A KAZHDAN GROUP WITH AN INFINITE OUTER AUTOMORPHISM GROUP

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Abstract. D. Kazhdan has introduced in 1967 the Property (T) for local compact groups (see [3]). In this article we prove that for \( n \geq 3 \) and \( m \in \mathbb{N} \) the group \( SL_n(K) \ltimes M_{n,m}(K) \) is a Kazhdan group having the outer automorphism group infinite.

Definition 1. ([1]) Let \((\pi, \mathcal{H})\) be a unitary representation of a topological group \( G \).

(i) For a subset \( Q \) of \( G \) and real number \( \varepsilon > 0 \), a vector \( \xi \in \mathcal{H} \) is \((Q, \varepsilon)\)-invariant if:

\[
\sup_{x \in Q} ||\pi(x)\xi - \xi|| < \varepsilon ||\xi||.
\]

(ii) The representation \((\pi, \mathcal{H})\) almost has invariant vectors if it has \((Q, \varepsilon)\) - invariant vectors for every compact subset \( Q \) of \( G \) and every \( \varepsilon > 0 \). If this holds, we write \( 1_G \preceq \pi \).

(iii) The representation \((\pi, \mathcal{H})\) has non-zero invariant vectors if there exists \( \xi \neq 0 \) in \( \mathcal{H} \) such that \( \pi(x)\xi = \xi \) for all \( g \in G \). If this holds, we write \( 1_G \subset \pi \).

Definition 2. ([3]) Let \( G \) be a topological group.

\( G \) has Kazhdan’s Property (T), or is a Kazhdan group, if there exists a compact subset \( Q \) of \( G \) and \( \varepsilon > 0 \) such that, whenever a unitary representation \( \pi \) of \( G \) has a \((Q, \varepsilon)\) - invariant vector, then \( \pi \) has a non-zero invariant vector.

Proposition 3. ([1]) Let \( G \) be a topological group. The following statements are equivalent:

(i) \( G \) has Kazhdan’s Property(T);

(ii) whenever a unitary representation \((\pi, \mathcal{H})\) of \( G \) weakly contains \( 1_G \), it contains \( 1_G \) (in symbols: \( 1_G \preceq \pi \) implies \( 1_G \subset \pi \)).

Definition 4. Let \( K \) be a field. An absolute value on \( K \) is a real - valued function \( x \to |x| \) such that, for all \( x \) and \( y \) in \( K \):

(i) \( |x| \geq 0 \) and \( |x| = 0 \iff x = 0 \)

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(ii) $|xy| = |x||y|
(iii) |x + y| \leq |x| + |y|.

An absolute value defines a topology on $K$ given by the metric

$$d(x, y) = |x - y|.$$  

Definition 5. A field $K$ is a local field if $K$ can be equipped with an absolute value for which $K$ is locally compact and not discrete.

Example 6. $K = \mathbb{R}$ and $K = \mathbb{C}$ with the usual absolute value are local fields.

Example 7. ([1] and [2]) Groups with Property (T):

a) Compact groups, $SL_n(\mathbb{Z})$ for $n \geq 3$.

b) $SL_n(K)$ for $n \geq 3$ and $K$ a local field.

Lemma 8. (Mautner’s lemma) ([1])

Let $G$ be a topological group, and let $(\pi, \mathcal{H})$ be a unitary representation of $G$. Let $x \in G$ and assume that there exists a net $(y_i)_i$ in $G$ such that $\lim_{i} y_i x y_i^{-1} = e$. If $\xi$ is a vector in $\mathcal{H}$ which is fixed by $y_i$ for all $i$, then $\xi$ is fixed by $x$.

Theorem 9. Let $K$ be a local field. The group $SL_n(K)$ acts on $M_{n,m}(K)$ by left multiplication $(g, A) \rightarrow gA, g \in SL_n(K)$ and $A \in M_{n,m}(K)$.

Then the semi-direct product $SL_n(K) \ltimes M_{n,m}(K)$ has Property (T) for $(\forall)n \geq 3$ and $(\forall)m \in \mathbb{N}$.

Proof. Let $(\pi, \mathcal{H})$ be a unitary representation of $G = SL_n(K) \ltimes M_{n,m}(K)$ almost having invariant vectors. Since $SL_n(K)$ has Property (T), there exists a non-zero vector $\xi \in \mathcal{H}$ which is $SL_n(K)$ - invariant.

Since $K$ is non-discret, there exists a net $(\lambda_i)_i$ in $K$ with $\lambda_i \neq 0$ and such that $\lim \lambda_i = 0$.

Let $\Delta_{pq}(x) \in M_{n,m}(K)$ the matrix with $x$ as $(p, q)$ - entry and 0 elsewhere and $(A_i)_{\alpha, \beta} \in SL_n(K)$ the matrix:

$$
(A_i)_{\alpha, \beta} = \begin{cases} 
\lambda_i & \text{if } \alpha = \beta \text{ and } \alpha = p \\
\lambda_i^{-1} & \text{if } \alpha = \beta \text{ and } \alpha = (p + 1) \text{mod}(n + 1) + \lfloor p/n \rfloor \\
1 & \text{if } \alpha = \beta \text{ and } \alpha \notin \{p, (p + 1) \text{mod}(n + 1) + \lfloor p/n \rfloor\} \\
0 & \text{if } \alpha \neq \beta
\end{cases} \quad (0.1)
\Rightarrow A_i \Delta_{pq}(x) = \delta_{pq}(\lambda_i x), \text{ where } \delta_{pq}(\lambda_i x) \in M_{n,m}(K) \text{ is the matrix with } \lambda_i x \text{ as } (p, q) - \text{entry and 0 elsewhere.}

Then $\lim A_i \Delta_{pq}(x) = 0_{n,m}$.

Since in $G$ we have

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\[(A_i, 0_{n,m})(I_n, \Delta_{pq}(x))(A_i, 0_{n,m})^{-1} = (I_n, A_i \Delta_{pq}(x))\]
and since \(\xi \in \mathcal{H}\) is \((A_i, 0_{n,m})\) - invariant \(\Rightarrow\)
an from Mautner’s Lemma that \(\xi\) is \(\Delta_{pq}(x)\) - invariant.
Since \(\Delta_{pq}(x)\) generates the group \(\mathcal{M}_{n,m}(K)\) \(\Rightarrow\) \(\xi\) is \(G\) - invariant and \(G\) has Property \((T)\). \(\square\)

**Corollary 10.** The groups \(SL_n(K) \ltimes K^n\) and \(SL_n(R) \ltimes \mathcal{M}_n(R)\) has Property \((T)\), \((\forall)n \geq 3.\)

**Proposition 11.** For \(\delta \in SL_n(Z)\), let \(S_\delta : \Gamma \to \Gamma\), \(S_\delta((\alpha, A)) = (\alpha, A\delta), (\forall)(\alpha, A) \in \Gamma.\) Then:
\(a)\ S_\delta \in Aut(\Gamma)\)
\(b)\ \Phi : SL_n(Z) \to Aut(\Gamma), \Phi(\delta) = S_\delta\) is a group homomorphism.
\(c)\ S_\delta \in Int(\Gamma)\) if and only if \(\delta \in \{\pm I\}\). In particular, the outer automorphism of \(\Gamma\) is infinite.

**Proof.** a) \(S_\delta((\alpha_1, A_1) \cdot (\alpha_2, A_2)) = S_\delta((\alpha_1, A_1)) \cdot S_\delta((\alpha_2, A_2)) \iff\)
\(\iff S_\delta((\alpha_1\alpha_2, A_1 + A_2A_1)) = (\alpha_1, A_1\delta) \cdot (\alpha_2, A_2\delta) \iff\)
\(\iff (\alpha_1\alpha_2, A_1 + A_2A_1)\delta = (\alpha_1\alpha_2, A_1\delta + A_2A_1\delta)\)

Analogous \(S_{\delta^{-1}}\) is mormism and \(S_\delta \cdot S_{\delta^{-1}} = S_{\delta^{-1}} \cdot S_\delta = I_\Gamma.\)
b) \(\Phi(\delta_1 \cdot \delta_2) = \Phi(\delta_1) \cdot \Phi(\delta_2) \iff S_{\delta_1 \cdot \delta_2} = S_{\delta_1} \cdot S_{\delta_2}.\)
c) Assume that \(S_\delta \in Int(\Gamma)\) \(\Rightarrow\) \((\exists)(\alpha_0, A_0) \in \Gamma\) such that
\(S_\delta((\alpha, A)) = (\alpha_0, A_0)((\alpha, A)(\alpha_0, A_0)^{-1}, (\forall)(\alpha, A) \in \Gamma).\)
\(\Rightarrow (\alpha, A\delta) = (\alpha_0\alpha_0^{-1}, A_0 + \alpha_0A - \alpha_0\alpha_0^{-1}A_0) \Rightarrow\)
\(\Rightarrow i)\ \alpha = \alpha_0\alpha_0^{-1}, (\forall)\alpha \in SL_n(Z) \Rightarrow \alpha \in \{\pm I_n\}\)
\(\Rightarrow ii)\ A\delta = A_0 \pm A - \alpha A_0, (\forall)A \in SL_n(Z), (\forall)A \in \mathcal{M}_n(Z) \Rightarrow A_0 = 0_n\) and \(\delta = \pm I_n.\)
\(\Rightarrow Out(\Gamma) = Aut(\Gamma)/Int(\Gamma)\) is infinite. \(\square\)

**References**


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