TOEPLITZ OPERATORS AND MULTIPLICATION OPERATORS IN THE COMMUTANT OF A COMPOSITION OPERATOR ON WEIGHTED BERGMAN SPACES

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Abstract. Let \( \varphi \) be an analytic self-map of \( D \). We investigate which Toeplitz operators and multiplication operators commute with a given composition operator \( C_\varphi \) on \( A^p_\alpha(D) \) for \( 1 < p < \infty \) and \(-1 < \alpha < \infty\). Let \( S \) be a bounded linear operator in the commutant of \( C_\varphi \). We show that under a certain condition on \( S \), \( S \) is a polynomial in \( C_\varphi \).

1 Introduction

Let \( D \) denote the open unit disc in the complex plane and let \( dA \) be the normalized area measure on \( D \). For \( 0 < p < \infty \) and \(-1 < \alpha < \infty\), the weighted Bergman space \( A^p_\alpha(D) = A^p_\alpha \) is the space of analytic functions in \( L^p(D,dA_\alpha) \), where

\[
dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).
\]

If \( f \) is in \( L^p(D,dA_\alpha) \), we note that

\[
\|f\|_{p,\alpha} = \left( \int_D |f(z)|^p dA_\alpha(z) \right)^{\frac{1}{p}}.
\]

When \( 1 \leq p < \infty \), the space \( L^p(D,dA_\alpha) \) is a Banach space and the weighted Bergman space \( A^p_\alpha \) is closed in \( L^p(D,dA_\alpha) \). So \( A^p_\alpha \) is a Banach space. Let \( L^\infty(D) \) denote the space of essentially bounded functions on \( D \). For \( f \in L^\infty(D) \), we define

\[
\|f\|_\infty = \text{esssup}\{|f(z)| : z \in D\}.
\]

The space \( L^\infty(D) \) is a Banach space with the above norm. As usual, let \( H^\infty(D) = H^\infty \) denote the space of bounded analytic functions on \( D \). It is clear that \( H^\infty \) is closed in \( L^\infty(D) \) and hence is a Banach space.

2010 Mathematics Subject Classification: 47B33; 47B38.

Keywords: Toeplitz operator; Weighted Bergman spaces; Composition operator; Commutant; Multiplication operators.

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Let $\varphi$ be an analytic self-map of the unit disc, $1 < p < \infty$ and $-1 < \alpha < \infty$. The composition operator $C_\varphi$ on $A_p^\alpha$, is defined by the rule $C_\varphi(f) = f \circ \varphi$. Every composition operator $C_\varphi$ on $A_p^\alpha$ is bounded (see, e.g., [9]).

Let for each $1 < p < \infty$, $P_\alpha : L^p(D, dA_\alpha) \to A_p^\alpha$ be the Bergman projection. We note that $P_\alpha$ is an integral operator represented by

$$P_\alpha g(z) = \int_D K(z, w) g(w) dA_\alpha(w),$$

where

$$K(z, w) = \frac{1}{(1 - zw)^{2+\alpha}} = \sum_{n=0}^\infty \frac{\Gamma(n + 2 + \alpha)}{n! \Gamma(2 + \alpha)} (zw)^n.$$

For each $f \in L^\infty(D)$ and $1 < p < \infty$, we define the Toeplitz operator $T_f$ on $A_p^\alpha$ with symbol $f$ by $T_f(g) = P_\alpha(fg)$. If we define $M_f : L^p(D, dA_\alpha) \to L^p(D, dA_\alpha)$ by $M_f(g) = fg$, it is obvious that $M_f$ is bounded. Since the Bergman projection is bounded (see, e.g., [8]), we conclude that $T_f$ is a bounded operator.

If $f$ is a bounded complex valued harmonic function defined on $D$, then there are holomorphic functions $f_1$ and $f_2$ such that $f = f_1 + f_2$. This decomposition is unique if we require $f_2(0) = 0$. Of course $f_1$ and $f_2$ are not necessarily bounded, but they are certainly Bloch functions and they are in $A_p^\alpha$ for $1 \leq p \leq \infty$ (see, e.g., [1]). Throughout this paper, we write $\varphi^{[j]}$ to denote the $j$th iterate of $\varphi$, that is, $\varphi^{[0]}$ is the identity map on $D$ and $\varphi^{[j+1]} = \varphi \circ \varphi^{[j]}$.

Suppose that $\varphi$ is an analytic self-map of $D$ which is not the identity and not an elliptic disc automorphism. Then there is a point $a$ in $\overline{D}$ such that iterates of $\varphi$ converges to $a$ uniformly on compact subsets of $D$. We note that for each fixed positive integer $l$, $\{(\varphi^{[n]})^l\}$ converges weakly to $a^l$ as $n \to \infty$ (see, e.g., [6]). For each $1 < p < \infty$ and $w$ in $D$, let $\lambda_w$ be the point evaluation function at $w$, that is, $\lambda_w(g) = g(w)$, where $g \in A_p^\alpha$. It is well-known that point evaluations at the points of $D$ are all continuous on $A_p^\alpha$ (see, e.g., [8]).

Given a fixed operator $A$, we say that an operator $B$ commutes with $A$ if $AB = BA$. The set of all operators which commute with a fixed operator $A$ is called the commutant of $A$. The commutant of a particular operator is known in a few cases. For further information about commutant of a composition operator, see [2], [3] and [7]. Also in [5], Carl Cowen showed that if $f$ is a covering map of $D$ onto a bounded domain in the complex plane, then the commutant of the Toeplitz operator $T_f$ is generated by composition operators induced by linear fractional transformation $\varphi$.
that satisfy \( f \circ \varphi = f \) and by Toeplitz operators. Also in [4], Bruce Clod determined which Toeplitz operators are in the commutant of a given composition operator \( C_\varphi \) on \( H^2 \).

In this paper, under certain conditions on \( \varphi \) we investigate which Toeplitz operators and Multiplication operators commute with \( C_\varphi \) on \( A^p_\alpha \) for \( 1 < p < \infty \).

## 2 Toeplitz operators in the commutant of a composition operator

Throughout this section, \( C_\varphi \) denotes a bounded composition operator on \( A^p_\alpha \) for \( 1 < p < \infty \) and \(-1 < \alpha < \infty \). Our goal is to find information about the commutant of \( C_\varphi \).

**Theorem 1.** Let \( f \) be a harmonic function in \( L^\infty(\mathbb{D}) \), and let \( \varphi \) be an analytic self-map of \( \mathbb{D} \) which is neither an elliptic disc automorphism of finite periodicity nor the identity mapping. If \( C_\varphi T_f = T_f C_\varphi \), then \( f \) is an analytic function.

**Proof.** Let \( f = f_1 + \overline{f_2} \) such that \( f_1 \) and \( f_2 \) belong to \( A^p_\alpha \). \( f_2(0) = 0 \), \( f_1(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( f_2(z) = \sum_{n=1}^{\infty} b_n z^n \). Since \( \varphi \) is an analytic map which is not an elliptic disc automorphism of finite periodicity, \( \varphi \) is a constant function or \( \varphi \) is an elliptic automorphism of infinite periodicity or \( \varphi \) is neither an elliptic disc automorphism nor a constant.

**Case(1):** Let \( \varphi \) be a constant. Then \( \varphi(z) = b \) for all \( z \in \mathbb{D} \), where \( |b| < 1 \). Since \( T_f C_\varphi(1) = C_\varphi T_f(1) \), we have \( f_1(z) = f_1(b) \). Thus \( f_1 \) is a constant, let \( f_1 = c \). For every \( g \) in \( A^p_\alpha \), \( T_f C_\varphi(g) = C_\varphi T_f(g) \) which implies that

\[
 cg(b) = P(\overline{f_2}g)(b) + cg(b).
\]

So \( P(\overline{f_2}g)(b) = 0 \). In particular, if \( g(z) = z^k \), then \( b_k = 0 \) for all \( k \in \mathbb{N} \). Hence \( f = f_1 = c \) is analytic.

**Case(2):** Suppose that \( \varphi \) is an elliptic disc automorphism of infinite periodicity. If \( \varphi(0) = 0 \), then Schwarz’s Lemma implies that \( \varphi(z) = e^{i\theta}z \), where \( e^{i\theta} \neq 1 \) for all integers \( n \neq 0 \). Since \( C_\varphi T_f(1) = T_f C_\varphi(1) \), we have \( f_1(e^{i\theta}z) = f_1(z) \) and so \( f_1 = a_0 \). Now by induction, we show that \( f_2 = 0 \). Since \( T_f C_\varphi(z) = C_\varphi T_f(z) \), we have \( \overline{b_1} = e^{i\theta} \overline{b_1} \), so \( b_1 = 0 \). Let \( b_1 = b_2 = \cdots = b_{l-1} = 0 \). We show that \( b_l = 0 \). Since \( C_\varphi T_f(z_l) = T_f C_\varphi(z_l) \), we have \( \overline{b_l} = e^{i\theta} \overline{b_l} \) and so \( b_l = 0 \). Hence \( f \) must be a constant function.

Now let \( b \neq 0 \) be the fixed point of \( \varphi \). Since \( T_f C_\varphi(1) = C_\varphi T_f(1) \), we have \( f_1 = f_1 \circ \varphi \). Since \( \varphi \) has infinite periodicity, we conclude that \( f_1 \) is a constant. Hence \( f_2 \) induces a Toeplitz operator which commutes with \( C_\varphi \). We claim that

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Surveys in Mathematics and its Applications 9 (2014), 139 – 147

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\[ f_2 = 0. \] Let \( \alpha(z) = \frac{b - z}{1 - \overline{b}z} \), note that \( \alpha^{-1} = \alpha \). Since \( T_{f_2} \) commutes with \( C_\varphi \), \( A = C_\alpha T_{f_2} C_\alpha \) commutes with \( C_\alpha C_\varphi C_\alpha = C_{\alpha \varphi \alpha} \). The function \( \alpha \circ \varphi \circ \alpha \) is an elliptic disc automorphism of infinite periodicity with fixed point 0. Thus there exists \( \{ \lambda_n \}_{n=1}^\infty \) such that \( A(z^n) = \lambda_n z^n \) and \( T_{f_2} = C_\alpha AC_\alpha \) (If \( C_\varphi T = TC_\varphi \) and \( \varphi(z) = e^{i\theta}z \), then there exists \( \{ \lambda_n \}_{n=1}^\infty \) such that \( T(z^n) = \lambda_n z^n \)). Set \( g = A(\alpha) \), we have

\[ g(z) = \lambda_0 b + \sum_{k=1}^{\infty} \lambda_k (\overline{b})^k - 1 (|b|^2 - 1) z^k. \]

Since \( T_{f_2} (z) = \frac{z - 1}{z + 1} \), we see that \( g \circ \alpha \) is a constant. Hence \( g \) is a constant which implies that \( \lambda_k = 0 \) for \( k \geq 1 \). On the other hand, \( \lambda_0 = 0 \). Thus \( A = 0 \) and hence \( f_2 = 0 \).

Case(3): Let \( \varphi \) be neither an elliptic disc automorphism nor a constant. Suppose that \( a \) is the Denjoy-Wolff point of \( \varphi \). Since \( T_f C_\varphi = C_\varphi T_f \), we have

\[ T_f C_{\varphi[n]}(z) = C_{\varphi[n]} T_f(z). \]

Therefore

\[ C_{\varphi[n]} T_f(z) = C_{\varphi[n]} P(z f_1 + z T_{f_2}) = \left( \frac{2}{2 + \alpha} \overline{b_1} + z f_1 \right) \circ \varphi^n, \]

and \( T_f C_\varphi(1) = C_\varphi T_f(1) \) which implies that \( f_1 \circ \varphi = f_1 \). Hence

\[ T_f C_{\varphi[n]}(z) = \frac{2}{2 + \alpha} \overline{b_1} + f_1 \varphi^n. \]

Now if we apply \( \lambda_0 \) on \( T_f C_{\varphi[n]} \), then we obtain

\[ \lambda_0(T_f C_{\varphi[n]}(z)) = \frac{2}{2 + \alpha} \overline{b_1} + a_0 \varphi^n(0). \]

Hence \( \{ \lambda_0(T_f C_{\varphi[n]}) \} \) converges to \( \frac{2}{2 + \alpha} \overline{b_1} + a_0 a \) as \( n \to \infty \). Since \( \{ \varphi^n \} \) converges weakly to \( a \) as \( n \to \infty \), \( \{ T_f(\varphi^n) \} \) converges weakly to \( T_f(a) = af_1 \) as \( n \to \infty \). So \( \{ \lambda_0(T_f C_{\varphi[n]}) \} \) converges to \( a_0 a \) as \( n \to \infty \). Thus \( b_1 = 0 \).

Now let \( b_1 = b_2 = \cdots = b_{l-1} = 0 \). Consider \( T_f(z^l) \) in the above argument, we have

\[ T_f((\varphi^n)^l) = \frac{\Gamma(l + 1)\Gamma(\alpha + 2)}{\Gamma(l + 2 + \alpha)} \overline{b_1} + f_1 (\varphi^n)^l. \]

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Surveys in Mathematics and its Applications 9 (2014), 139 – 147
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By applying $\lambda_0$ on $T_f((\varphi_1[n])^l)$ and since \( \{T_f((\varphi_1[n])^l)\} \) converges weakly to $T_f(a^l)$ as $n \to \infty$, we get
\[
a^l a_0 = \frac{\Gamma(l + 1)\Gamma(\alpha + 2)}{\Gamma(l + 2 + \alpha)} b_l + a^l a_0.
\]
Thus $b_l = 0$. Hence by the strong induction, $b_n = 0$ for all $n \geq 1$, that is, $f$ is analytic.

**Remark 2.** If $\varphi(z) = \frac{1}{z}$, then $\varphi$ is loxodromic and $\varphi$ is not an elliptic disc automorphism. Also let $f(z) = |z|^2$, we have $f$ is bounded and $f$ is not a harmonic function. Since for every $n \in \mathbb{N}$,
\[
T_f C_\varphi(z^n) = C_\varphi T_f(z^n) = \frac{n + 1}{2^n(n + 2 + \alpha)} z^n,
\]
we have $C_\varphi T_f = T_f C_\varphi$ and $f$ is not analytic. This example shows that Theorem 1 is not true in general without $f$ being harmonic.

The following theorem shows that Theorem 1 is not true for all elliptic disc automorphisms.

**Theorem 3.** Let $f$ be a harmonic function in $L^\infty(\mathbb{D})$, and let $\varphi$ be an elliptic disc automorphism of period $q$, where $q \geq 2$ with $\varphi(0) = 0$. Then $T_f C_\varphi = C_\varphi T_f$ if and only if $f(z) = \sum_{n=0}^\infty a_n z^n + \sum_{n=1}^\infty b_n z^n$.

**Proof.** By hypothesis, \( \varphi(z) = e^{i\theta}z \) with $\theta = 2\pi q$, where $p$ is an integer, $q$ is a natural number and g.c.d($p,q) = 1$. Let $f = f_1 + f_2$ such that $f_1$ and $f_2$ belong to $A^p_\infty$, $f_2(0) = 0$, $f_1(z) = \sum_{n=0}^\infty a_n z^n$ and $f_2(z) = \sum_{n=1}^\infty b_n z^n$. Since $T_f C_\varphi(1) = C_\varphi 1 = 1$, we have
\[
f_1(z) = f_1(e^{2\pi i p q} z) = \sum_{n=0}^\infty a_n z^n = \sum_{n=0}^\infty a_n (e^{2\pi i p q} z)^n.
\]
So if $q \nmid n$, $a_n = 0$. Hence $f_1(z) = \sum_{n=0}^\infty a_n z^n$. Since $T_f C_\varphi(1) = C_\varphi 1$, we have
\[
2 + \alpha \sum_{n=0}^\infty a_n z^n + \sum_{n=1}^\infty b_n z^n f_1(z) = z e^{2\pi i p q} f_1(z) + \frac{2}{2 + \alpha} b_1.
\]
Therefore $b_1 = 0$. For $n$ such that $q \nmid n$ assume by induction that if $m < n$ and $q \nmid m$, then $b_m = 0$. Since
\[
T_f C_\varphi(e^{2\pi i p q} z^n) = C_\varphi e^{2\pi i p q} T_f(z^n),
\]
by a similar argument, we can prove that $b_n = 0$ which we omit the details.

Conversely, if $f(z) = \sum_{n=0}^\infty a_n z^n + \sum_{n=1}^\infty b_n z^n$, then by straightforward calculation $T_f$ commutes with $C_\varphi$. □
In Theorems 1 and 3 we have shown that except for elliptic disc automorphisms of finite periodicity, the Toeplitz operators which commute with $C_\varphi$ must be analytic, that is, symbol of the Toeplitz operator must be analytic. Now let $f$ be in $H^\infty$. Then $T_f = M_f$ and in this case $M_f$ commutes with $C_\varphi$ is equivalent to $f \circ \varphi = f$. We will determine which multiplication operators commute with $C_\varphi$ for certain composition operator $C_\varphi$.

**Lemma 4.** Let $f$ be in $H^\infty$, and let $\alpha$ be a disc automorphism. Then $C_\alpha M_f C_{\alpha^{-1}} = M_{f \circ \alpha}$.

**Proof.** Let $g$ be in $A_\alpha^\infty$. Then

$$C_\alpha M_f C_{\alpha^{-1}}(g) = C_\alpha M_f (g \circ \alpha^{-1})$$

$$= C_\alpha (g \circ \alpha^{-1} \cdot f)$$

$$= (g \circ \alpha^{-1} \cdot f) \circ \alpha$$

$$= g \cdot f \circ \alpha$$

$$= M_{f \circ \alpha}(g).$$

\[ \Box \]

**Proposition 5.** Let $\varphi$ be an elliptic disc automorphism with fixed point $b$, and let $f \in H^\infty$. Then

(a) If $\varphi$ is of infinite periodicity, then the multiplication operator $M_f$ commutes with $C_\varphi$ if and only if $f$ is a constant.

(b) If $\varphi$ is of period $q$, then $M_f$ commutes with $C_\varphi$ if and only if $f$ is of the form $f(z) = \sum_{n=0}^{\infty} a_n q (\frac{b - z}{1 - bz})^n$.

**Proof.** (a) The proof follows from Theorem 1 case (2).

(b) If $f \in H^\infty$ and $\alpha(z) = \frac{b - z}{1 - bz}$, then $\alpha \circ \varphi \circ \alpha$ is an elliptic disc automorphism of period $q$, with fixed point 0 and we have $M_f$ commutes with $C_\varphi$ if and only if $C_\alpha M_f C_\alpha$ commutes with $C_\alpha C_\varphi C_\alpha = C_{\alpha \circ \varphi \circ \alpha}$ if and only if (by Lemma 4) $M_{f \circ \alpha}$ commutes with $C_{\alpha \circ \varphi \circ \alpha}$ if and only if (by Theorem 3) $f \circ \alpha(z) = \sum_{n=0}^{\infty} a_n z^n q$ if and only if $f(z) = \sum_{n=0}^{\infty} a_n q (\frac{b - z}{1 - bz})^n$.

\[ \Box \]

**Proposition 6.** Let $\varphi$ be a self-map of $\mathbb{D}$, and let $f \in H^\infty$. Also suppose that $\varphi$ is neither an elliptic disc automorphism nor the identity mapping, and $\varphi$ has an interior fixed point. If $M_f$ commutes with $C_\varphi$, then $f$ is a constant.

**Proof.** Let $a \in \mathbb{D}$ and $\varphi(a) = a$. Since $f \circ \varphi = f$, we have $f(\varphi^n(z)) = f(z)$ for each $z \in \mathbb{D}$ and all $n \in \mathbb{N}$. From this, we have $f(z) = f(a)$ for all $z \in \mathbb{D}$, because $\{\varphi^n(z)\}$ converges to $a$ as $n \to \infty$ for every $z \in \mathbb{D}$.

\[ \Box \]
3 Some properties of the commutant of composition operators on weighted Bergman spaces

In this section, we consider the commutant of composition operator $C_\varphi$ on $A^p_\alpha$ for $1 < p < \infty$ and $-1 < \alpha < \infty$, where $\varphi$ is an analytic self-map of $\mathbb{D}$ which is neither an elliptic disc automorphism nor the identity and a constant. Also we assume that $\varphi(a) = a$ for some $a \in \mathbb{D}$.

Lemma 7. There exists a point $z_0$ in $\mathbb{D}$ such that the iterates of $\varphi$ at $z_0$ are distinct.

Proof. See [10].

Lemma 8. Let $z_0$ satisfy the properties of Lemma 7. Then the linear span of reproducing kernels, $\{K_{\varphi^{[n]}(z_0)} : n \geq 0\}$ is dense in $A^p_\alpha$ for $1 < p < \infty$.

Proof. Let $A$ be the linear span of $\{K_{\varphi^{[n]}(z_0)} : n \geq 0\}$. Suppose that $x^*$ is a bounded linear function on $A^p_\alpha$ for $1 < p < \infty$. If $\frac{1}{p} + \frac{1}{q} = 1$, then there is $g \in A^q_\alpha$ such that $x^* = Fg$ and $Fg$ define by

$$Fg(f) = \int_\mathbb{D} f(z) \overline{g(z)} dA(z)$$

for each $f \in A^p_\alpha$ (see, e.g., [8]). Hence

$$A^\perp = \{Fg : Fg(K_{\varphi^{[n]}(z_0)}) = 0 \ (\forall n)\} = \{Fg : g(\varphi^{[n]}(z_0)) = 0 \ (\forall n)\}.$$

By the Denjoy-Wolff Theorem, the sequence $\{\varphi^{[n]}(z_0)\}_{n=0}^\infty$ has a limit point in $\mathbb{D}$. Then $A^\perp = \{0\}$ and $A^\perp = A = A^p_\alpha$, so the proof is complete.

Proposition 9. $C_\varphi^*$ is cyclic.

Proof. Since $C_\varphi^*(K_{\varphi^{[n]}(z_0)}) = K_{\varphi^{[n+1]}(z_0)}$, by Lemmas 7 and 8, the proof is complete.

Remark 10. If the Denjoy-Wolff point of $\varphi$ is in the boundary of $\mathbb{D}$, then Lemma 8 is not true in general. For example, if $\varphi(z) = az + b$, where $a, b \neq 0$ and $|a| + |b| = 1$, then the sequence $\{\varphi^{[n]}(0)\}_{n=0}^\infty$ has distinct elements and each Blaschke product with zeros $\{\varphi^{[n]}(0)\}_{n=0}^\infty$ is in $A^\perp$. So $A$ is not dense in $A^p_\alpha$.

By Lemma 8, we can answer to some questions about the commutant of $C_\varphi$.

Theorem 11. Let $S$ be a bounded operator such that $SC_\varphi = C_\varphi S$ and $S^*K_{z_0} = \sum_{j=0}^m a_j K_{\varphi^{[j]}(z_0)}$ for some $z_0$ in $\mathbb{D}$ for which $\{\varphi^{[n]}(z_0)\}_{n=0}^\infty$ are distinct. Then $S$ is a polynomial in $C_\varphi$. 

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Surveys in Mathematics and its Applications 9 (2014), 139 – 147
http://www.utgjiu.ro/math/sma
Proof. Let \( p(z) = \sum_{j=0}^{m} a_j z^j \), we show that \( p(C^*_\varphi) = S^* \). By an easy computation, we have \( p(C^*_\varphi)K_{z_0} = S^*K_{z_0} \). Let \( \varepsilon > 0 \) and \( f \in A_\alpha^p \). Since the linear span of \( \{K_{\varphi^n(z_0)} : n \geq 0\} \) is dense in \( A_\alpha^p \), there is \( g = \sum_{k=0}^{n} g_k K_{\varphi^k(z_0)} \) such that
\[
\|f - g\|_{p,\alpha} < \varepsilon/(1 + \|p(C^*_\varphi) - S^*\|).
\]
Since \( C^*_\varphi K_{z_0} = K_{\varphi^k(z_0)} \), we have
\[
\|p(C^*_\varphi) - S^*\|f\|_{p,\alpha} \leq \|p(C^*_\varphi) - S^*\|(f - g)\|_{p,\alpha} + \|p(C^*_\varphi) - S^*\|g\|_{p,\alpha} \leq \varepsilon + \sum_{k=0}^{n} g_k C^*_\varphi K_{z_0}\|p(C^*_\varphi) - S^*\|_{p,\alpha} \leq \varepsilon.
\]
Hence \( p(C^*_\varphi) = S^* \) and so the proof is complete. \( \square \)

Corollary 12. Let iterates of \( \varphi \) at zero be distinct, and let \( S \) be a bounded operator such that \( SC_\varphi = C_\varphi S \) and \( S^*(1) = \lambda I \). Then \( S \) is a multiple of the identity.

Proof. Since \( K_0 = I \), by Theorem 11, we have \( S^* = \lambda I \). \( \square \)

Theorem 13. Let \( S \) be a bounded operator such that \( SC_\varphi = C_\varphi S \). Then there is a dense subset on which \( S \) can be approximated by polynomials in \( C_\varphi \).

Proof. Assume \( \varphi \) and \( z_0 \) are as in the Lemma 7 and \( S^*K_{z_0} = f \). Since the linear span of \( \{K_{\varphi^n(z_0)} : n \geq 0\} \) is dense in \( A_\alpha^p \), there exists \( f_j = \sum_{k=0}^{m} a_{j,k} K_{\varphi^k(z_0)} \) such that \( \|f - f_j\|_{p,\alpha} \to 0 \) as \( j \to \infty \). If \( p_j = \sum_{k=0}^{m} a_{j,k} z^k \), then we show that \( p_j(C^*_\varphi) \) approximate \( S^* \) on the linear span of \( \{K_{\varphi^n(z_0)} : n \geq 0\} \). Let \( g = \sum_{n=0}^{m} g_n K_{\varphi^n(z_0)} \). Since \( C^*_\varphi K_{z_0} = K_{\varphi^n(z_0)} \) and \( S^*C^*_\varphi = C^*_\varphi S \), by an easy computation, we have
\[
S^*g = \sum_{n=0}^{m} g_n C^*_\varphi(z_0)\phi_f
\]
and
\[
p_j(C^*_\varphi)g = \sum_{k=0}^{m} \sum_{n=0}^{m} a_{j,k} g_n K_{\varphi^{k+n}(z_0)} = \sum_{n=0}^{m} g_n C^*_\varphi(z_0)\phi_f.
\]
Since \( \{\varphi^n(0)\} \) converges to the Denjoy-Wolff point in the disc as \( n \to \infty \), by using similar arguments as the proof of [9, Theorem 2.3], we have
\[
\|C^*_\varphi\| \leq \left(\frac{1 + |\varphi^n(0)|}{1 - |\varphi^n(0)|}\right)^{\frac{2+\alpha}{p}} \leq b,
\]
where \( b \) is independent of \( n \) on \( A_\alpha^p \) and so we have
\[
\|S^* - p_j(C^*_\varphi)\|_{p,\alpha} \leq \sum_{n=0}^{m} g_n C^*_\varphi(z_0)(f - f_j)\|_{p,\alpha} \leq b\|f - f_j\|_{p,\alpha} \sum_{n=0}^{m} |g_n|,
\]
which converges to zero as $j \to 0$.

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