A FUNCTIONAL CALCULUS FOR QUOTIENT BOUNDED OPERATORS

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Abstract. If \((X, \mathcal{P})\) is a sequentially locally convex space, then a quotient bounded operator \(T \in Q\mathcal{P}(X)\) is regular (in the sense of Waelbroeck) if and only if it is a bounded element (in the sense of Allan) of algebra \(Q\mathcal{P}(X)\). The classic functional calculus for bounded operators on Banach space is generalized for bounded elements of algebra \(Q\mathcal{P}(X)\).

1 Introduction

It is well-known that if \(X\) is a Banach space and \(\mathcal{L}(X)\) is Banach algebra of bounded operators on \(X\), then formula

\[
 f(T) = \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z, T) \, dz,
\]

( where \(f\) is an analytic function on some neighborhood of \(\sigma(T)\), \(\Gamma\) is a closed rectifiable Jordan curve whose interior domain \(D\) is such that \(\sigma(T) \subset D\), and \(f\) is analytic on \(D\) and continuous on \(D \cup \Gamma\) defines a homomorphism \(f \rightarrow f(T)\) from the set of all analytic functions on some neighborhood of \(\sigma(T)\) into \(\mathcal{L}(X)\), with very useful properties.

Through this paper all locally convex spaces will be assumed Hausdorff, over complex field \(\mathbb{C}\), and all operators will be linear. If \(X\) and \(Y\) are topological vector spaces we denote by \(L(X, Y)\) (\(\mathcal{L}(X, Y)\)) the algebra of linear operators (continuous operators) from \(X\) to \(Y\).

Any family \(\mathcal{P}\) of seminorms which generate the topology of locally convex space \(X\) (in the sense that the topology of \(X\) is the coarsest with respect to which all seminorms of \(\mathcal{P}\) are continuous) will be called a calibration on \(X\). A calibration \(\mathcal{P}\) is characterized by the property that the collection of all sets

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constitute a neighborhoods sub-base at 0. A calibration on $X$ will be principal if it is directed. The set of all calibrations for $X$ is denoted by $\mathcal{C}(X)$ and the set of all principal calibration by $\mathcal{C}_0(X)$.

If $(X, \mathcal{P})$ is a locally convex algebra and each seminorms $p \in \mathcal{P}$ is submultiplicative then $(X, \mathcal{P})$ is locally multiplicative convex algebra or l.m.c.-algebra.

Any family of seminorms on a linear space is partially ordered by relation “$\leq$”, where

$$p \leq q \Leftrightarrow p(x) \leq q(x), \text{ (\forall) } x \in X.$$ 

A family of seminorms is preorder by relation “$\prec$”, where

$$p \prec q \Leftrightarrow \text{ there exists some } r > 0 \text{ such that } p(x) \leq rq(x), \text{ for all } x \in X.$$ 

If $p \prec q$ and $q \prec p$, we write $p \approx q$.

**Definition 1.** Two families $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on a linear space are called $Q$-equivalent (denoted $\mathcal{P}_1 \approx \mathcal{P}_2$) provided:

1. for each $p_1 \in \mathcal{P}_1$ there exists $p_2 \in \mathcal{P}_2$ such that $p_1 \approx p_2$;
2. for each $p_2 \in \mathcal{P}_2$ there exists $p_1 \in \mathcal{P}_1$ such that $p_2 \approx p_1$.

It is obvious that two $Q$-equivalent and separating families of seminorms on a linear space generate the same locally convex topology.

**Definition 2.** If $(X, \mathcal{P})$, $(Y, \mathcal{Q})$ are locally convex spaces, then for each $p, q \in \mathcal{P}$ the application $m_{pq}: L(X, Y) \to \mathbb{R} \cup \{\infty\}$, defined by

$$m_{pq}(T) = \sup_{p(x) \neq 0} \frac{q(Tx)}{p(x)}, \text{ (\forall) } T \in L(X, Y).$$ 

is called the mixed operator seminorm of $T$ associated with $p$ and $q$. When $X = Y$ and $p = q$ we use notation $\hat{p} = m_{pp}$.

**Lemma 3** ([9]). If $(X, \mathcal{P})$, $(Y, \mathcal{Q})$ are locally convex spaces and $T \in L(X, Y)$, then

1. $m_{pq}(T) = \sup_{p(x) = 1} q(Tx) = \sup_{p(x) \leq 1} q(Tx), \text{ (\forall) } p \in \mathcal{P}, (\forall) q \in \mathcal{Q};$
2. $q(Tx) \leq m_{pq}(T)p(x), \text{ (\forall) } x \in X, \text{ whenever } m_{pq}(T) < \infty.$
3. $m_{pq}(T) = \inf \{ M > 0 | q(Tx) \leq Mp(x), \text{ (\forall) } x \in X \}, \text{ whenever } m_{pq}(T) < \infty.$

**Definition 4.** An operator $T$ on a locally convex space $X$ is quotient bounded with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ if for every seminorm $p \in \mathcal{P}$ there exists some $c_p > 0$ such that

$$p(Tx) \leq c_pp(x), \text{ (\forall) } x \in X.$$ 

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The class of quotient bounded operators with respect to a calibration \( \mathcal{P} \in \mathcal{C}(X) \) is denoted by \( Q_\mathcal{P}(X) \). If \( X \) is a locally convex space and \( \mathcal{P} \in \mathcal{C}(X) \), then for every \( p \in \mathcal{P} \) the application \( \hat{\rho} : Q_\mathcal{P}(X) \to \mathbb{R} \) defined by

\[
\hat{\rho}(T) = \inf \{ r > 0 \mid p(Tx) \leq r \, p(x), (\forall) \, x \in X \},
\]

is a submultiplicative seminorm on \( Q_\mathcal{P}(X) \), satisfying \( \hat{\rho}(I) = 1 \). We denote by \( \hat{\mathcal{P}} \) the family \{ \( \mathcal{P} \mid p \in \mathcal{P} \) \}.

**Lemma 5 ([8]).** If \( X \) is a sequentially complete convex space, then \( Q_\mathcal{P}(X) \) is a sequentially complete \( m \)-convex algebra for all \( \mathcal{P} \in \mathcal{C}(X) \).

**Definition 6.** Let \( X \) be a locally convex space and \( T \in Q_\mathcal{P}(X) \). We say that \( \lambda \in \rho(Q_\mathcal{P}, T) \) if the inverse of \( \lambda I - T \) exists and \( (\lambda I - T)^{-1} \in Q_\mathcal{P}(X) \). Spectral sets \( \sigma(Q_\mathcal{P}, T) \) are defined to be complements of resolvent sets \( \rho(Q_\mathcal{P}, T) \).

Let \( (X, \mathcal{P}) \) be a locally convex space and \( T \in Q_\mathcal{P}(X) \). We have said that \( T \) is bounded element of the algebra \( Q_\mathcal{P}(X) \) if it is bounded element in the sens of G.R.Allan [1], i.e. some scalar multiple of it generates a bounded semigroup. The class of bounded element of \( Q_\mathcal{P}(X) \) is denoted by \( (Q_\mathcal{P}(X))_0 \).

**Proposition 7 ([5]).** Let \( X \) is a locally convex space and \( \mathcal{P} \in \mathcal{C}(X) \).

1. \( Q_\mathcal{P}(X) \) is a unital subalgebra of the algebra of continuous linear operators on \( X \)
2. \( Q_\mathcal{P}(X) \) is a unitary l.m.c.-algebra with respect to the topology determined by \( \mathcal{P} \)
3. If \( \mathcal{P}' \in \mathcal{C}(X) \) such that \( \mathcal{P} \approx \mathcal{P}' \), then \( Q_{\mathcal{P}'}(X) = Q_\mathcal{P}(X) \) and \( \hat{\mathcal{P}} \approx \hat{\mathcal{P}}' \)
4. The topology generated by \( \hat{\mathcal{P}} \) on \( Q_\mathcal{P}(X) \) is finer than the topology of uniform convergence on bounded subsets of \( X \)

**Definition 8.** If \( (X, \mathcal{P}) \) is a locally convex space and \( T \in Q_\mathcal{P}(X) \) we denote by \( r_\mathcal{P}(T) \) the radius of boundness of operator \( T \) in \( Q_\mathcal{P}(X) \), i.e.

\[
r_\mathcal{P}(T) = \inf \{ \alpha > 0 \mid \alpha^{-1} \, T \text{ generates a bounded semigroup in } Q_\mathcal{P}(X) \}.
\]

We have said that \( r_\mathcal{P}(T) \) is the \( \mathcal{P} \)-spectral radius of the operator \( T \).

**Proposition 9 ([8]).** Let \( X \) be a sequentially complete locally convex space and \( \mathcal{P} \in \mathcal{C}(X) \). If \( T \in Q_\mathcal{P}(X) \), then \( | \sigma(Q_\mathcal{P}, T) | = r_\mathcal{P}(T) \).

**Definition 10.** Let \( \mathcal{P} \) be a calibration on \( X \). A linear operator \( T : X \to X \) is universally bounded on \( (X, \mathcal{P}) \) if exists a constant \( c_0 > 0 \) such that

\[
p(Tx) \leq c_0 \, p(x), (\forall) \, x \in X.
\]

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Denote by $B_P(X)$ the collection of all universally bounded operators on $(X, P)$. It is obvious that $Q_P(X) \subset B_P(X) \subset L(X)$.

**Lemma 11.** If $P$ a calibration on $X$, then $B_P(X)$ is a unital normed algebra with respect to the norm $\|\cdot\|_P$ defined by

$$
\|T\|_P = \inf\{M > 0 \mid p(Tx) \leq Mp(x), (\forall) x \in X, (\forall) p \in P\}.
$$

**Corollary 12.** If $P \in C(X)$, then for each $T \in B_P(X)$ we have

$$
\|T\|_P = \sup\{m_{pp}(T) \mid p \in P\}, (\forall) T \in B_P(X).
$$

**Proposition 13 ([5]).** Let $X$ be a locally convex space and $P \in C(X)$. Then:

1. $B_P(X)$ is a subalgebra of $L(X)$;
2. $(B_P(X), \|\cdot\|_P)$ is unitary normed algebra;
3. for each $P' \in C(X)$ with the property $P \approx P'$, we have

$$
B_P(X) = B_{P'}(X) \text{ and } \|\cdot\|_P = \|\cdot\|_{P'}.
$$

**Proposition 14 ([2]).** Let $X$ be a locally convex space and $P \in C(X)$. Then:

1. the topology given by the norm $\|\cdot\|_P$ on the algebra $B_P(X)$ is finer than the topology of uniform convergence;
2. if $(T_n)_n$ is a Cauchy sequence in $(B_P(X), \|\cdot\|_P)$ which converges to an operator $T$, we have $T \in B_P(X)$;
3. the algebra $(B_P(X), \|\cdot\|_P)$ is complete if $X$ is sequentially complete.

**Proposition 15 ([5]).** Let $(X, P)$ be a locally convex space. An operator $T \in Q_P(X)$ is bounded in the algebra $Q_P(X)$ if and only if there exists some calibration $P' \in C(X)$ such that $P \approx P'$ and $T \in B_{P'}(X)$.

**Definition 16.** Let $(X, P)$ be a locally convex space and $T \in B_P(X)$. We said that $\alpha \in C$ is in resolvent set $\rho(B_P, T)$ if there exists $(\alpha I - T)^{-1} \in B_P(X)$. The spectral set $\sigma(B_P, T)$ will be the complementary set of $\rho(B_P, T)$.

**Remark 17.** It is obvious that we have the following inclusions

$$
\sigma(T) \subset \sigma(Q_P, T) \subset \sigma(B_P, T).
$$

**Proposition 18.** Proposition If $(X, P$ is a locally convex space and $T \in B_P(X)$, then the set $\sigma(B_P, T)$ is compact.
Proposition 19. Let \((X, \mathcal{P})\) be a locally convex space. Then an operator \(T \in Q_{\mathcal{P}}(X)\) is regular if and only if \(T \in (Q_{\mathcal{P}}(X))_0\).

Proof. Assume that \(T \in Q_{\mathcal{P}}(X)\) is bounded element of \(Q_{\mathcal{P}}(X)\). It folows from proposition (15) that there is some calibration \(\mathcal{P}' \in C(X)\) such that \(\mathcal{P} \approx \mathcal{P}'\), and \(T \in B_{\mathcal{P}'}(X)\). Moreover, \(Q_{\mathcal{P}}(X) = Q_{\mathcal{P}'}(X)\).

If \(|\lambda| > 2 \|T\|_{\mathcal{P}}\), then Neumann series \(\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}\) converges in \(B_{\mathcal{P}'}(X)\) and its sum is \(R(\lambda, T)\). This means that the operator \(\lambda I - T\) is invertible in \(Q_{\mathcal{P}}(X)\) for all \(|\lambda| > 2 \|T\|_{\mathcal{P}'}\). Moreover, for each \(\epsilon > 0\) there exists an index \(n_\varepsilon \in \mathbb{N}\) such that

\[
\left\| R(\lambda, T) - \sum_{k=0}^{n_\varepsilon} \frac{T^k}{\lambda^{k+1}} \right\|_{\mathcal{P}} < \varepsilon, \quad (\forall) \quad n \geq n_\varepsilon,
\]

which implies that for each \(n \geq n_\varepsilon\) we have

\[
\left\| R(\lambda, T) - \sum_{k=0}^{n_\varepsilon} \frac{T^k}{\lambda^{k+1}} \right\|_{\mathcal{P}} < \varepsilon + \left( \frac{1}{\lambda} \right)^{-1} \sum_{k=0}^{n_\varepsilon} 2^{-k} < \epsilon + (\|T\|_{\mathcal{P}'})^{-1} \sum_{k=0}^{n_\varepsilon} 2^{-k} < \epsilon + (\|T\|_{\mathcal{P}'})^{-1}.
\]

Since \(\epsilon > 0\) is arbitrarily chosen, we have that

\[
\left\| R(\lambda, T) \right\|_{\mathcal{P}} < (\|T\|_{\mathcal{P}'})^{-1}, \quad (\forall) \quad |\lambda| > 2 \|T\|_{\mathcal{P}'},
\]

From definition of norm \(\| \cdot \|_{\mathcal{P}}\) it follows that

\[
\hat{\rho}(R(\lambda, T)) < (\|T\|_{\mathcal{P}'})^{-1},
\]

for any \(\mathcal{P}'\) and for each \(|\lambda| > 2 \|T\|_{\mathcal{P}'},\) which means that the set

\[
\{ R(\lambda, T) \mid |\lambda| > 2 \|T\|_{\mathcal{P}'},\}
\]

is bounded in \(Q_{\mathcal{P}}(X)\). Therefore, \(T\) is regular.

Now suppose that \(T \in Q_{\mathcal{P}}(X)\) is regular, but it is not bounded in \(Q_{\mathcal{P}}(X)\). By proposition (9) this means that

\[
|\sigma(Q_{\mathcal{P}}, T)| = r_{\mathcal{P}}(T) = \infty,
\]

which contradicts the assumption we have made. Therefore, \(T\) is bounded element of \(Q_{\mathcal{P}}(X)\). \(\square\)
2 A functional calculus

In this section we assume that $X$ will be sequentially complete locally convex space. We show that we can develop a functional for bounded elements of algebra $\mathcal{Q}_P(X)$, where $P \in C(X)$.

**Definition 20.** Let $(X, \mathcal{P})$ be a locally convex space. The Waelbroeck resolvent set of an operator $T \in \mathcal{Q}_P(X)$, denoted by $\rho_W(\mathcal{Q}_P, T)$, is the subset of elements of $\lambda_0 \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, for which there exists a neighborhood $V \in \mathcal{V}_{(\lambda_0)}$ such that:

1. the operator $\lambda I - T$ is invertible in $\mathcal{Q}_P(X)$ for all $\lambda \in V \setminus \{\infty\}$
2. the set \( \{ (\lambda I - T)^{-1} | \lambda \in V \setminus \{\infty\} \} \) is bounded in $\mathcal{Q}_P(X)$.

The Waelbroeck spectrum of $T$, denoted by $\sigma_W(\mathcal{Q}_P, T)$, is the complementary set of $\rho_W(\mathcal{Q}_P, T)$ in $\mathbb{C}_\infty$. It is obvious that $\sigma(\mathcal{Q}_P, T) \subset \sigma_W(\mathcal{Q}_P, T)$.

**Definition 21.** Let $(X, \mathcal{P})$ be a locally convex space. An operator $T \in \mathcal{Q}_P(X)$ is regular if $\sigma_W(\mathcal{Q}_P, T)$, i.e. there exists some $t > 0$ such that:

1. the operator $\lambda I - T$ is invertible in $\mathcal{Q}_P(X)$, for all $|\lambda| > t$
2. the set $\{ R(\lambda, T) | |\lambda| > t \}$ is bounded in $\mathcal{Q}_P(X)$.

Let $P \in C(X)$ be arbitrary chosen and $D \subset C$ a relatively compact open set.

**Lemma 22.** Let $p \in P$ then the application $|f|_{p,D}: \mathcal{O}(D, \mathcal{Q}_P(X)) \to \mathbb{R}$ given by,

$$|f|_{p,D} = \sup_{z \in D} p(f(z)), \forall f \in \mathcal{O}(D, \mathcal{Q}_P(X)),$$

is a submultiplicative seminorm on $\mathcal{O}(D, \mathcal{Q}_P(X))$.

If we denote by $\sigma_{\mathcal{P}, D}$ the topology defined by the family $\{ |f|_{p,D} | p \in \mathcal{P} \}$ on $\mathcal{O}(D, \mathcal{Q}_P(X))$, then $(\mathcal{O}(D, \mathcal{Q}_P(X)), \sigma_{\mathcal{P}, D})$ is a l.m.c.-algebra.

Let $K \subset C$ be a compact set arbitrary chosen. We define the set

$$\mathcal{O}(K, \mathcal{Q}_P(X)) = \cup \{ \mathcal{O}(D, \mathcal{Q}_P(X)) | D \subset C \text{ is relatively compact open} \}$$

We need the following lemma from complex analysis.

**Lemma 23.** For each compact set $K \subset C$ and each relatively compact open set $D \supset K$ there exists some open set $G$ such that:

1. $K \subset G \subset \overline{G} \subset D$;
2. $G$ has a finite number of conex components $(G_i)_{i=1}^n$, the closure of which are pairwise disjoint;

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3. the boundary $\partial G_i$ of $G_i$, $i = 1, n$, consists of a finite positive number of closed rectifiable Jordan curves $(\Gamma_{ij})_{j=1,m_i}$, no two of which intersect;

4. $K \cap \Gamma_{ij} = \emptyset$, for each $i = 1, n$ and every $j = 1, m_i$.

**Definition 24.** Let $K$ and $D$ be like in the previous lemma. An open set $G$ is called Cauchy domain for pair $(K, D)$ if it has the properties 1-4 of the previous lemma. The boundary 

$$
\Gamma = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \Gamma_{ij}
$$

of $G$ is called Cauchy boundary for pair $(K, D)$.

Using some results from I. Colojoara [3] we can develop a functional calculus for bounded elements of locally $m$-convex algebra $Q_P(X)$.

**Theorem 25.** If $P \in C_0(X)$ and $T \in (Q_P(X))_0$, then for each relatively compact open set $D \supset \sigma_W(Q_P, T)$, the application $F_{T,D} : O(D) \to Q_P(X)$ defined by 

$$
F_{T,D}(f) = \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z, T) \, dz, \forall f \in O(D),
$$

where $\Gamma$ is a Cauchy boundary for pair $(\sigma_W(Q_P, T), D)$, is a unitary continuous homomorphism. Moreover, 

$$
F_{T,D}(z) = T,
$$

where $z$ is the identity function.

Like in the Banach case we make the following notation $f(T) = F_{T,D}(f)$.

The following theorem represents the analogous of spectral mapping theorem for Banach spaces.

**Theorem 26.** If $P \in C_0(X)$, $T \in (Q_P(X))_0$ and $f$ is a holomorphic function on an open set $D \supset \sigma_W(Q_P, T)$, then 

$$
\sigma_W(Q_P, f(T)) = f(\sigma_W(Q_P, T)).
$$

**Theorem 27.** Assume that $P \in C_0(X)$ and $T \in (Q_P(X))_0$. If $f$ is holomorphic function on the open set $D \supset \sigma_W(Q_P, T)$ and $g \in O(D_g)$, such that $D_g \supset f(D)$, then $(g \circ f)(T) = g(f(T))$.

**Lemma 28.** Assume that $P \in C_0(X)$ and $T \in (Q_P(X))_0$. If $f$ is holomorphic function on the open set $D \supset \sigma_W(Q_P, T)$ and $f(\lambda) = \sum_{k=0}^{\infty} \alpha_k \lambda^k$ on $D$, then $f(T) = \sum_{k=0}^{\infty} \alpha_k T^k$. 

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Corollary 29. If $P \in C_0(X)$ and $T \in (Q_P(X))_0$, then $\exp T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$.

Theorem 30. Let $P \in C_0(X)$ and $T \in (Q_P(X))_0$. If $D$ is an open relatively compact set which contains the set $\sigma_W(Q_P, T), f \in \mathcal{O}(D)$ and $S \in (Q_P(X))_0$, such that $r_P(S) < \text{dist} (\sigma_W(Q_P, T), C \setminus D)$, and $TS = ST$, then we have

1. $\sigma_W(Q_P, T + S) \subset D$;
2. $f(T + S) = \sum_{n \geq 0} \frac{f^{(n)}(T)}{n!} S^n$.

References


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