty) \) by

\[
\phi(x) = \begin{cases}
\frac{x}{2} & \text{if } x > 0 \\
0 & \text{otherwise}.
\end{cases}
\]
Take $H(x) = \frac{x}{4}$ and $f(x, y) = x - y$ for each $x, y \in X$. Then, for each $x \in X$ and $u \in Tx$ with $\alpha(x, u) = 1$, there exists $v \in Tu$ such that
\[
H(d(u, v)) \leq f(\phi(x), \phi(u)).
\]
Therefore, $T$ is $(\alpha, H_{LC}, f_{RC})$-Caristi type contraction mapping. It is easy to see that the rest of the conditions of Theorem 11 hold. Thus, $T$ has a fixed point. Note that Theorem 2.1 of [1] is not applicable here, to see consider $x = \frac{1}{3}$ and $u = \frac{1}{4} \in Tx$.

**Definition 14.** Let $(X, d)$ be a metric space, $\alpha : X \times X \to [0, \infty)$ and $\phi : X \to [0, \infty)$ be two mappings, further, $f$ is a $RC$-class and $H$ is a $LC$-class function. A mapping $T : X \to CL(X)$ is said to be an $(\alpha_T, H_{LC}, f_{RC})$-Caristi type contraction if for each $x \in X$ and $u \in Tx$ there exists $v \in Tu$ such that
\[
H(d(u, v)) \leq f(\phi(x), \phi(u)) \quad \text{whenever } \alpha(u, v) \geq 1.
\]

**Theorem 15.** Let $(X, d)$ be a complete metric space and let $T : X \to CL(X)$ be an $(\alpha_T, H_{LC}, f_{RC})$-Caristi type contraction mapping. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(ii) $T$ is $\alpha_*$-admissible;

(iii) $T$ is $\alpha$-continuous.

Then $T$ has a fixed point.

**Proof.** The proof of this theorem can be obtained on the same lines as the proof of last theorem is done. \qed

3 Consequence

In this section we list some fixed point theorems which can be obtained from our results:

**Theorem 16.** Let $(X, d, \preceq)$ be a complete ordered metric space and let $T : X \to CL(X)$ be a mapping such that for each $x \in X$ and $u \in Tx$ with $x \preceq u$ there exists $v \in Tu$ satisfying
\[
H(d(u, v)) \leq f(\phi(x), \phi(u))
\]
where $\phi : X \to [0, \infty)$ be a function. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$;

(ii) for each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for each $z \in Ty$;

******************************************************************************


http://www.utgjiu.ro/math/sma
(iii) $T$ is ordered-continuous, that is, for each sequence $\{x_n\}$ in $X$ such that $x_n \to x$ and $x_n \leq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, we have $Tx_n \to Tx$.

Then $T$ has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 
1 & \text{if } x \leq y \\
0 & \text{otherwise} 
\end{cases}.$$ 

Then by the hypothesis of this theorem, it is easy to see that all conditions of Theorem 11 hold. Thus, $T$ has a fixed point.

In following result, we assume that $(X, d)$ is a metric space and $G = (V(G), E(G))$ is a directed graph such that the set of its vertices $V(G)$ coincides with $X$ (i.e., $V(G) = X$) and the set of its edges $E(G)$ is such that $E(G) \supseteq \Delta$, where $\Delta = \{(x, x) : x \in X\}$. Further assume that $G$ has no parallel edges.

**Theorem 17.** Let $(X, d)$ be a complete metric space endowed with the graph $G$ and let $T : X \to CL(X)$ be a mapping such that for each $x \in X$ and $u \in Tx$ with $(x, u) \in E(G)$ there exists $v \in Tu$ satisfying

$$H(d(u, v)) \leq f(\phi(x), \phi(u))$$

where $\phi : X \to [0, \infty)$ be a function. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;

(ii) for each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for each $z \in Ty$;

(iii) $T$ is $G$-continuous, that is, for each sequence $\{x_n\}$ in $X$ such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N} \cup \{0\}$, we have $Tx_n \to Tx$.

Then $T$ has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in E(G) \\
0 & \text{otherwise} 
\end{cases}.$$ 

Then by the hypothesis of this theorem, it is easy to see that all the conditions of Theorem 11 hold. Thus, $T$ has a fixed point.

**Acknowledgment.** Authors are thankful to the reviewer for his useful comments.
References


******************************************************************************


http://www.utgjiu.ro/math/sma


Arslan Hojat Ansari  
Department of Mathematics  
Karaj Branch, Islamic Azad University  
Karaj, Iran.  
e-mail:analisisamirmath2@gmail.com

Muhammad Usman Ali  
Department of Mathematics  
COMSATS Institute of Information Technology  
Attock, Pakistan.  
e-mail:muh_usman.ali@yahoo.com

**************************************************************************

http://www.utgjiu.ro/math/sma
License

This work is licensed under a Creative Commons Attribution 4.0 International License.