GLOBAL EXISTENCE OF SOLUTION FOR REACTION DIFFUSION SYSTEMS WITH A FULL MATRIX OF DIFFUSION COEFFICIENTS

K. Boukerrioua

Abstract. The goal of this work is to study the global existence in time of solutions for some coupled systems of reaction diffusion which describe the spread within a population of infectious disease. We consider a full matrix of diffusion coefficients and we show the global existence of the solutions.

1 Introduction

We are mainly interested in global existence in time of solutions to reaction-diffusion system of the form

\[ \frac{\partial u}{\partial t} - a \Delta u - b \Delta v = \Pi - f(u, v) - \sigma u \quad \text{in } ]0, +\infty[ \times \Omega \] (1.1)

\[ \frac{\partial v}{\partial t} - c \Delta u - a \Delta v = f(u, v) - \sigma v \quad \text{in } ]0, +\infty[ \times \Omega \] (1.2)

with the following boundary conditions

\[ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{in } ]0, +\infty[ \times \partial \Omega \] (1.3)

and the initial data

\[ u(0, x) = u_0, \quad v(0, x) = v_0 \quad \text{in } \Omega. \] (1.4)

where \( \Omega \) is an open bounded domain in \( \mathbb{R}^n \) with boundary \( \partial \Omega \) of class \( C^1 \), \( \frac{\partial}{\partial \eta} \) denotes the outwards normal derivative on \( \partial \Omega \), \( \Delta \) denotes the Laplacian operator with respect to the \( x \) variable, \( a, b, c, \sigma \) are positive constants, \( c \geq 0 \) satisfying the condition \( (b + c) < 2a \) which reflects the parabolicity of the system, \( \Pi \geq 0 \).

2010 Mathematics Subject Classification: 35K45; 35K57.

Keywords: global existence; reaction diffusion systems; Lyapunov functional.

This work was supported by the CNEPRU-MESRS Research Program B01120120103.

http://www.utgjiu.ro/math/sma
We assume that $b < c$, and the initial data are assumed to be in the following region

$$
\Sigma = \left\{ (u_0, v_0) \in \mathbb{R}^2 \text{ such that } v_0 \geq \sqrt{\frac{c}{b}} |u_0| \right\}.
$$

For more details, one may consult [6].

The function $f$ is nonnegative and continuously differentiable function on $\Sigma$ such that

$$
f(-\sqrt{\frac{b}{c}} \eta, \eta) = 0 \text{ and } f(\sqrt{\frac{b}{c}} \eta, \eta) \geq \frac{\Pi \sqrt{c}}{\sqrt{c} + \sqrt{b}}, \text{ for all } \eta \geq 0.
$$

In addition we suppose that

$$
(\xi, \eta) \in \Sigma \implies 0 \leq f(\xi, \eta) \leq \varphi(\xi)(1 + \eta)^\beta,
$$

where $\beta \geq 1$ and $\varphi$ is nonnegative function of class $C(\mathbb{R})$ such that

$$
\lim_{\xi \to -\infty} \frac{\varphi(\xi)}{\xi} = 0.
$$

B. Rebai [10] has proved the global existence of solutions for system (1.1)-(1.4), in the case $b = 0,c > 0$ (triangular matrix). The present investigation is a continuation of results obtained in [10].

In this study, we will treat the case of a general full matrix of diffusion coefficients satisfying $a = d$. Here, we make use of the Lyapunov function techniques and present an approach similar to that developed in [8] under the assumptions (1.6)-(1.7).

The components $u(t, x)$ and $v(t, x)$ represent either chemical concentrations or biological population densities and system (1.1)-(1.4) is a mathematical model describing various chemical and biological phenomena (see, e.g. Cussler [3]).

## 2 Local Existence and Invariant Regions

Throughout the text we shall denote by $\| \|_p$ the norm in $L^p(\Omega), \| \|_\infty$ the norm in $L^\infty(\Omega)$ or $C(\Omega)$.

For any initial data in $C(\Omega)$ or $L^p(\Omega), p \in [1, +\infty[$, local existence and uniqueness of solutions to the initial value problem (1.1)-(1.4) follow from the basic existence theory for abstract semilinear differential equations (see D. Henry [5] and A. Pazy [9]). The solutions are classical on $]0; T^*[$, where $T^*$ denotes the eventual blowing-up time in $L^\infty(\Omega)$.

Furthermore, if $T^* < +\infty$, then

Surveys in Mathematics and its Applications 9 (2014), 105 – 115

http://www.utgjiu.ro/math/sma
Therefore, if there exists a positive constant $C$ such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C, \forall t \in [0, T^*],$$

then $T^* = +\infty$.

Multiplying equation (1.1) through by $\sqrt{c}$ and equation (1.2) by $\sqrt{b}$, subtracting the resulting equations one time and adding them an other time we get

$$\frac{\partial w}{\partial t} - \left(a + \sqrt{bc}\right) \Delta w = \sqrt{c}\Pi - \left(\sqrt{c} - \sqrt{b}\right) F(w, z) - \sigma w \quad \text{in } ]0, T^*[ \times \Omega, (2.1)$$

$$\frac{\partial z}{\partial t} - \left(a - \sqrt{bc}\right) \Delta z = - \sqrt{c}\Pi + \left(\sqrt{c} + \sqrt{b}\right) F(w, z) - \sigma z \quad \text{in } ]0, T^*[ \times \Omega, (2.2)$$

with the boundary conditions

$$\frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0 \quad \text{in } ]0, T^*[ \times \partial \Omega, (2.3)$$

and the initial data

$$w(0, x) = w_0(x), z(0, x) = z_0(x) \quad \text{in } \Omega, (2.4)$$

where,

$$w(t, x) = \sqrt{c}u(t, x) + \sqrt{b}v(t, x), \quad (2.5)$$

$$z(t, x) = -\sqrt{c}u(t, x) + \sqrt{b}v(t, x),$$

for any $(t, x)$ in $]0, T^*[ \times \Omega$ and

$$F(w, z) = f(u, v) \quad \text{for all } (u, v) \text{ in } \Sigma. (2.6)$$

To prove that $\Sigma$ is an invariant region for system (1.1)–(1.4) it suffices to prove that the region

$$\Sigma_1 = \{ (w_0, z_0) \in \mathbb{R}^2 \text{ such that } w_0 \geq 0, \ z_0 \geq 0 \}.$$

is invariant for system (2.1)–(2.4).

Now, to prove that the region $\Sigma_1$ is invariant for system (2.1)–(2.4), it suffices to show that $(\sqrt{c}\Pi - (\sqrt{c} - \sqrt{b}) F(0, z)) \geq 0$ for $z \geq 0$ and $(\sqrt{c}\Pi + (\sqrt{c} + \sqrt{b}) F(w, 0)) \geq 0$, for $w \geq 0$, see [10].
From (1.5), it is clear that the region $\Sigma_1$ is invariant for system (2.1)–(2.4) and from (2.5) we have

$$
\begin{align*}
    u(t, x) &= \frac{1}{2\sqrt{c}}(w(t, x) - z(t, x)), \\
    v(t, x) &= \frac{1}{2\sqrt{b}}(w(t, x) + z(t, x)).
\end{align*}
$$

3 Existence of global solutions

By a simple application of comparison theorem [[10], Theorem 10.1] to system (2.1)–(2.4) implies that for any initial conditions $w_0 \geq 0$ and $z_0 \geq 0$, we have

$$
0 \leq w(t, x) \leq \max(\|w_0\|_{\infty}, \sigma) = K,
$$

To prove the global existence of the solutions of problem (1.1)–(1.4), one needs to prove it for problem (2.1)–(2.4). To this subject, it is well known that, it suffices to derive a uniform estimate of the quantity $\| -\sqrt{c}\Pi + (\sqrt{c} + \sqrt{b})F(w, z) - \sigma z \|_p$ for some $p > \frac{n}{2}$, i.e.

$$
\left\| -\sqrt{c}\Pi + (\sqrt{c} + \sqrt{b})F(w, z) - \sigma z \right\|_p \leq C,
$$

where $C$ is a nonnegative constant independent of $t$.

From the assumptions (1.6) and (1.7), we are led to establish the uniform boundedness of the $\|z\|_p$ on $[0, T^*[$ in order to get that of $\|z\|_{\infty}$ on $[0, T^*[$.

For $p \geq 2$, we put

$$
\alpha = \frac{bc}{(a^2 - bc)}, \quad \alpha(p) = \frac{p\alpha + 1}{p - 1}, \quad M_p = K + \frac{\sqrt{c}\Pi}{\sigma\alpha(p)},
$$

We firstly introduce the following lemmas, which are useful in our main results.

**Lemma 1.** Let $(w, z)$ be a solution of (2.1)–(2.4). Then

$$
\frac{d}{dt} \int_{\Omega} w dx + (\sqrt{c} - \sqrt{b}) \int_{\Omega} F(w, z) dx + \sigma \int_{\Omega} w dx = \sqrt{c}\Pi |\Omega|.
$$

**Proof.** We integrate both sides of (2.1) satisfied by $w$, which is positive and then we obtain

$$
\frac{d}{dt} \int_{\Omega} w dx = \sqrt{c}\Pi |\Omega| - (\sqrt{c} - \sqrt{b}) \int_{\Omega} F(w, z) dx - \sigma \int_{\Omega} w dx.
$$

\[\square\]
Lemma 2. Assume that $p \geq 2$ and let

$$G_q(t) = \int_\Omega \left[ qw + \exp\left( -\frac{p-1}{p\alpha+1} \ln(\alpha(p)(M_p - w)) \right) z^p \right] dt,$$

where $(w, z)$ is the solution of (2.1)-(2.4) on $]0, T^*[$. Then under the assumptions (1.6) - (1.7) there exist two positive constants $q > 0$ and $s > 0$ such that

$$\frac{d}{dt} G_q(t) \leq -(p-1)\sigma G_q + s.$$

Proof. The proof is similar to that in Melkemi et al [8].

Let

$$h(w) = -\frac{p-1}{p\alpha+1} \ln(\alpha(p)(M_p - w)).$$

Then

$$G_q(t) = q \int_\Omega wdx + N(t),$$

where

$$N(t) = \int_\Omega e^{h(w)} z^p dx.$$

Differentiating $N(t)$ with respect to $t$ and using the Green formula one obtains

$$\frac{d}{dt} N = H + S,$$

where

$$H = -\left( a + \sqrt{bc} \right) \int_\Omega \left( (h'(w))^2 + h''(w) \right) e^{h(w)} z^p (\nabla w)^2 dx$$

$$-2pa \int_\Omega h'(w) e^{h(w)} z^{p-1} \nabla w \nabla z dx$$

$$- \left( a - \sqrt{bc} \right) \int_\Omega p(p-1) e^{h(w)} z^{p-2} (\nabla z)^2 dx,$$

and

$$S = \sqrt{c} \Pi \int_\Omega h'(w) e^{h(w)} z^p dx +$$

$$\int_\Omega \left[ pz^{p-1}(\sqrt{c} + \sqrt{b}) F(w, z) - (\sqrt{c} - \sqrt{b}) h'(w) z^p F(w, z) \right] e^{h(w)} dx$$

$$-p\sigma \int_\Omega h'(w) w e^{h(w)} z^p dx - p\sigma \int_\Omega e^{h(w)} z^p dx - p\sqrt{c} \Pi \int_\Omega e^{h(w)} z^{p-1} dx.$$

We observe that $H$ is given by

$$H = -\int_\Omega Q e^{h(w)} dx,$$
where
\[ Q = \left( a + \sqrt{bc} \right) ((h'(w))^2 + h''(w))z^p(\nabla w)^2 + 2pah'(w)z^{p-1}\nabla w \nabla z + \left( a - \sqrt{bc} \right) p(p-1)z^{p-2}(\nabla z)^2 \]
is a quadratic form with respect to \( \nabla w \) and \( \nabla z \), which is nonnegative if
\[
(2pah'(w)z^{p-1})^2 - 4(a^2 - bc)p(p-1)((h'(w))^2 + h''(w))z^{2p-2} \leq 0. \tag{3.7}
\]
We have chosen \( h(w) \) such that
\[
h'(w) = \frac{1}{\alpha(p)(M_p - w)}, \quad h''(w) = \frac{\alpha(p)}{(\alpha(p)(M_p - w))^2}.
\]
It is easy to see that the left-hand side of (3.7) can be written as
\[
4(a^2 - bc)pz^{2p-2} \left\{ p \left[ \frac{1}{(\alpha(p)(M_p - w))^2} - \frac{\alpha(p)}{(\alpha(p)(M_p - w))^2} \right] + \frac{1 + \alpha(p)}{(\alpha(p)(M_p - w))^2} \right\} = 0.
\]
Since
\[
p\alpha - p\alpha(p) + 1 + \alpha(p) = 0,
\]
the inequality (3.7) holds, \( Q \geq 0 \) and we have
\[
H = - \int_{\Omega} Q e^{h(w)} dx \leq 0,
\]
the second term \( S \) can be estimate as
\[
S \leq \int_{\Omega} (\sqrt{\Pi}h'(w) - \sigma p)e^{h(w)}z^p dx + \int_{\Omega} \left[ pz^{p-1}(\sqrt{c} + \sqrt{b})F(w, z) - h'(w)z^p(\sqrt{c} - \sqrt{b})F(w, z) \right] e^{h(w)} dx
\]
\[
\leq -(p-1)\sigma \int_{\Omega} e^{h(w)}z^p dx + \int_{\Omega} \left[ (\sqrt{c} + \sqrt{b})pz^{p-1}F(w, z) - (\sqrt{c} - \sqrt{b})h'(w)z^pF(w, z) \right] e^{h(w)} dx,
\]
We have
\[
h'(w) = \frac{1}{\alpha(p)(M_p - w)} \leq \frac{1}{\alpha(p)(M_p - K)} = \frac{\sigma}{\sqrt{\Pi}},
\]
and
\[
-h'(w) = \frac{-1}{\alpha(p)(M_p - w)} \leq \frac{-1}{\alpha(p)M_p}, \tag{3.9}
\]
\[
h(w) \leq \frac{-1}{\alpha(p)} \ln \frac{\sqrt{\Pi}}{\sigma}.
\]
Taking into account the fact that $z \geq 0$, and from (3.9), we observe that
\[
\begin{align*}
px^{-1}(\sqrt{c} + \sqrt{b})F(w, z) - h'(w)z^p(\sqrt{c} - \sqrt{b})F(w, z) \\
\leq (p(\sqrt{c} + \sqrt{b})z^p - \frac{1}{\alpha(p)M_p}(\sqrt{c} - \sqrt{b})z^p)F(w, z).
\end{align*}
\]
Then for $\eta_0 = p(\sqrt{c} + \sqrt{b})\eta_0 = p(\sqrt{c} + \sqrt{b})(\sqrt{c} - \sqrt{b})(\alpha(p)M_p + 1) > 0$, and for $0 \leq \xi \leq K, \eta \geq \eta_0$, we have
\[
(px^{-1}(\sqrt{c} + \sqrt{b}) - \frac{1}{\alpha(p)M_p}(\sqrt{c} - \sqrt{b})\eta^p)F(\xi, \eta) \\
= \left[ p(\sqrt{c} + \sqrt{b}) - \frac{(\sqrt{c} - \sqrt{b})}{\alpha(p)M_p} \right] \eta^p F(\xi, \eta) \leq 0,
\]
on the other hand, we deduce that the function $(\xi, \eta) \rightarrow p(\sqrt{c} + \sqrt{b})\eta^p - \frac{1}{\alpha(p)M_p}(\sqrt{c} - \sqrt{b})\eta^p$ is bounded on the compact interval $[0, \eta_0]$, then there exists $c_1 > 0$ such that
\[
px^{-1}(\sqrt{c} + \sqrt{b})F(w, z) - (\sqrt{c} - \sqrt{b})h'(w)z^pF(w, z) \leq c_1 F(w, z). \tag{3.10}
\]
From (3.5), (3.8) and (3.10), we deduce immediately the following inequality
\[
S \leq -(p-1)\sigma N + c_1 \int_{\Omega} F(w, z) e^{h(w)} dx \leq -(p-1)\sigma N + c_1 e^{\frac{1}{\alpha(p)} \ln \frac{\sqrt{c}}{\sigma}} \int_{\Omega} F(w, z) dx,
\]
we put
\[
q = c_1 e^{\frac{1}{\alpha(p)} \ln \frac{\sqrt{c}}{\sigma}},
\]
by (3.2), we have
\[
S \leq -(p-1)\sigma N + q\sqrt{c} H \mid \Omega \mid - q \frac{d}{dt} \int_{\Omega} w(t, x) dx,
\]
and from (3.4), it follows that
\[
S \leq -(p-1)\sigma G_q + q((p-1)\sigma K + \sqrt{c} H) \mid \Omega \mid - q \frac{d}{dt} \int_{\Omega} w(t, x) dx,
\]
and from (3.4) and (3.6), we conclude that
\[
\frac{d}{dt} G_q \leq -(p-1)\sigma G_q + s, \tag{3.11}
\]
where
\[
s = q((p-1)\sigma K + \sqrt{c} H) \mid \Omega \mid.
\]
Now we can establish the global existence and uniform boundedness of the solutions of (2.1)-(2.4).

**Theorem 3.** Under the assumptions (1.6) and (1.7), the solutions of (2.1)-(2.4) are global and uniformly bounded on $[0, +\infty] \times \Omega$.

**Proof.** Multiplying the inequality (3.11) by $e^{(p-1)\sigma t}$ and then integrating, we deduce that there exists a positive constants $C > 0$ independent of $t$ such that:

$$G_q(t) \leq C.$$

From (3.3), we observe that

$$e^{h(w)} \geq e^{\frac{1}{\alpha(p)} \ln(\alpha(p)M_p)},$$

it follows from (3.1) that for all $p \geq 2$,

$$\int_{\Omega} z^p dx \leq e^{\frac{1}{\alpha(p)} \ln(K\alpha(p) + \sqrt{\Pi})} G_q(t) \leq C_1(p),$$

where

$$C_1(p) = Ce^{\frac{1}{\alpha(p)} \ln(K\alpha(p) + \sqrt{\Pi})},$$

select $p > \frac{n}{2}$ and proceed to bounds $\| -\sqrt{\Pi} + (\sqrt{c} + \sqrt{b})F(w, z) - \sigma z \|_p$.

Let

$$A = \max_{\xi_0 \leq \xi \leq K_1} \varphi(\xi),$$

where

$$K_1 = \frac{1}{2\sqrt{c}} K,$$

and $\xi_0$ is such that

$$\xi \leq \xi_0 \implies \varphi(\xi) \prec |\xi|,$$

since $\lim_{\xi \to -\infty} \frac{\varphi(\xi)}{\xi} = 0$ $\iff \forall \varepsilon > 0$, there exists $\xi_0$ such that for $\xi \leq \xi_0$, we have $|\varphi(\xi)/\xi| < \varepsilon$, using (1.6) and (2.6), we deduce that

$$F(w, z) = f(u, v) \leq \varphi(u)(1 + v)^b,$$

which implies,

$$\int_{\Omega} F^p(w, z) dx \leq \int_{\Omega} (\varphi(u))^p(1 + v)^{bp} dx =$$

$$\int_{u \leq \xi_0} (\varphi(u))^p(1 + v)^{bp} dx + \int_{\xi_0 \leq u} (\varphi(u))^p(1 + v)^{bp} dx$$

$$\leq \int_{u \leq \xi_0} |u|^p(1 + v)^{bp} dx + A^p \int_{\xi_0 \leq u} (1 + v)^{bp} dx$$

**************************************************************************

Surveys in Mathematics and its Applications 9 (2014), 105 – 115

http://www.utgjiu.ro/math/sma
using (2.7), we have
\[ |u|^p = \left| \frac{1}{2\sqrt{c}}(w(t, x) - z(t, x)) \right|^p \leq \left( \frac{1}{2\sqrt{c}} \right)^p (w(t, x) + z(t, x))^p, \]
then
\[ \int_{\Omega} F^p (w, z) \, dx \leq \int_{u \leq \xi_0} \left( \frac{1}{2\sqrt{c}} \right)^p (w + z)^p (1 + \frac{1}{2\sqrt{b}} (w + z))^\beta p \, dx \]
\[ + A^p \int_{\xi_0 \leq u} \left( 1 + \frac{1}{2\sqrt{b}} (w + z) \right)^\beta p \, dx \]
\[ \leq \max(A^p, \left( \frac{1}{2\sqrt{c}} \right)^p) \int_{u \leq \xi_0} (w + z)^p (1 + \frac{1}{2\sqrt{b}} (w + z))^\beta p \, dx \]
\[ + \int_{\xi_0 \leq u} \left( 1 + \frac{1}{2\sqrt{b}} (w + z) \right)^\beta p \, dx \]
\[ \leq \max(A^p, \left( \frac{1}{2\sqrt{c}} \right)^p) \int_{\Omega} (w + z)^p (1 + \frac{1}{2\sqrt{b}} (w + z))^\beta p \, dx \]
\[ + \int_{\Omega} \left( 1 + \frac{1}{2\sqrt{b}} (w + z) \right)^\beta p \, dx. \]

We also have
\[ \int_{\Omega} (w + z)^p (1 + \frac{1}{2\sqrt{b}} (w + z))^\beta p \, dx \]
\[ \leq 2^{\beta p - 1} \int_{\Omega} (w + z)^p \, dx + \left( \frac{1}{2\sqrt{b}} \right)^\beta p \int_{\Omega} (w + z)^p (\beta + 1) \, dx \]
\[ \leq 2^{(\beta + 1)p - 2} (K^p |\Omega| + C_1(p)) \]
\[ + 2^{2(\beta + 1)p - 2} \left( \frac{1}{2\sqrt{b}} \right)^\beta p (K^{(\beta + 1)p} |\Omega| + C_1((\beta + 1)p)) \]
\[ = C_2(\beta, p, K, \Omega), \]
and
\[ \int_{\Omega} \left( 1 + \frac{1}{2\sqrt{b}} (w + z) \right)^\beta p \, dx \leq \]
\[ 2^{\beta p - 1} (|\Omega| + \left( \frac{1}{2\sqrt{b}} \right)^\beta p \times 2^{\beta p - 1} (K^{\beta p} |\Omega| + C_1(\beta p))) = C_3(\beta, p, K, \Omega) \]
Consequently,
\[ \int_{\Omega} F^p (w, z) \, dx \leq C_4(A, \beta, p, K, \Omega). \]
Finally

\[ \left\| -\sqrt{c}I + (\sqrt{c} + \sqrt{b})F(w, z) - \sigma z \right\|_p \]
\[ \leq (\sqrt{c} + \sqrt{b}) \| F(w, z) \|_p + \sigma \| z \|_p + \sqrt{c} |\Omega| \]
\[ \leq (\sqrt{c} + \sqrt{b}) \sqrt{C_4(A, \beta, p, K)} + \sigma \sqrt{C_1(p)} + \sqrt{c} |\Omega| \]
\[ = C_5(A, \beta, p, K, \Omega, \sigma). \]

Using the regularity results for solutions of parabolic equations in [5], we conclude that the solutions of the problem (2.1)-(2.4) are uniformly bounded on \([0, +\infty] \times \Omega\).

By (2.7), it’s easy to see that the solutions of the problem (1.1)-(1.4) are also uniformly bounded on \([0, +\infty] \times \Omega\].

\[ \square \]

**Remark 4.** The condition of parabolicity implies that \( \det (A) = a^2 - bc > 0 \), where \( A \) is the matrix of diffusion.

**Remark 5.** Noting that if \( (\xi, \eta) \in \Sigma \), then \( \xi \in \mathbb{R} \) and \( \eta \geq 0 \).

**Remark 6.** Because \( 0 \leq w(t, x) \leq K \) and \( z(t, x) \geq 0 \), we deduce that

\[ -\infty \leq u(t, x) = \frac{1}{2\sqrt{c}}(w(t, x) - z(t, x)) \leq \frac{1}{2\sqrt{c}}K = K_1. \]

**References**


K. Boukerrioua
University of Guelma, Guelma, Algeria.
E-mail: khaledv2004@yahoo.fr

**************************************************************************************************

Surveys in Mathematics and its Applications 9 (2014), 105 – 115
http://www.utgjiu.ro/math/sma