ON SECOND HANKEL DETERMINANT FOR TWO NEW SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. In this paper, we obtain sharp upper bounds for the function \(|a_2a_4 - a_3^2|\) for functions belonging to \(S^*(\alpha, \beta)\) and \(C(\alpha, \beta)\). Our results extend corresponding previously known results.

1 Introduction

Let \(S\) denote the class of normalized analytic univalent functions \(f(z)\) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]  

(1.1)

where \(z \in E : \{z : |z| < 1\}\).

In 1976, Noonan and Thomas [9] defined the \(q^{th}\) Hankel determinant for \(q \geq 1\) and \(n \geq 0\) by

\[
H_q(n) = \begin{vmatrix}
 a_n & a_{n+1} & \cdots & a_{n+q-1} \\
 a_{n+1} & \cdots \\
 \vdots & \ddots \\
 a_{n+q-1} & \cdots & a_{n+2q-2}
\end{vmatrix}
\]

This determinant has also been considered by several authors. For example, Noor in [10], determined the rate of growth of \(H_q(n)\) as \(n \rightarrow \infty\) for functions of the form (1.1) with bounded boundary. In particular, sharp bounds on \(H_2(2)\) were obtained by the authors of articles [1], [3], [5], [6], [12] for different classes of functions.

One can observe that the Fekete-Szego functional is \(H_2(1)\). Also they generalized the estimate \(|a_3 - \mu a_2^2|\), where \(\mu\) is real and \(f(z) \in S\).

2010 Mathematics Subject Classification: Primary 30C80; Secondary 30C45.

Keywords: Coefficient bounds; Fekete-Szego functional; Hankel determinant.

This work was supported by UGC, under the grant F.MRP-4117/12 (MRP/UGC-SERO) of the second author.

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In this paper, we consider the second Hankel determinant for $q = 2$ and $n = 2$, 
$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$ and obtain an upper bound for the functional $|a_2a_4 - a_3^2|$ for functions belonging to the classes $S^*(\alpha, \beta)$ and $C(\alpha, \beta)$ which are defined as follows:

**Definition 1.** Let $f(z)$ be given by (1.1). Then $f(z) \in S^*(\alpha, \beta)$ if and only if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} + \alpha \frac{zf''(z)}{f'(z)} \right\} > \beta, \ z \in E \text{ for some } \beta \ (0 \leq \beta < 1) \text{ and } \alpha \geq 0.$$ 

**Remark 2.** The choice $\alpha = 0$ yields $\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \ z \in E$, so that we get $S^*(0, \beta)$, the class of starlike functions of order $\beta$ [11].

**Remark 3.** When $\alpha = 0$, $\beta = 0$, we get the class $S^*$, the class of starlike functions [11].

**Remark 4.** When $\beta = 0$, we get the corresponding result of Shanmugam [13].

**Definition 5.** Let $f(z)$ be given by (1.1). Then $f(z) \in C(\alpha, \beta)$ if and only if

$$\text{Re} \left\{ \frac{zf'(z) + \alpha zf''(z)}{f'(z)} \right\} > \beta, \ z \in E, \text{ for some } \beta \ (0 \leq \beta < 1) \text{ and } \alpha \geq 0.$$ 

**Remark 6.** The choice $\alpha = 0$ yields $\text{Re} \left\{ \frac{1+zf'(z)}{f'(z)} \right\} > \beta, \ z \in E$, so that we get $C(0, \beta)$, the class of convex functions of order $\beta$ [11].

**Remark 7.** When $\alpha = 0$, $\beta = 0$, we get the class $C^*$, the class of convex functions [11].

**Remark 8.** When $\beta = 0$, we get the corresponding result of Shanmugam [13].

## 2 Preliminary Results

Let $P$ be the family of all functions $p(z)$ analytic in $E$ for which $\text{Re}\{p(z)\} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

(2.1)

for $z \in E$.

To prove the main results we shall need the following lemmas. Throughout this paper, we assume that $p(z)$ is given by (2.1) and $f(z)$ is given by (1.1).

**Lemma 9.** [2] If $p(z) \in P$, then $|c_k| \leq 2$ for each $k \in N$.

**Lemma 10.** ([7, 8]) Let $p(z) \in P$, then

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

(2.2)

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y$$

(2.3)

for some value of $x, y$ such that $|x| \leq 1$ and $|y| \leq 1$. 

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Theorem 11. [4] Let \( f(z) \in S^* \). Then
\[
|a_2a_4 - a_3^2| \leq 1.
\]
The result obtained is sharp.

Theorem 12. [4] Let \( f(z) \in C \). Then
\[
|a_2a_4 - a_3^2| \leq \frac{1}{8}.
\]
The result obtained is sharp.

3 Main Results

Theorem 13. Let \( f(z) \in S^*(\alpha, \beta) \), then
\[
|a_2a_4 - a_3^2| \leq \frac{(1 - \beta)^2}{(1 + 3\alpha)^2}.
\]
The result obtained is sharp.

Proof. Let \( f(z) \in S^*(\alpha, \beta) \). Then there exists a \( p(z) \in P \), such that
\[
zf'(z) + \alpha z^2 f''(z) = f(z)[(1 - \beta)p(z) + \beta]
\]
for some \( z \in E \).

Equating the coefficients in (3.1), we get
\[
a_2 = \frac{c_1(1 - \beta)}{1 + 2\alpha},
a_3 = \frac{c_2(1 - \beta)}{2(1 + 3\alpha)} + \frac{c_2^2(1 - \beta)^2}{2(1 + 2\alpha)(1 + 3\alpha)},
a_4 = \frac{c_3(1 - \beta)}{3(1 + 4\alpha)} + \frac{c_1c_2(1 - \beta)^2(3 + 8\alpha)}{6(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)} + \frac{c_3^2(1 - \beta)^3}{6(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)}. \tag{3.2}
\]

From (3.2), it is easily established that
\[
|a_2a_4 - a_3^2| = \left| -\frac{c_1c_3(1 - \beta)^2}{3(1 + 2\alpha)(1 + 4\alpha)} - \frac{c_2^2(1 - \beta)^2}{4(1 + 3\alpha)^2} - \frac{c_1^2(1 - \beta)^4(1 + 6\alpha)}{12(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)^2} - \frac{\alpha c_1^2c_2(1 - \beta)^3}{6(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} \right| \tag{3.3}
\]
Substituting for $c_2$ and $c_3$ from (2.2) and (2.3) and since $|c_1| \leq 2$, by Lemma 9, let $c_1 = c$ and assume without restriction that $c \in [0,2]$. We obtain

$$|a_{2a} - a_3^2| = \left| \frac{(1 - \beta)^2[c^4 + 2(4 - c^2)c^2x - (4 - c^2)c^2x^2 + 2c(4 - c^2)(1 - c^2)y]}{12(1 + 2\alpha)(1 + 4\alpha)} - \frac{(1 - \beta)^2c^4 + (4 - c^2)^2x^2 + 2c^2x(4 - c^2)}{16(1 + 3\alpha)^2} \right|$$

$$- \frac{(1 - \beta)^4c^2(1 + 6\alpha)}{12(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} - \frac{(1 - \beta)^2c^4 + (4 - c^2)c^2}{12(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} \right| \quad (3.4)$$

By triangle inequality,

$$|a_{2a} - a_3^2| \leq \frac{(1 - \beta)^2c^4 + 2(4 - c^2)c^2\rho + 2c(4 - c^2) + c(c - 2)(4 - c^2)\rho^2}{12(1 + 2\alpha)(1 + 4\alpha)} + \frac{(1 - \beta)^2c^4 + (4 - c^2)^2\rho^2 + 2c\rho(4 - c^2)}{16(1 + 3\alpha)^2}$$

$$+ \frac{(1 - \beta)^4c^2(1 + 6\alpha)}{12(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} + \frac{(1 - \beta)^3c^4 + c^2\rho(4 - c^2)}{12(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} \right| \quad (3.5)$$

with $\rho = |x| \leq 1$. Furthermore

$$F'(\rho) = \frac{(1 - \beta)^2[2c^2(4 - c^2) + 2c\rho(c - 2)(4 - c^2)]}{12(1 + 2\alpha)(1 + 4\alpha)} + \frac{(1 - \beta)^2[2c(4 - c^2)^2\rho + 2c(4 - c^2)]}{16(1 + 3\alpha)^2}$$

$$+ \frac{(1 - \beta)^3c^2(4 - c^2)}{12(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} \right|$$

and with elementary calculus, we can show that $F'(\rho) > 0$ for $\rho > 0$.

This implies that $F$ is an increasing function and thus the upper bound for (3.4) corresponds to $\rho = 1$ and $c = 0$ gives

$$|a_{2a} - a_3^2| \leq \frac{(1 - \beta)^2}{(1 + 3\alpha)^2}.$$}

It follows from (2.3) that if $c_1 = c = 0$ and $|x| = \rho = 1$ then $c_3 = 0$.

If $p(z) \in P$ with $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ then we obtain

$$p(z) = \frac{1 + z^2}{1 - z^2} = 1 + 2z^2 + 2z^4 + \cdots \in P,$$

which shows that the result is sharp.

**Remark 14.** When we replace $\beta$ by $0$, we get the corresponding result of Shanmugam et al. [13].

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Remark 15. When we replace \( \beta \) by \( 0 \) and \( \alpha \) by \( 0 \), then we get the corresponding result of Janteng et al. \cite{Janteng2012}.

Theorem 16. Let \( f(z) \in C(\alpha, \beta) \), then
\[
|a_2a_4 - a_3^2| \leq \frac{1}{144} \left| \frac{M}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} \right|,
\]
where \( M = (1 - \beta)^2(280\alpha^3 + 332\alpha^2 + 128\alpha + 16) + (1 - \beta)^4(1 + 7\alpha) + (1 - \beta)^3(8\alpha^2 + 3\alpha + 1) \). The result obtained is sharp.

Proof. Let \( f(z) \in C(\alpha, \beta) \)
Then there exists a \( p(z) \in P \), such that
\[
f'(z) + zf''(z) + \alpha z^2f'''(z) + 2\alpha z f''(z) = f'(z)[(1 - \beta)p(z) + \beta]
\]
for some \( z \in E \).

Equating the coefficients in (3.6), we get
\[
a_2 = \frac{c_1(1 - \beta)}{2(1 + 2\alpha)}
\]
\[
a_3 = \frac{c_1^2(1 - \beta)^2}{6(1 + 2\alpha)(1 + 3\alpha)} \frac{c_2(1 - \beta)}{6(1 + 3\alpha)}
\]
\[
a_4 = \frac{c_1^2(1 - \beta)^3}{24(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)} + \frac{c_1c_2(1 - \beta)^2(3 + 8\alpha)}{24(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)} + \frac{c_3(1 - \beta)}{12(1 + 4\alpha)}.
\]
(3.7)

From (3.7),
\[
|a_2a_4 - a_3^2| = \frac{1}{144} \left| \frac{6c_1c_3(1 - \beta)^2}{(1 + 2\alpha)(1 + 4\alpha)} - \frac{4c_2(1 - \beta)^2}{(1 + 3\alpha)^2} - \frac{c_1^2(1 - \beta)^4(1 + 7\alpha)}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} + \frac{c_1^2c_2(1 - \beta)^3(8\alpha^2 + 3\alpha + 1)}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} \right|
\]
(3.8)

Now assuming \( c_1 = c \) (0 \( \leq c \leq 2 \)) and using (2.2) and (2.3), we get
\[
= \frac{1}{144} \left| (1 - \beta)^2[6c^4 + 12c(4 - c^2)x - 6c^2(4 - c^2)x^2 + 12c(4 - c^2)](1 - |x|^2)y \right|
\]
\[
- \frac{(1 - \beta)^2[c^2 + x(4 - c^2)]^2}{(1 + 3\alpha)^2} - \frac{(1 - \beta)^4c^4(1 + 7\alpha)}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}
\]
\[
+ \frac{(1 - \beta)^3c^2[c^2 + x(4 - c^2)](8\alpha^2 + 3\alpha + 1)}{2(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} \right|
\]
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Trivially, \( \rho \) with 

\[
\frac{(1 - \beta)^2[6c^4 + 12c^2\rho(4 - c^2) + 6c(\ell - 2)\rho^2(4 - c^2) + 12c(4 - c^2)]}{4(1 + 2\alpha)(1 + 4\alpha)}
\]

\[
+ \frac{(1 - \beta)^2[4 + \rho^2(4 - c^2)\rho^2(4 - c^2)]}{(1 + 3\alpha)^2}
\]

\[
+ \frac{(1 - \beta)^4c^4(1 + 7\alpha)}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)} + \frac{(1 - \beta)^3[c^4 + \rho(4 - c^2)](8\alpha^2 + 3\alpha + 1)}{2(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}
\]

\[
= F(\rho)
\]

(3.9)

with \( \rho = |x| \leq 1 \).

Furthermore,

\[
F'(\rho) = \frac{(1 - \beta)^2[4\rho(4 - c^2) + c(c - 2)(4 - c^2)]}{(1 + 2\alpha)(1 + 4\alpha)}
\]

\[
+ \frac{(1 - \beta)^3c^2(4 - c^2)(8\alpha^2 + 3\alpha + 1)}{2(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}
\]

\[
+ \frac{(1 - \beta)^2[2\rho(4 - c^2)^2]}{(1 + 3\alpha)^2}
\]

Using elementary calculus, we can show that \( F'(\rho) > 0 \) for \( \rho > 0 \). This shows that \( F \) is an increasing function and \( \max_{\rho \leq 1} F(\rho) = F(1) \).

Now, let

\[
G(c) = F(1) = \frac{3(1 - \beta)^2[c^2(4 - c^2) + c(c - 2)(4 - c^2)]}{(1 + 2\alpha)(1 + 4\alpha)}
\]

\[
+ \frac{(1 - \beta)^2c^2(4 - c^2)(8\alpha^2 + 3\alpha + 1)}{2(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}
\]

\[
+ \frac{2(1 - \beta)^2[c^2(4 - c^2) + (4 - c^2)^2]}{(1 + 3\alpha)^2}
\]

Trivially, \( G \) attains its maximum at \( c = 1 \). Thus the upper bound for (3.9) corresponds to \( \rho = 1 \) and \( c = 1 \), gives

\[
\left| \frac{(1 - \beta)^2c^2}{(1 + 2\alpha)(1 + 4\alpha)} - \frac{(1 - \beta)^2c^2}{(1 + 3\alpha)^2} \right| - \frac{(1 - \beta)^4c^4(1 + 7\alpha)}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}
\]

\[
+ \frac{(1 - \beta)^3(8\alpha^2 + 3\alpha + 1)}{2(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}
\]

\[
\leq \frac{15(1 - \beta)^2}{(1 + 2\alpha)(1 + 4\alpha)} + \frac{(1 - \beta)^2c^2}{(1 + 3\alpha)^2} + \frac{(1 - \beta)^4c^4(1 + 7\alpha)}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}
\]

\[
+ \frac{2(8\alpha^2 + 3\alpha + 1)}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}
\].
If $c_1 = 1$, $c_2 = -1$ and $c_3 = -2$ then we know
\[ p(z) = \frac{1 - z^2}{1 - z + z^2} = 1 + z - z^2 - 2z^3 + z^4 + \cdots \in P, \]
which shows that the result is sharp.

**Remark 17.** When we replace $\beta$ by 0, we get
\[
|a_2a_4 - a_3^2| \leq \frac{15}{(1 + 2\alpha)(1 + 4\alpha)} + \frac{2(8\alpha^2 + 3\alpha + 1)}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}
\]
\[
+ \frac{16}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)},
\]
a result obtained by Shanmugam et al. [13].

**Remark 18.** When we replace $\beta$ by 0 and $\alpha$ by 0, we get
\[
|a_2a_4 - a_3^2| \leq \frac{1}{8},
\]
the sharp result obtained by Janteng et al. [4].

**Acknowledgement.** The authors thank the referee for very useful comments, especially, relating to the sharpness of the results in Theorems 13 and 16, which helped to revise and improve the paper.

**References**


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