GENERALIZED COMPATIBILITY IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we introduce the notion of generalized compatibility of a pair of mappings $F, G : X \times X \to X$, where $(X, d)$ is a partially ordered metric space. We use this concept to prove a coupled coincidence point theorem for nonlinear contractions in partially ordered metric spaces. Our work extends the paper of Choudhury and Kundu [B.S. Choudhury and A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal. 73 (2010) 2524-2531]. Some examples are also given to illustrate the new concepts and the obtained result.

1 Introduction

Fixed point problems of contractive mappings in metric spaces endowed with a partial order have been studied by many authors (see [12, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 10, 13, 14]). In [12], some applications to matrix equations are presented and in [8, 11] some applications to ordinary differential equations are given. Bhaskar and Lakshmikantham [4] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and studied the problems of the uniqueness of a coupled fixed point in partially ordered metric spaces and applied their theorems to problems of the existence and uniqueness of solution for a periodic boundary value problem. In [9], Lakshmikantham and Ćirić introduced the concept of a coupled coincidence point for mappings $F : X \times X \to X$ and $g : X \to X$, and proved some nice coupled coincidence point theorems for nonlinear contractions in partially ordered metric spaces under the hypotheses that $g$ is continuous and commutes with $F$. In 2011, Choudhury and Kundu [5] introduced the notion of compatible mappings $F : X \times X \to X$ and $g : X \to X$, and obtained coupled coincidence point results under the hypotheses $g$ is continuous and the pair $\{F, g\}$ is compatible.

In this paper, we consider mappings $F, G : X \times X \to X$, where $(X, d)$ is a partially ordered metric space. We introduce a new concept of generalized compatibility of

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the pair \( \{ F, G \} \) and we prove a coupled coincidence point theorem for nonlinear contractions in partially ordered metric spaces. The presented theorem extends the recent result of Choudhury and Kundu [5] and some examples are also considered.

2 Mathematical preliminaries

Let \((X, \preceq)\) be a partially ordered set. The concept of a mixed monotone property of the mapping \( F : X \times X \rightarrow X \) has been introduced by Bhaskar and Lakshmikantham in [4].

**Definition 1.** (see Bhaskar and Lakshmikantham [4]). Let \((X, \preceq)\) be a partially ordered set and \( F : X \times X \rightarrow X \). Then the map \( F \) is said to have mixed monotone property if \( F(x, y) \) is monotone non-decreasing in \( x \) and is monotone non-increasing in \( y \); that is, for any \( x, y \in X \),

\[
x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y)
\]

and

\[
y_1 \preceq y_2 \text{ implies } F(x, y_2) \preceq F(x, y_1).
\]

Lakshmikantham and Ćirić in [9] introduced the concept of a \( g \)-mixed monotone mapping.

**Definition 2.** (see Lakshmikantham and Ćirić [9]). Let \((X, \preceq)\) be a partially ordered set, \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \). Then the map \( F \) is said to have mixed \( g \)-monotone property if \( F(x, y) \) is monotone \( g \)-non-decreasing in \( x \) and is monotone \( g \)-non-increasing in \( y \); that is, for any \( x, y \in X \),

\[
gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y)
\]

and

\[
gy_1 \preceq gy_2 \text{ implies } F(x, y_2) \preceq F(x, y_1).
\]

**Definition 3.** (see Bhaskar and Lakshmikantham [4]). An element \((x, y) \in X \times X\) is called a coupled fixed point of a mapping \( F : X \times X \rightarrow X \) if

\[
F(x, y) = x \text{ and } F(y, x) = y.
\]

**Definition 4.** (see Lakshmikantham and Ćirić [9]). An element \((x, y) \in X \times X\) is called a coupled coincidence point of the mappings \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \) if

\[
F(x, y) = gx \text{ and } F(y, x) = gy.
\]

**Definition 5.** (see Lakshmikantham and Ćirić [9]). Let \( X \) be a non-empty set. Then we say that the mappings \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \) are commutative if

\[
g(F(x, y)) = F(gx, gy).
\]
Lakshmikantham and Ćirić in [9] proved the following nice result.

**Theorem 6.** (see Lakshmikantham and Ćirić [9]). Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Assume there is a function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(t) < t$ and $\lim_{r \to t^+} \phi(r) < t$ for each $t > 0$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that $F$ has the mixed $g$-monotone property and

$$d(F(x, y), F(u, v)) \leq \phi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right)$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gy \preceq gv$. Assume that $F(X \times X) \subseteq g(X)$, $g$ is continuous and commutes with $F$ and also suppose either $F$ is continuous or $X$ has the following properties:

1. if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all $n$,
2. if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all $n$.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$ then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is, $F$ and $g$ have a coupled coincidence point.

Choudhury and Kundu in [5] introduced the notion of compatibility.

**Definition 7.** (see Choudhury and Kundu [5]). The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if

$$\lim_{n \to +\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n \to +\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0,$$

whenever $(x_n)$ and $(y_n)$ are sequences in $X$, such that

$$\lim_{n \to +\infty} F(x_n, y_n) = \lim_{n \to +\infty} gx_n = x$$

and

$$\lim_{n \to +\infty} F(y_n, x_n) = \lim_{n \to +\infty} gy_n = y,$$

for all $x, y \in X$ are satisfied.

Using the concept of compatibility, Choudhury and Kundu proved the following interesting result.

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Theorem 8. (see Choudhury and Kundu [5]). Let \((X, \preceq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(\phi : [0, +\infty) \to [0, +\infty)\) be such that \(\phi(t) < t\) and \(\lim_{r \to t^+} \phi(r) < t\) for all \(t > 0\). Suppose \(F : X \times X \to X\) and \(g : X \to X\) be two mappings such that \(F\) has the mixed \(g\)-monotone property and satisfy

\[
d(F(x, y), F(u, v)) \leq \phi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right)
\]

for all \(x, y, u, v \in X\) with \(gx \preceq gu\) and \(gv \preceq gy\). Let \(F(X \times X) \subseteq g(X)\), \(g\) be continuous and monotone increasing and \(F\) and \(g\) be compatible mappings. Also suppose either \(F\) is continuous or \(X\) has the following properties:

1. if a non-decreasing sequence \(x_n \to x\), then \(x_n \preceq x\) for all \(n\),
2. if a non-increasing sequence \(x_n \to x\), then \(x \preceq x_n\) for all \(n\).

If there exist \(x_0, y_0 \in X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(F(y_0, x_0) \preceq gy_0\) then there exist \(x, y \in X\) such that \(gx = F(x, y)\) and \(gy = F(y, x)\), that is, \(F\) and \(g\) have a coupled coincidence point.

Now, we introduce the following new concepts. Let \((X, \preceq)\) be a partially ordered set endowed with a metric \(d\). We consider two mappings \(F, G : X \times X \to X\).

Definition 9. \(F\) is said to be \(G\)-increasing with respect to \(\preceq\) if for all \(x, y, u, v \in X\), we have

\[
G(x, y) \preceq G(u, v) \text{ implies } F(x, y) \preceq F(u, v).
\]

We present three examples illustrating Definition 9.

Example 10. Let \(X = (0, +\infty)\) endowed with the natural ordering of real numbers \(\leq\). Define the mappings \(F, G : X \times X \to X\) by

\[
F(x, y) = \ln(x + y) \quad \text{and} \quad G(x, y) = x + y
\]

for all \((x, y) \in X \times X\). Then \(F\) is \(G\)-increasing with respect to \(\preceq\).

Example 11. Let \(X = \mathbb{N}\) endowed with the partial order \(\preceq\) defined by

\[
x, y \in X, \quad x \preceq y \text{ if and only if } y \text{ divides } x.
\]

Define the mappings \(F, G : X \times X \to X\) by

\[
F(x, y) = x^2y^2 \quad \text{and} \quad G(x, y) = xy
\]

for all \((x, y) \in X \times X\). Then \(F\) is \(G\)-increasing with respect to \(\preceq\).
Example 12. Let \( X \) be the set of all subsets of \( \mathbb{N} \). We endow \( X \) with the partial order \( \preceq \) defined by
\[
A, B \in X, \quad A \preceq B \quad \text{if and only if} \quad A \subseteq B.
\]

Define the mappings \( F, G : X \times X \to X \) by
\[
F(A, B) = A \cup B \cup \{0\} \quad \text{and} \quad G(A, B) = A \cup B
\]
for all \( A, B \in X \). Then \( F \) is \( G \)-increasing with respect to \( \preceq \).

Definition 13. An element \( (x, y) \in X \times X \) is called a coupled coincidence point of \( F \) and \( G \) if
\[
F(x, y) = G(x, y) \quad \text{and} \quad F(y, x) = G(y, x).
\]

Example 14. Let \( X = \mathbb{R} \) and \( F, G : X \times X \to X \) defined by
\[
F(x, y) = xy \quad \text{and} \quad G(x, y) = \frac{2}{3}(x + y)
\]
for all \( x, y \in X \). Then \((0,0), (1,2)\) and \((2,1)\) are coupled coincidence points of \( F \) and \( G \).

Definition 15. We say that the pair \( \{F, G\} \) satisfies the generalized compatibility if
\[
\begin{align*}
\lim_{n \to +\infty} d(F(y_n, x_n), G(y_n, x_n)) & = 0, \\
\lim_{n \to +\infty} d(F(x_n, y_n), G(x_n, y_n)) & = 0,
\end{align*}
\]
whenever \( (x_n) \) and \( (y_n) \) are sequences in \( X \) such that
\[
\begin{align*}
F(x_n, y_n) & \to t_1 \quad G(x_n, y_n) \to t_1 \quad \text{as} \quad n \to +\infty; \\
F(y_n, x_n) & \to t_2 \quad G(y_n, x_n) \to t_2 \quad \text{as} \quad n \to +\infty.
\end{align*}
\]

The following examples illustrate the concept of generalized compatibility.

Example 16. Let \( X = \mathbb{R} \) endowed with the standard metric \( d(x, y) = |x - y| \) for all \( x, y \in X \). Define \( F, G : X \times X \to X \) by
\[
F(x, y) = x^2 - y^2 \quad \text{and} \quad G(x, y) = x^2 + y^2
\]
for all \( x, y \in X \). Let \( (x_n) \) and \( (y_n) \) two sequences in \( X \) such that
\[
\begin{align*}
F(x_n, y_n) & \to t_1 \quad G(x_n, y_n) \to t_1 \quad \text{as} \quad n \to +\infty; \\
F(y_n, x_n) & \to t_2 \quad G(y_n, x_n) \to t_2 \quad \text{as} \quad n \to +\infty.
\end{align*}
\]

We can prove easily that \( t_1 = t_2 = 0 \) and
\[
\begin{align*}
\lim_{n \to +\infty} d(F(x_n, y_n), G(x_n, y_n)) & = 0, \\
\lim_{n \to +\infty} d(F(y_n, x_n), G(y_n, x_n)) & = 0.
\end{align*}
\]

Then the pair \( \{F, G\} \) satisfies the generalized compatibility.
Example 17. Let \((X, d)\) be a metric space, \(F : X \times X \to X\) and \(g : X \to X\). Define the mapping \(G : X \times X \to X\) by
\[
G(x, y) = gx, \quad \forall (x, y) \in X \times X.
\]
It is easy to show that if \(\{F, g\}\) is compatible, then \(\{F, G\}\) satisfies the generalized compatibility.

3 Main result

First, denote by \(\Phi\) be the set of functions \(\varphi : [0, +\infty) \to [0, +\infty)\) satisfying

(i) \(\varphi\) is non-decreasing,

(ii) \(\varphi(t) < t\) for all \(t > 0\),

(iii) \(\lim_{r \to t^+} \varphi(r) < t\) for all \(t > 0\).

Lemma 18. Let \(\varphi \in \Phi\) and \((u_n)\) be a given sequence such that \(u_n \to 0^+\) as \(n \to +\infty\). Then, \(\varphi(u_n) \to 0^+\) as \(n \to +\infty\).

Proof. Let \(\varepsilon > 0\). Since \(u_n \to 0^+\) as \(n \to +\infty\), there exists \(N \in \mathbb{N}\) such that \(0 \leq u_n < \varepsilon\) for all \(n \geq N\).

Using (i) and (ii), we get
\[
\varphi(u_n) \leq \varphi(\varepsilon) < \varepsilon\text{ for all } n \geq N.
\]
Thus we proved that \(\varphi(u_n) \to 0^+\) as \(n \to +\infty\).

Theorem 19. Let \((X, \preceq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(F, G : X \times X \to X\) be two mappings such that \(F\) is \(G\)-increasing with respect to \(\preceq\), and satisfy
\[
d(F(x, y), F(u, v)) \leq \varphi \left( \frac{d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))}{2} \right), \tag{3.1}
\]
for all \(x, y, u, v \in X\), with \(G(x, y) \preceq G(u, v)\) and \(G(v, u) \preceq G(y, x)\), where \(\varphi \in \Phi\).

Suppose that for any \(x, y \in X\), there exist \(u, v \in X\) such that
\[
\begin{align*}
F(x, y) &= G(u, v) \\
F(y, x) &= G(v, u).
\end{align*} \tag{3.2}
\]
Suppose that \(G\) is continuous and has the mixed monotone property, and the pair \(\{F, G\}\) satisfies the generalized compatibility. Also suppose either \(F\) is continuous or \(X\) has the following properties:
(a) if a non-decreasing sequence \( x_n \to x \), then \( x_n \preceq x \) for all \( n \),

(b) if a non-increasing sequence \( x_n \to x \), then \( x \preceq x_n \) for all \( n \).

If there exist \( x_0, y_0 \in X \) such that \( G(x_0, y_0) \preceq F(x_0, y_0) \) and \( F(y_0, x_0) \preceq G(y_0, x_0) \), then \( F \) and \( G \) have a coupled coincidence point.

**Proof.** Let \( x_0, y_0 \in X \) such that \( G(x_0, y_0) \preceq F(x_0, y_0) \) and \( F(y_0, x_0) \preceq G(y_0, x_0) \) (such points exist by hypothesis). Thanks to (3.2), there exists \((x_1, y_1) \in X \times X \) such that

\[
F(x_0, y_0) = G(x_1, y_1) \quad \text{and} \quad F(y_0, x_0) = G(y_1, x_1).
\]

Continuing this process, we can construct two sequences \((x_n)\) and \((y_n)\) in \( X \) such that

\[
F(x_n, y_n) = G(x_{n+1}, y_{n+1}), \quad F(y_n, x_n) = G(y_{n+1}, x_{n+1}), \quad \text{for all } n \in \mathbb{N}. \tag{3.3}
\]

We will show that for all \( n \in \mathbb{N} \), we have

\[
G(x_n, y_n) \preceq G(x_{n+1}, y_{n+1}) \quad \text{and} \quad G(y_{n+1}, x_{n+1}) \preceq G(y_n, x_n). \tag{3.4}
\]

We shall use the mathematical induction. Since \( G(x_0, y_0) \preceq F(x_0, y_0) \) and \( F(y_0, x_0) \preceq G(y_0, x_0) \), and as \( G(x_1, y_1) = F(x_0, y_0) \) and \( G(y_1, x_1) = F(y_0, x_0) \), we have

\[
G(x_0, y_0) \preceq G(x_1, y_1) \quad \text{and} \quad G(y_1, x_1) \preceq G(y_0, x_0).
\]

Thus (3.4) holds for \( n = 0 \). Suppose now that (3.4) holds for some fixed \( n \in \mathbb{N} \). Since \( F \) is \( G \)-increasing with respect to \( \preceq \), we have

\[
G(x_{n+1}, y_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, y_{n+1}) = G(x_{n+2}, y_{n+2})
\]

and

\[
F(y_{n+1}, x_{n+1}) = G(y_{n+2}, x_{n+2}) \preceq F(y_n, x_n) = G(y_{n+1}, x_{n+1}).
\]

Thus we proved that (3.4) holds for all \( n \in \mathbb{N} \).

For all \( n \in \mathbb{N} \), denote

\[
\delta_n = d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1})). \tag{3.5}
\]

We can suppose that \( \delta_n > 0 \) for all \( n \in \mathbb{N} \), if not, \((x_n, y_n)\) will be a coincidence point and the proof is finished. We claim that for any \( n \in \mathbb{N} \), we have

\[
\delta_{n+1} \leq 2\varphi\left(\frac{\delta_n}{2}\right). \tag{3.6}
\]

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Since \(G(x_n, y_n) \leq G(x_{n+1}, y_{n+1})\) and \(G(y_n, x_n) \geq G(y_{n+1}, x_{n+1})\), letting \(x = x_n, y = y_n, u = x_{n+1}\) and \(v = y_{n+1}\) in (3.1), and using (3.3), we get
\[
\begin{align*}
\delta & = \lim_{n \to +\infty} \delta_n \\
& \leq 2 \lim_{n \to +\infty} \varphi \left( \frac{\delta_n - 1}{2} \right) \\
& = 2 \lim_{n \to +\infty} \varphi \left( \frac{\delta_n - 1}{2} \right) < 2 \frac{\delta}{2} = \delta,
\end{align*}
\]
which is a contradiction. Thus \(\delta = 0\), that is,
\[
\lim_{n \to +\infty} d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1})) = \lim_{n \to +\infty} \delta_n = 0.
\] (3.9)

We shall prove that \(((G(x_n, y_n), G(y_n, x_n)))\) is a Cauchy sequence in \(X \times X\) endowed with the metric \(\eta\) defined by
\[
\eta((x, y), (u, v)) = d(x, u) + d(y, v)
\]
for all \((x, y), (u, v) \in X \times X\). We argue by contradiction. Suppose that \(((G(x_n, y_n), G(y_n, x_n)))\) is not a Cauchy sequence in \((X \times X, \eta)\). Then, there exists \(\varepsilon > 0\) for which we can

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find two sequences of positive integers \((m(k))\) and \((n(k))\) such that for all positive integer \(k\) with \(n(k) > m(k) > k\), we have

\[
\begin{align*}
\eta((Gx_{m(k)}, Gy_{m(k)}), (Gy_{m(k)}, Gx_{m(k)})), ((Gx_{n(k)}, Gy_{n(k)}), (Gy_{n(k)}, Gx_{n(k)}))) &> \varepsilon, \\
\eta((Gx_{m(k)}, Gy_{m(k)}), (Gy_{n(k)}, Gx_{m(k)})), ((Gx_{n(k)-1}, Gy_{n(k)-1}), (Gy_{n(k)-1}, Gx_{n(k)-1}))) &\leq \varepsilon.
\end{align*}
\]

(3.10)

By definition of the metric \(\eta\), we have

\[
d_k = d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)}, Gy_{n(k)})) + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{n(k)}, Gx_{n(k)})) > \varepsilon
\]

(3.11)

and

\[
d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)-1}, Gy_{n(k)-1})) + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{n(k)-1}, Gx_{n(k)-1})) \leq \varepsilon.
\]

(3.12)

Further from (3.11) and (3.12), for all \(k \geq 0\), we have

\[
\varepsilon < d_k \leq d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)-1}, Gy_{n(k)-1})) + d((Gx_{n(k)-1}, Gy_{n(k)-1}), (Gx_{n(k)}, Gy_{n(k)}))
\]

\[
+ d((Gy_{n(k)}, Gx_{m(k)}), (Gy_{n(k)-1}, Gx_{n(k)-1}))) + d((Gy_{n(k)-1}, Gx_{n(k)-1}), (Gy_{n(k)}, Gx_{n(k)})) \leq \varepsilon + \delta_{n(k)-1}.
\]

Taking the limit as \(k \to +\infty\) in the above inequality, we have by (3.9),

\[
\lim_{k \to +\infty} d_k = \varepsilon^+.
\]

(3.13)

Again, for all \(k \geq 0\), we have

\[
d_k = d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)}, Gy_{n(k)})) + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{n(k)}, Gx_{n(k)}))
\]

\[
\leq d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)+1}, Gy_{n(k)+1}))) + d((Gx_{n(k)+1}, Gy_{n(k)+1}), (Gx_{n(k)+1}, Gy_{n(k)+1})))
\]

\[
+ d((Gy_{n(k)+1}, Gx_{m(k)+1}), (Gy_{n(k)+1}, Gx_{m(k)+1}))) + d((Gy_{n(k)+1}, Gx_{m(k)+1}), (Gy_{n(k)+1}, Gx_{m(k)+1})))
\]

\[
= d((Gx_{m(k)}, Gy_{m(k)}), (Gx_{n(k)+1}, Gy_{n(k)+1}))) + d((Gy_{m(k)}, Gx_{m(k)}), (Gy_{n(k)+1}, Gx_{m(k)+1})))
\]

\[
+ d((Gy_{n(k)+1}, Gx_{m(k)+1}), (Gy_{n(k)+1}, Gx_{m(k)+1}))) + d((Gy_{n(k)+1}, Gx_{n(k)+1}), (Gy_{n(k)+1}, Gx_{n(k)+1})))
\]

Hence, for all \(k \geq 0\),

\[
d_k \leq \delta_{m(k)} + \delta_{n(k)}
\]

\[
+ d((Gx_{m(k)+1}, Gy_{m(k)+1}), (Gx_{n(k)+1}, Gy_{n(k)+1}))) + d((Gy_{m(k)+1}, Gx_{m(k)+1}), (Gy_{n(k)+1}, Gx_{n(k)+1}))).
\]

(3.14)
From (3.1), (3.4) and (3.11), for all \( k \geq 0 \), we have
\[
d((Gx_{m(k)+1}, Gy_{m(k)+1}), (Gx_{n(k)+1}, Gy_{n(k)+1})) = d((Fx_{m(k)}, Fy_{m(k)}), (Fx_{n(k)}, Fy_{n(k)})) \\
\leq \varphi \left( \frac{d(G(x_{m(k)}, y_{m(k)}), G(x_{n(k)}, y_{n(k)})) + d(G(y_{m(k)}, x_{m(k)}), G(y_{n(k)}, x_{n(k)}))}{2} \right) \\
= \varphi \left( \frac{d_k}{2} \right).
\]  
(3.15)

Also, from (3.1), (3.4) and (3.11), for all \( k \geq 0 \), we have
\[
d((Gy_{m(k)+1}, Gx_{m(k)+1}), (Gy_{n(k)+1}, Gx_{n(k)+1})) = d((Fy_{m(k)}, Fx_{m(k)}), (Fy_{n(k)}, Fx_{n(k)})) \\
\leq \varphi \left( \frac{d(G(y_{m(k)}, x_{m(k)}), G(y_{n(k)}, x_{n(k)})) + d(G(x_{m(k)}, y_{m(k)}), G(x_{n(k)}, y_{n(k)}))}{2} \right) \\
= \varphi \left( \frac{d_k}{2} \right).
\]  
(3.16)

Putting (3.15) and (3.16) in (3.14), we get
\[
d_k \leq \delta_{m(k)} + \delta_{n(k)} + 2\varphi \left( \frac{d_k}{2} \right).
\]

Letting \( k \to +\infty \) in the above inequality and using (3.9) and (3.13), we obtain
\[
\varepsilon \leq 2 \lim_{k \to +\infty} \varphi \left( \frac{d_k}{2} \right) = 2 \lim_{d_k \to +\varepsilon} \varphi \left( \frac{d_k}{2} \right) < \frac{2\varepsilon}{2} = \varepsilon,
\]  
(3.17)

which is a contradiction. Thus we proved that \( ((G(x_n, y_n), G(y_n, x_n)) \) is a Cauchy sequence in \( (X \times X, \eta) \), which implies that \( ((G(x_n, y_n)) \) and \( (G(y_n, x_n)) \) are Cauchy sequences in \( (X, d) \).

Now, since \( (X, d) \) is complete, there exist \( x, y \in X \) such that
\[
\lim_{n \to +\infty} G(x_n, y_n) = \lim_{n \to +\infty} F(x_n, y_n) = x \quad \text{and} \quad \lim_{n \to +\infty} G(y_n, x_n) = \lim_{n \to +\infty} F(y_n, x_n) = y.
\]  
(3.18)

Since the pair \( \{F, G\} \) satisfies the generalized compatibility, from (3.18), we get
\[
\lim_{n \to +\infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) = 0
\]  
(3.19)

and
\[
\lim_{n \to +\infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) = 0.
\]  
(3.20)

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Suppose that \( F \) is continuous.
For all \( n \geq 0 \), we have
\[
d(G(x, y), F(G(x_n, y_n), G(y_n, x_n))) \leq d(G(x, y), G(F(x_n, y_n), F(y_n, x_n))) \\
+ d(G(F(x_n, y_n), F(y_n, x_n)), F(G(x_n, y_n), G(y_n, x_n))).
\]
Taking the limit as \( n \to +\infty \), using (3.18), (3.19) and the fact that \( F \) and \( G \) are continuous, we have
\[
G(x, y) = F(x, y).
\]
(3.21)
Similarly, using (3.18), (3.20) and the fact that \( F \) and \( G \) are continuous, we have
\[
G(y, x) = F(y, x).
\]
(3.22)
Thus, we proved that \((x, y)\) is a coupled coincidence point of \( F \) and \( G \).
Now, suppose that (a) and (b) hold.
By (3.4) and (3.18), we have \((G(x_n, y_n))\) is non-decreasing sequence, \(G(x_n, y_n) \to x\) and \((G(y_n, x_n))\) is non-increasing sequence, \(G(y_n, x_n) \to y\) as \( n \to +\infty \). Then by (a) and (b), for all \( n \in \mathbb{N} \), we have
\[
G(x_n, y_n) \preceq x \quad \text{and} \quad G(y_n, x_n) \succeq y.
\]
(3.23)
Since the pair \( \{F, G\} \) satisfies the generalized compatibility and \( G \) is continuous, by (3.19) and (3.20), we have
\[
\lim_{n \to +\infty} G(G(x_n, y_n), G(y_n, x_n)) = G(x, y) \\
= \lim_{n \to +\infty} G(F(x_n, y_n), F(y_n, x_n))
\]
(3.24)
and
\[
\lim_{n \to +\infty} G(G(y_n, x_n), G(x_n, y_n)) = G(y, x) \\
= \lim_{n \to +\infty} G(F(y_n, x_n), F(x_n, y_n))
\]
(3.25)
Now, we have
\[
d(G(x, y), F(x, y)) \leq d(G(x, y), G(G(x_{n+1}, y_{n+1}), G(y_{n+1}, x_{n+1}))) \\
+ d(G(F(x_n, y_n), F(y_n, x_n)), F(x, y)).
\]
Letting \( n \to +\infty \) in the above inequality and using (3.24), we get
\[
d(G(x, y), F(x, y)) \leq \lim_{n \to +\infty} d(G(F(x_n, y_n), F(y_n, x_n)), F(x, y)) \\
= \lim_{n \to +\infty} d(F(G(x_n, y_n), G(y_n, x_n)), F(x, y)).
\]
Since $G$ has the mixed monotone property, it follows from (3.23) that

$$G(G(x_n, y_n), G(y_n, x_n)) \preceq G(x, y) \quad \text{and} \quad G(G(y_n, x_n), G(x_n, y_n)) \succeq G(y, x).$$

Then, using (3.1), (3.24), (3.25) and Lemma 18, we get

$$d(G(x, y), F(x, y)) \leq \lim_{n \to +\infty} \varphi \left( \frac{d(G(x_n, y_n), G(y_n, x_n)), G(x, y)) + d(G(y_n, x_n), G(x_n, y_n)), G(y, x))}{2} \right) = 0.$$ 

Then we get

$$G(x, y) = F(x, y).$$

Similarly, we can show that

$$G(y, x) = F(y, x).$$

Thus we proved that $(x, y)$ is a coupled coincidence point of $F$ and $G$. This completes the proof of the Theorem 19.

Now, we deduce an analogous result to Theorem 8 of Choudhury and Kundu [5]. At first, we introduce the following definition.

**Definition 20.** Let $(X, \preceq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. We say that $F$ is $g$-increasing with respect to $\preceq$ if for any $x, y \in X$,

$$gx_1 \preceq gx_2 \quad \text{implies} \quad F(x_1, y) \preceq F(x_2, y)$$

and

$$gy_1 \preceq gy_2 \quad \text{implies} \quad F(x, y_1) \preceq F(x, y_2).$$

**Corollary 21.** Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ is $g$-increasing with respect to $\preceq$, and satisfy

$$d(F(x, y), F(u, v)) \leq \varphi \left( \frac{d(gx, gu) + d(gy, gv)}{2} \right),$$

for all $x, y, u, v \in X$, with $gx \preceq gu$ and $gy \preceq gy$, where $\varphi \in \Phi$. Suppose that $F(X \times X) \subseteq g(X)$, $g$ is continuous and monotone increasing with respect to $\preceq$, and the pair $\{F, g\}$ is compatible. Also suppose either $F$ is continuous or $X$ has the following properties:

(a) if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all $n$,

(b) if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all $n$. 

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If there exist \( x_0, y_0 \in X \) such that \( gx_0 \leq F(x_0, y_0) \) and \( F(y_0, x_0) \leq gy_0 \), then \( F \) and \( g \) have a coupled coincidence point, that is, there exists \( (x, y) \in X \times X \) such that \( gx = F(x, y) \) and \( gy = F(y, x) \).

**Proof.** Taking \( G : X \times X \to X, (x, y) \mapsto G(x, y) = gx \) in Theorem 19, we obtain Corollary 21.

Now, we present an example to illustrate our obtained result given by Theorem 19.

**Example 22.** Let \( X = [0, 1] \) endowed with the natural ordering of real numbers. We endow \( X \) with the standard metric \( d(x, y) = |x - y| \) for all \( x, y \in X \). Then \( (X, d) \) is a complete metric space. Define the mappings \( G, F : X \times X \to X \) by

\[
G(x, y) = \begin{cases} 
  x - y & \text{if } x \geq y \\
  0 & \text{if } x < y
\end{cases} \quad \text{and} \quad F(x, y) = \begin{cases} 
  \frac{x - y}{3} & \text{if } x \geq y \\
  0 & \text{if } x < y
\end{cases}.
\]

Let us prove that \( F \) is \( G \)-increasing.

Let \( (x, y), (u, v) \in X \times X \) with \( G(x, y) \leq G(u, v) \). We consider the following cases.

**Case-1:** \( x < y \).
In this case, we have \( F(x, y) = 0 \leq F(u, v) \).
If \( u \geq v \), we get

\[
G(x, y) \leq G(u, v) \Rightarrow x - y \leq u - v \Rightarrow \frac{x - y}{3} \leq \frac{u - v}{3} \Rightarrow F(x, y) \leq F(u, v).
\]

If \( u < v \), we get

\[
G(x, y) \leq G(u, v) \Rightarrow 0 \leq x - y \leq 0 \Rightarrow x = y \Rightarrow F(x, y) = 0 \leq F(u, v).
\]

Thus we proved that \( F \) is \( G \)-increasing.

Let us prove that for any \( x, y \in X \), there exist \( u, v \in X \) such that

\[
\begin{cases} 
  F(x, y) = G(u, v) \\
  F(y, x) = G(v, u)
\end{cases}.
\]

Let \( (x, y) \in X \times X \) be fixed. We consider the following cases:

**Case-1:** \( x = y \).
In this case, \( F(x, y) = 0 = G(x, y) \) and \( F(y, x) = 0 = G(y, x) \).

**Case-2:** \( x > y \).
In this case, we have

\[
F(x, y) = \frac{x - y}{3} = G(x/3, y/3) \quad \text{and} \quad F(y, x) = 0 = G(y/3, x/3).
\]

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Case-3: $x < y$.

In this case, we have

$$F(x, y) = 0 = G(x/3, y/3) \quad \text{and} \quad F(y, x) = \frac{y - x}{3} = G(y/3, x/3).$$

$G$ is continuous and has the mixed monotone property.

Clearly $G$ is continuous. Let $(x, y) \in X \times X$ be fixed. Suppose that $x_1, x_2 \in X$ are such that $x_1 < x_2$. We distinguish the following cases.

Case-1: $x_1 < y$.

In this case, we have

$$G(x_1, y) = 0 \leq G(x_2, y).$$

Case-2: $x_2 > x_1 \geq y$.

In this case, we have

$$G(x_1, y) = x_1 - y \leq x_2 - y = G(x_2, y).$$

Similarly, we can show that if $y_1, y_2 \in X$ are such that $y_1 < y_2$, then $G(x, y_1) \geq G(x, y_2)$.

Now, we prove that the pair $\{F, G\}$ satisfies the generalized compatibility hypothesis.

Let $(x_n)$ and $(y_n)$ be two sequences in $X$ such that

$$t_1 = \lim_{n \to +\infty} G(x_n, y_n) = \lim_{n \to +\infty} F(x_n, y_n)$$

and

$$t_2 = \lim_{n \to +\infty} G(y_n, x_n) = \lim_{n \to +\infty} F(y_n, x_n).$$

Then obviously, $t_1 = t_2 = 0$. It follows easily that

$$\lim_{n \to +\infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) = 0$$

and

$$\lim_{n \to +\infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) = 0.$$

There exists $(x_0, y_0) \in X \times X$ such that $G(x_0, y_0) \leq F(x_0, y_0)$ and $G(y_0, x_0) \geq F(y_0, x_0)$.

We have

$$G(0, 1/2) = 0 = F(0, 1/2) \quad \text{and} \quad G(1/2, 0) = 1/2 \geq 1/6 = F(1/2, 0).$$

Now, let $\varphi : [0, +\infty) \to [0, +\infty)$ be defined as

$$\varphi(t) = \frac{2t}{3} \text{ for all } t \geq 0.$$

Clearly $\varphi \in \Phi$. Let us prove that inequality (3.1) is satisfied for all $x, y, u, v \in X$, with $G(x, y) \leq G(u, v)$ and $G(v, u) \leq G(y, x)$.
Let $x, y, u, v \in X$, with $G(x, y) \preceq G(u, v)$ and $G(v, u) \preceq G(y, x)$. We have
\[
d(F(x, y), F(u, v)) = |F(x, y) - F(u, v)| \\
= \frac{1}{3}|G(x, y) - G(u, v)| \\
= \frac{2}{3} \left( \frac{|G(x, y) - G(u, v)|}{2} \right) \\
\leq \frac{2}{3} \left( \frac{|G(x, y) - G(u, v)| + |G(y, x) - G(v, u)|}{2} \right) \\
= \varphi \left( d(G(x, y), G(u, v)) + d(G(y, x), G(v, u)) \right).
\]

Then, inequality (3.1) is satisfied.

Now, all the required hypotheses of Theorem 19 are satisfied. Thus we deduce the existence of a coupled coincidence point of $F$ and $G$. Here, $(0, 0)$ is a coupled coincidence point of $F$ and $G$.

References


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