SOME ABSOLUTELY CONTINUOUS REPRESENTATIONS OF FUNCTION ALGEBRAS

Adina Juratoni

Abstract. In this paper we study some absolutely continuous representations of function algebras, which are weak ρ-spectral in the sense of [5] and [6], for a scalar ρ > 0. Precisely we investigate certain conditions for the existence of a spectral ρ-dilation of such representation. Among others we obtain different results which generalize the corresponding theorems of D. Gaspar [3].

1 Preliminaries

Let X be a compact Hausdorff space, C(X) (respectively C_R(X)) be the Banach algebra of all complex (real) valued continuous functions on X.

Let A be a function algebra on X (that is a closed subalgebra of C(X) containing the constants and separating the points of X) and A be the set of the complex conjugates of the functions from A. Denote by M(A) the set of all nonzero complex homomorphisms of A and for γ ∈ M(A) we put A_γ = ker γ. Clearly, any γ ∈ M(A) can be extended to a bounded linear functional on A + A̅, also denoted by γ, which satisfies for f, g ∈ A:

γ(f + g) = γ(f) + γ(g), \quad |γ(f + g)| ≤ 2∥f + g∥.

Two homomorphisms γ_0, γ_1 ∈ M(A) is called Gleason equivalent if

∥γ_0 − γ_1∥ < 2.

The Gleason equivalence is a relation of equivalence in M(A), and the corresponding equivalence classes are called the Gleason parts of A ([1], [9]).

If γ ∈ M(A) we denote by M_γ the set of all representing measures for γ, that is a positive Borel measures on X satisfying

γ(f) = ∫ f dm \quad (f ∈ A).

2000 Mathematics Subject Classification: 46J25, 47A20, 46J10
Keywords: Uniform algebra, representation, spectral dilation.

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Denote by $\text{Bor}(X)$ the family of all Borel sets of $X$, and $M$ a set of positive Borel measures on $X$. A Borel measure $\nu$ on $X$ is called $M$-absolutely continuous (M a.c.) if $\nu(\sigma) = 0$ for any $M$-null set $\sigma \in \text{Bor}(X)$ (that is with $\mu(\sigma) = 0$ for every $\mu \in M$). Also, on says that $\nu$ is M-singular (M s.) if $\nu$ is supported on a $M$-null set. It is known [1] that each Borel measure $\nu$ on $X$ has a unique decomposition of the form

$$\nu = \nu_a + \nu_s$$

where $\nu_a$ is M a.c. and $\nu_s$ is M s. This decomposition is called the $M$ - decomposition of $\nu$. A measure $\nu$ is completely singular if it is M s. where $M = \bigcup \bar{M}_{\gamma}$. We recall ([1]) that if $\gamma_0$ and $\gamma_1$ are Gleason equivalent, than the $M_{\gamma_j}$ decompositions of $\nu$ coincide for $j = 0, 1$. When $\gamma_0$ and $\gamma_1$ are in different Gleason parts, than the $M_{\gamma_j}$ a.c. component of $\nu$ is $M_{\gamma_{j-1}}$ s., for $j = 0, 1$.

Let $H$ be a complex Hilbert space, and $B(H)$ be the Banach algebra of all bounded linear operators on $H$.

A representation of a function algebra $A$ on $H$ is a multiplicative linear map $\Phi$ of $A$ into $B(H)$ with $\Phi(1) = I$, the identity operator, and

$$\|\Phi(f)\| \leq c \|f\| \quad (f \in A),$$

for some constant $c > 0$. When $c = 1$, $\Phi$ is a contractive representation.

If $\Phi$ is a representations of $A$ on $H$, then by Hahn - Banach and Riesz - Kakutani theorems it follows that, for each $x, y \in H$ there exists a measure $\mu_{x,y}$ on $X$ such that $\|\mu_{x,y}\| \leq c \|x\| \|y\|$ and

$$< \Phi(f)x, y > = \int fd\mu_{x,y} \quad (f \in A).$$

Such measures $\mu_{x,y}$ $(x, y \in H)$ are called elementary measures for $\Phi$. Also, if $M \subset M_{\gamma}$ for some $\gamma \in M(A)$, one says that $\Phi$ is $M$ - absolutely continuous (M a.c.), respectively $\Phi$ is $M$ - singular (M s.), if there exist M a.c. respectively M s. elementary measures $\mu_{x,y}$ of $\Phi$ for any $x, y \in H$. When $\Phi$ is $M_{\gamma}$ s. for every $\gamma \in M(A)$, $\Phi$ is called completely singular.

For $\rho > 0$, and $\gamma \in M(A)$ a contractive representation $\tilde{\Phi}$ of $C(X)$ on a Hilbert space $K \supset H$ is called a $\gamma-$ spectral $\rho-$ dilation of a representation $\Phi$ of $A$ on $H$ if

$$\Phi(f) = \rho P_{\mu} \tilde{\Phi}(f) |H \quad (f \in A_{\gamma}),$$

where $P_{\mu}$ is the orthogonal projection of $K$ on $H$. When $\rho = 1$, such a representation $\tilde{\Phi}$ is called a spectral dilation of $\Phi$ (that is $\tilde{\Phi}$ is a $\varphi-$ spectral 1 - dilation of $\Phi$, for any $\varphi \in M(A)$).

According to [2], one says that a representation $\Phi$ of $A$ on $H$ is of class $C_{\rho}(A_{\gamma}, H)$ if $\Phi$ has a $\gamma-$ spectral $\rho$ - dilation. Clearly, if $\Phi$ is of class $C_{\rho}(A_{\gamma}, H)$ then

$$\|\Phi(f)\| \leq \|\rho f + (1 - \rho) \gamma(f)\| \quad (f \in A),$$

but the converse assertion is not true, in general (even if $\rho = 1$). D. Gaspar
(2) and (3)) obtains certain conditions under which (3) assures the existence of a \( \gamma \)-spectral \( \rho \)-dilation for \( \Phi \). This happens for instance, if \( A \) is a Dirichlet algebra on \( X \) (that is \( A + \overline{A} \) is dense in \( C(X) \)), or more general, when \( \gamma \) has a unique representing measure \( m \) and \( \Phi \) is \( m \) a.c. Also, T. Nakazi [7], [8] gives other equivalent conditions with the existence of a \( \gamma \)-spectral \( \rho \)-dilation, if \( A \) is a hypo-Dirichlet algebra (that is \( A + \overline{A} \) has finite codimension in \( C(X) \)).

In this paper we generalize some results of D. Gaspar [3] by investigating a weakly condition than (3), namely the condition

\[
\omega(\Phi(f)) \leq \|\rho f + (1 - \rho)\gamma(f)\| \quad (f \in A),
\]

here \( \omega(T) \) is the numerical radius for \( T \in B(H) \).

A representation \( \Phi \) of \( A \) on \( H \) satisfying (4) is called weak \( \rho \)-spectral with respect to \( \gamma \). When \( \Phi \) satisfies (3) it simply called spectral with respect to \( \gamma \).

In [5] and [6] were given different characterizations for that a representation \( \Phi \) to be weak \( \rho \)-spectral with respect to \( \gamma \). This happens if and only if for any \( x \in H \) there exists a positive measure \( \mu_x \) on \( X \) with \( \mu_x(X) = \|x\|^2 \) such that

\[
< \Phi(f)x, x > = \int (\rho f + (1 - \rho)\gamma(f)) \, d\mu_x \quad (f \in A).
\]

Such a measure \( \mu_x \) is called a weak \( \rho \)-spectral measure attached to \( x \) by \( \Phi \) and \( \gamma \).

The aim of this paper is to further investigate the weak \( \rho \)-spectral representations for the weak*-Dirichlet function algebras. Recall [10] that \( m \) is a probability measure on \( X \) and \( A \subset L^\infty(m) \) is a subalgebra, then \( A \) is called a weak*-Dirichlet algebra in \( L^\infty(m) \) if \( m \) is multiplicative on \( A \) and \( A + \overline{A} \) is weak*-dense in \( L^\infty(m) \).

2 Representations with Spectral \( \rho \)-Dilations

In this section we refer to some weak \( \rho \)-spectral representations which have spectral \( \rho \)-dilations. In fact we generalize certain results concerning the \( \rho \)-spectral representations in the case of unique representing measure ([3]).

We begin with the following

**Theorem 1.** Let \( A \) be a function algebra on \( X \) which is weak*-Dirichlet in \( L^\infty(m) \) for some representing measure \( m \) for \( \gamma \in M(A) \). If \( \Phi \) is a representing of \( A \) on \( H \) such that for any \( x \in H \) there exists a \( m \) a.c. weak \( \rho \)-spectral measure attached to \( x \) by \( \Phi \) and \( \gamma \), then \( \Phi \) has a \( \gamma \)-spectral \( \rho \)-dilation. Moreover, in this case there exists a unique \( B(H) \)-valued and \( m \) a.c. semispectral measure \( F \) on \( X \) satisfying

\[
< \Phi(f)x, y > = \int (\rho f(\xi) + (1 - \rho)\gamma(f)) \, d(F(\xi)x, y) \quad (f \in A, x, y \in H).
\]

**Proof.** Let \( \Phi \) a representation of \( A \) on \( H \) and we suppose that for \( x \in H \) there exists a \( m \) a.c. measure \( \mu_x \geq 0 \) with \( \mu_x(X) = \|x\|^2 \) and

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\[ \Phi(f), x, y = \int \left( \rho f + (1 - \rho) \gamma(f) \right) d\mu_x \quad (f \in A). \]

For \( f \in A \) and \( x, y \in H \) we have

\[
\int \left( \rho f + (1 - \rho) \gamma(f) \right) d \left( \mu_{x+y} + \mu_{x-y} \right)
= <\Phi(f)(x+y), x+y> + <\Phi(f)(x-y), x-y>
= 2 <\Phi(f)x, x> + <\Phi(f)y, y>
= 2 \int \left( \rho f + (1 - \rho) \gamma(f) \right) d \left( \mu_x + \mu_y \right),
\]

or equivalently

\[
\rho \int fd \left( \mu_{x+y} + \mu_{x-y} \right) + (1 - \rho) \gamma(f) \left( \|x+y\|^2 + \|x-y\|^2 \right)
= 2 \rho \int fd \left( \mu_x + \mu_y \right) + (1 - \rho) \gamma(f) \left[ 2 \left( \|x\|^2 + \|y\|^2 \right) \right].
\]

This yields for each \( x, y \in H \),

\[
\int fd \left( \mu_{x+y} + \mu_{x-y} \right) = \int fd \left( 2\mu_x + 2\mu_y \right) \quad (f \in A),
\]

and since the measures \( \mu_{x+y} + \mu_{x-y} \) and \( \mu_x + \mu_y \) are m.a.c., by Gleason - Whitney theorem \([10]\) it follows that

\[ \mu_{x+y} + \mu_{x-y} = 2 \left( \mu_x + \mu_y \right) \quad (x, y \in H). \]

Now, if we define the measure

\[ \mu_{x,y} = \frac{1}{4} \left[ \mu_{x+y} - \mu_{x-y} + i \left( \mu_{x+iy} - \mu_{x-iy} \right) \right], \]

then it is known \([9]\) that the \( B(H) \) valued measure \( F \) on \( X \) defined by

\[ <F(\sigma)x, y> = \mu_{x,y}(\sigma) \]

for \( \sigma \in \text{Bor}(X) \) and \( x, y \in H \) is a semispectral measure which clearly satisfies

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\[
< \Phi(f) x, y > = \int (\rho f(\xi) + (1 - \rho) \gamma(f)) d(F(\xi) x, y) \quad (f \in A).
\]

Next by Naimark dilation theorem (see [9]) there exists a contractive representation \( \tilde{\Phi} \) of \( C(X) \) on a Hilbert space \( K \supset H \) such that

\[
< \tilde{\Phi}(g) x, y > = \int g(\xi) d(F(\xi) x, y) \quad (g \in C(X), x, y \in \mathcal{H}).
\]

Thus for \( f \in A_\gamma \) and \( x, y \in H \) one infers

\[
< \Phi(f) x, y > = \rho \int f(\xi) d(F(\xi) x, x) = \rho < \tilde{\Phi}(f) x, y >,
\]

whence we get

\[
\Phi(f) = \rho P_H \tilde{\Phi}(f) |_{\mathcal{H}} \quad (f \in A_\gamma).
\]

Hence \( \tilde{\Phi} \) is a \( \gamma \)-spectral \( \rho \)-dilation of \( \Phi \).

As an application the following result can be obtained, which completes the [3, Theorem 2] of D. Gaspar (the equivalence (ii) \( \Leftrightarrow \) (i) below).

**Theorem 2.** Let \( A \) be a function algebra on \( X \) and \( \gamma \in M(A) \) such that \( \gamma \) has a unique representing measure \( m \). Then for a \( m \) a.c. representation \( \Phi \) of \( A \) on \( H \) the following statements are equivalent:

(i) \( \Phi \) has a \( \gamma \)-spectral \( \rho \)-dilation;

(ii) \( \Phi \) is a \( \rho \)-spectral with respect to \( \gamma \);

(iii) \( \Phi \) is weak \( \rho \)-spectral with respect to \( \gamma \).

**Proof.** Since the implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) are trivial, it remains to prove the implication (iii) \( \Rightarrow \) (i).

Suppose that the statement (iii) holds and let \( \mu_x \) be a weak \( \rho \)-spectral attached to \( x \in H \) by \( \Phi \) and \( \gamma \). As \( \Phi \) is a \( m \) a.c. representation there exists a system \( \{ \nu_{x,y} \}_{x,y \in \mathcal{H}} \) of \( m \) a.c. elementary measures for \( \Phi \). If \( \nu_x = \nu_{x,x} \) (\( x \in \mathcal{H} \)) then it follows that

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that is $\nu_x - \rho \mu_x$ is orthogonal to $A_\gamma$. Now if $\mu_x = \mu_x^a + \mu_x^s$ is m decomposition of $\mu_x$ then by M. and F. Riesz theorem ([1] and [4]) one has that $\rho \mu_x^s$ is orthogonal to $A$, since $\mu_x^s \geq 0$ it results $\mu_x^a = 0$. Thus $\mu_x = \mu_x^a$ that is $\mu_x$ is m a.c. for any $x \in H$, and then by Theorem 1 the representation $\Phi$ has a $\gamma-$ spectral $\rho-$ dilation. This ends the proof.

From Theorem 1 we infer also the following

**Corollary 3.** Let $A$ be a function algebra on $X$ and $\xi \in X$ a peak point for $A$ such that $A$ is weak$^\ast-$ Dirichlet in $L^\infty(m)$ for some $m \in M_\xi$. Suppose that the Gleason part of $A$ containing $\xi$ is reduced to $\{\xi\}$. Then any m a.c. representation $\Phi$ of $A$ on $H$ which is weak $\rho-$ spectral with respect to $\xi$ is a contractive spectral representation. Moreover, we have

$$\Phi(f) = f(\xi) I \quad (f \in A).$$

**Proof.** Let $\Phi$ as above and $\{\nu_{x,y}\}_{x,y \in H}$ be a system of m a.c. measures for $\Phi$, where we denote $\nu_x = \nu_{x,x}$. Let also $\{\mu_x\}$ be a system of weak $\rho-$ spectral measures attached to the points $x \in H$ by $\Phi$ and $\xi$. If $\mu_x = \mu_x^a + \mu_x^s$ is the $M_\xi$ decomposition of $\mu_x$, then the $M_\xi$ decomposition of $\nu_x - \rho \mu_x$ is

$$\nu_x - \rho \mu_x = (\nu_x - \rho \mu_x^a) - \rho \mu_x^s$$

because $\nu_x$ being m a.c. it is also $M_\xi$ a.c. Since $\nu_x - \rho \mu_x$ is orthogonal to

$$A_\xi = \{f \in A : f(\xi) = 0\}$$

by M. and F. Riesz ([1] and [4]) we have that $\rho \mu_x^s$ is orthogonal to $A$, hence $\mu_x^s = 0$ because $\mu_x^s \geq 0$. Therefore $\mu_x = \mu_x^a$ is $M_\xi$ a.c. and also the measure

$$\nu_x - \rho \mu_x - (1 - \rho) \|x\|^2 m$$

is $M_\xi$ a.c. Since this measure is orthogonal to $A$ and by hypothesis $\xi$ is a peak point and $\{\xi\}$ is a Gleason part for $A$, from a result in [4] it follows that

$$\nu_x - \rho \mu_x - (1 - \rho) \|x\|^2 m = 0.$$

But this implies that $\mu_x$ is m a.c., for any $x \in H$ and by Theorem 1 there exists a m a.c. semispectral measure $F$ on $X$ satisfying

$$< \Phi(f) x, y > = \int (\rho f(\eta) + (1 - \rho) f(\xi)) d(F(\eta) x, y)$$

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for $f \in A$ and $x, y \in H$. Since $\xi$ is a peak point and $F$ is $m$ a.c. one infers that
\[
< \Phi(f)x, y > = (\rho f(\xi) + (1-\rho)f(\xi)) < F(\{\xi\})x, y > \\
= f(\xi)F(\{\xi\})x, y >
\]
and so $\Phi(f) = f(\xi)F(\{\xi\}), f \in A$. In particular it follows that $F(\{\xi\}) = I$ and consequently $\Phi(f) = f(\xi)I$, for $f \in A$. The proof is finished. \hfill \Box

Note that this corollary is a generalized version of the [3, Corollary 1] because our algebra $A$ is supposed to be weak*- Dirichlet in $L^\infty (m)$ and so that $m$ is not necessary the unique representing for the peak point $\xi$ for $A$. Also, we only assume that the representation $\Phi$ is weak $\rho-$ spectral with respect to $\xi$, a weaker condition than in [3], where $\Phi$ is $\rho$-spectral with respect to $\xi$. An example for which the above corollary can be applied is the following.

**Example.** Let $A_1 (T)$ be the algebra of all continuous functions on the unit circle $T$ which have analytic extensions $f$ to the open unit disc such that $f(0) = f(1)$. Then $A_1 (T)$ is a function algebra on $T$ which is weak*- Dirichlet in $L^\infty (m_1)$ where $m_1$ is the Haar measure on $T$. Clearly, the measure $m_1$, the Dirac measure $\delta_1$ which is supported in $\{1\}$ and also $\mu = \frac{1}{2} (m_1 + \delta_1)$ are representing measures for the homomorphism of evaluation at 1. But any point $\lambda \in T$ is a peak point for $A_1 (T)$ and the evaluation $e_\lambda$ at $\lambda \neq 1$ has a unique representing measure $m_\lambda$ relative to $A_1 (T)$. Also, $\{e_\lambda\}$ forms a Gleason part of $A_1 (T)$ for every $\lambda \in T$. Thus by Corollary 3 it follows that the only $m_\lambda$ a.e. representation of $A_1 (T)$ on $H$ which is weak $\rho-$ spectral with respect to $e_\lambda$ is $\Phi_\lambda$ given by $\Phi_\lambda(f) = f(\lambda)I$, $f \in A_1 (T)$, for any $\lambda \in T$.

Now we obtain in our context the following version of [3, Theorem 3].

**Theorem 4.** Let $A$ be a function algebra on $X$ which is weak*- Dirichlet in $L^\infty (m)$ for some $m \in M_+$ and $\gamma \in M (A)$. Suppose $\gamma' \in M (A)$ such that $\gamma'$ is not in the same Gleason part with $\gamma$. Then any $m$ a.e. representation $\Phi$ of $A$ on $H$ which is weak $\rho-$ spectral with respect to $\gamma'$ is a contractive and dilatate representation.

**Proof.** We use the idea from the proof of [3, Theorem 3]. Let $\Phi$ be a representation of $A$ on $H$ for which there exist a system $\{\nu_{x,y}\}_{x,y \in H}$ of $m$ a.e. elementary measures and a weak $\rho-$ spectral measure $\mu_x$ attached to every $x \in H$ by $\Phi$ and $\gamma'$. Putting $\nu_x = \nu_{x,x}, x \in H$ one has that $\nu_x - \rho \mu_x$ is orthogonal to $A_{\gamma'}$. If $\mu_x = \mu_x^a + \mu_x^s$ is the M. and F. Riesz theorem (1) and (4)) it follows that $\nu_x - \rho \mu_x^s$ is orthogonal to $A$, since $\nu_x$ being $m$ a.e. it is also $M_+$ a.e. and $\nu_x$ is $M_+$ s. because $\gamma$ and $\gamma'$ belong to different Gleason parts of $A$ (by [1, Theorem vi.2.2]).
Let now $\mu_x^s = \mu_x^{sa} + \mu_x^{ss}$ be the $M_7$ decomposition of $\mu_x^s$. Then applying also the M. and F. Riesz theorem we infer that the measures $\nu_x - \rho \mu_x^{sa}$ and $\rho \mu_x^{ss}$ are orthogonal to $A$, hence $\mu_x^{ss} = 0$ because $\mu_x^{sa} \geq 0$. Next, as $\nu_x$ is a Hahn - Banach extension to $C(X)$ of the functional $f \rightarrow <\Phi(f)x, x>$ on $A$, and since $\mu_x^{sa} \geq 0$ and $\int fd\nu_x = \rho \int fd\mu_x^{sa}$ we get

$$\|\nu_x\| = \sup_{f \in A} \left| \int f d\nu_x \right| = \rho \sup_{\|f\|_1=1} \left| \int f d\mu_x^{sa} \right| \leq \rho \|\mu_x^{sa}\|$$

whence

$$\|\nu_x\| = \nu_x(1) = \rho \|\mu_x^{sa}\|.$$  

This means that the measures $\nu_x$ are positive, for any $x \in H$.

Using the fact that $\nu_{x,y}$ are elementary measures for $\Phi$ we obtain for $x, x', y \in H$ and $\alpha, \beta \in C$ that

$$\int fd\nu_{x+\alpha x', y} = \int f (\alpha \nu_{x,y} + \beta \nu_{x', y}) \quad (f \in A).$$

But this implies by Gleason - Whitney theorem ([10]) that

$$\nu_{x+\alpha x', y} = \alpha \nu_{x,y} + \beta \nu_{x', y}$$

because the measures $\nu_{z,z'}$ are m.a.c. for any $z, z' \in H$. Similarly, one infers that

$$\nu_{x', \alpha x + \beta y} = \alpha \nu_{x', x} + \beta \nu_{x', y}.$$ 

Thus, for $\sigma \in \text{Bor}(X)$ the functional $(x, y) \rightarrow \nu_{x,y}(\sigma)$ is linear in $x \in H$, antilinear in $y \in H$ and also we have

$$|\nu_{x,y}(\sigma)| \leq \|\nu_{x,y}\| \leq \|\Phi\| \|x\| \|y\|$$

because $\nu_{x,y}$ is an elementary measure for $\Phi$. Hence we can define the map $F : \text{Bor}(X) \rightarrow B(H)$ by

$$< F(\sigma)x, y > = \nu_{x,y}(\sigma) \quad (\sigma \in \text{Bor}(X), x, y \in H)$$

and it is immediate that $F$ is a semispectral measure for $\Phi$. Finally, by the Naimark theorem ([9]), it follows that $\Phi$ has a spectral dilation, necessarily a contractive one. Consequently, $\Phi$ is a contractive representation, and the proof is finished.  

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As an application we have the following result which generalized [3, Theorem 3] because our hypothesis on $\Phi$ is weaker than the assumption from [3].

**Corollary 5.** Let $A$ be a function algebra on $X$ and $\gamma, \gamma' \in M(A)$ belonging to different Gleason parts of $A$, such that $\gamma$ has a unique representing measure. Then any $M_\gamma$ a.c. representation of $A$ on $H$ which is weak $\rho-$ spectral with respect to $\gamma'$ is a contractive representation and it has a spectral dilation.

**Proof.** If $M_\gamma = \{ m \}$ then $A$ is weak$^\ast-$ Dirichlet in $L^\infty(m)$. So we can apply Theorem 4 to any $m$ a.c. representation which is weak $\rho-$ spectral with respect to $\gamma'$ and the conclusion follows.

Finally, we prove the following

**Theorem 6.** Let $A$ be a function algebra on $X$ with the property that the only measure orthogonal to $A$ which is singular to all representing measures for the homomorphisms in $M(A)$ is the zero measure. Then every completely singular representation of $A$ on $H$ which is weak $\rho-$ spectral with respect to some $\gamma \in M(A)$ is a spectral one.

**Proof.** Let $\Phi$ and $\gamma$ as above, and for $x, y \in H$ let $\nu_{x,y}$ be a completely singular elementary measure for $\Phi$. If $\mu_x$ is a weak $\rho-$ spectral measure attached to $x \in H$ by $\Phi$ and $\gamma$ then $\nu_x - \rho \mu_x$ is orthogonal to $A_\gamma$. So, if $\mu_x = \mu_x^a + \mu_x^s$ is the $M_\gamma$ decomposition of $\mu_x$, by M. and F. Riesz theorem one has that $\nu_x - \rho \mu_x^s$ is orthogonal to $A$ because $\nu_x$ is also $M_\gamma$ s. (being completely singular). Next, as in the proof of Theorem 4 we deduce that $\nu_x$ is a positive measure. Also, using the hypothesis on $A$ we infer that the map $(x, y) \rightarrow \nu_{x,y}$ is linear in $x \in H$ and it is antilinear in $y \in H$. This leads (as in the proof of Theorem 1) to the fact $\Phi$ is a contractive representation which has a spectral dilation, and [3, Theorem 4] implies that $\Phi$ is even a spectral representation. This ends the proof.

**References**


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Politehnica University of Timișoara,
Romania.
e-mail: adinajuratoni@yahoo.com