A NOTE OF ZUK’S CRITERION

Traian Preda

Abstract. Zuk’s criterion give us a condition for a finitely generated group to have Property (T): the smallest non-zero eigenvalue of Laplace operator $\Delta_\mu$ corresponding to the simple random walk on $G(S)$ satisfies $\lambda_1(G) > \frac{1}{2}$. We present here two examples that prove that this condition cannot be improved.

Definition 1. (see [1] and [2])

i) A random walk or Markov kernel on a non-empty set $X$ is a kernel with non-negative values $\mu : X \times X \rightarrow \mathbb{R}_+$ such that:

$$\sum_{y \in X} \mu(x, y) = 1, \forall x \in X.$$  

ii) A stationary measure for a random walk $\mu$ is a function $\nu : X \rightarrow \mathbb{R}_+^*$ such that:

$$\nu(x)\mu(x, y) = \nu(y)\mu(y, x), \forall x, y \in X.$$  

Example 2. Let $G=(X,E)$ be a locally finite graph. For $x, y \in X$, set

$$\mu(x, y) = \begin{cases} 
\frac{1}{\text{deg}(x)} & \text{if } (x, y) \in E \\
0 & \text{otherwise}
\end{cases} \quad (0.1)$$

and $\text{deg}(x) = \text{card} \{y \in X | (x, y) \in E\}$ is the degree of a vertex $x \in X$. $\mu$ is called simple random walk on $X$ and $\nu$ is a stationary measure for $\mu$.

Consider the Hilbert space:

$$\Omega^0_C(X) = \{ f : X \rightarrow \mathbb{C} | \sum_{x \in X} |f(x)|^2\nu(x) < \infty \}.$$  

The Laplace operator $\Delta_\mu$ on $\Omega^0_C(X)$ is defined by

$$(\Delta_\mu f)(x) = f(x) - \sum_{x \sim y} f(y)\mu(x, y).$$

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Let $\Gamma$ be a group generated by a finite set $S$. We assume that $e \notin S$ and $S = S^{-1}$ (S is symmetric).

The graph $G(S)$ associated to $S$ has vertex set $S$ and the set of edges is the set of pairs $(s, t) \in S \times S$ such that $s^{-1}t \in S$.

**Theorem 3.** (Zuk’s criterion) (see [3])

Let $\Gamma$ be a group generated by a finite set $S$ with $e \notin S$. Let $G(S)$ be the graph associated to $S$. Assume that $G(S)$ is connected and that the smallest non-zero eigenvalue of the Laplace operator $\Delta_\mu$ corresponding to the simple random walk on $G(S)$ satisfies $\lambda_1(G(S)) > \frac{1}{2}$.

Then $\Gamma$ has Property (T).

We prove that the condition $\lambda_1(G(S)) > \frac{1}{2}$ cannot be improved, using two examples.

**Example 4.** Consider $S = \{1, -1, 2, -2\}$ a generating set of the group $\mathbb{Z}$ and let $G(S)$ be the finite graph associated to $S$. Then the graph $G(S)$ is the graph:

![Graph](attachment:image.png)

Since the Laplace operator $\Delta_\mu$ is defined by:

$$(\Delta_\mu f)(x) = f(x) - \sum_{x \sim y} f(y)\mu(x, y),$$

and

$$\mu(x, y) = \begin{cases} 1 & \text{if } (x, y) \in S \times S \\ \frac{\deg(x)}{2} & \text{otherwise} \end{cases}$$ (0.2)

Then the matrix of the Laplace operator $\Delta_\mu$ with respect to the basis $\{\delta_s | s \in S\}$

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is the following matrix:

\[
A = \begin{pmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
-1 & 1 & 0 & 0 \\
-\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\
0 & 0 & -1 & 1
\end{pmatrix}
\]  

(0.3)

Then \( \det(A - \alpha I_4) = (1 - \alpha)^2[(1 - \alpha)^2 - \frac{3}{4}] - \frac{1}{2}(1 - \alpha)^2 + \frac{1}{4} = 0 \)

\[\Rightarrow \alpha \in \{0, \frac{1}{2}, \frac{3}{2} \} \Rightarrow \lambda_1(\mathcal{G}(S)) = \frac{1}{2}.\]

But \( \mathbb{Z} \) does not have Property (T). (see [1])

**Example 5.** The group \( SL_2(\mathbb{Z}) \) is generated by the matrices

\[
A = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

We consider the following generating set of the group \( SL_2(\mathbb{Z}) \):

\[S = \{-I, A, B, -A, -B, A^{-1}, B^{-1}, -A^{-1}, -B^{-1}\}.\]

The graph \( \mathcal{G}(S) \) is:

\[
\begin{array}{c}
-1 \\
A \\
A^{-1} \\
B \\
B^{-1}
\end{array}
\]

Then the matrix of Laplace operator \( \Delta_\mu \) with respect to the basis \( \{\delta_s | s \in S\} \) is the following matrix:

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\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\
\end{bmatrix}
\] (0.4)

Computing \(\det(A - \alpha I_9) = [(1 - \alpha)^2 - \frac{1}{4}]^3 \left(\frac{3}{2} - \alpha\right)(\alpha^2 - \frac{5}{2}) = 0 \Rightarrow \alpha \in \{0, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\} \Rightarrow \lambda_1(G(S)) = \frac{1}{2} \)

But \(SL_2(\mathbb{Z})\) does not have Property \((T)\). (see [1])

These two examples shows that \(\frac{1}{2}\) is the best constant in Zuk’s criterion and cannot be improved.

References


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