Dessins d'enfants: bipartite maps and Galois groups

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Abstract. Belyĭ's Theorem implies that the Riemann surfaces defined over the field of algebraic numbers are precisely those which support bipartite maps; this provides a faithful representation of the Galois group of this field on these combinatorial objects.

My aim in this note is to show how combinatorics can play a central role in uniting such topics as Galois theory, algebraic number theory, Riemann surfaces, group theory and hyperbolic geometry. The relevant combinatorial objects are maps on surfaces, often called *dessins d'enfants* in view of the rather naïve appearance of some of the most common examples. For simplicity I will restrict attention to bipartite maps, though triangulations and hypermaps also play an important role in this theory. More detailed surveys can be found in [2, 5, 7], and for recent progress see [8].

A *bipartite map* $\mathcal{B}$ consists of a bipartite graph $\mathcal{G}$ imbedded (without crossings) in a compact, connected, oriented surface X, so that the faces (connected components of $X \setminus \mathcal{G}$) are simply connected. One can describe $\mathcal{B}$ by a pair of permutations g_0 and g_1 of its edge-set E: the vertices can be coloured black or white, so that each edge joins a black and a white vertex; the orientation of X then determines a cyclic ordering of the edges around each black or white vertex, and these are the disjoint cycles of g_0 and g_1 respectively. These two permutations generate a subgroup $G = \langle g_0, g_1 \rangle$ of the symmetric group S^E of all permutations of E, called the *monodromy group* of $\mathcal{B}$; the topological hypotheses imply that $\mathcal{G}$ has to be connected, so G acts transitively on E. Conversely, every 2-generator transitive group arises in this way from some bipartite map $\mathcal{B}$: the edges are the symbols permuted, the black and white vertices correspond to the cycles of the two generators g_0 and g_1, and the faces correspond to the cycles of $g_\infty = (g_0 g_1)^{-1}$. Isomorphism of maps (preserving orientation and vertex-colours) corresponds to conjugacy of pairs (g_0, g_1) in S^E, and the automorphism group of $\mathcal{B}$ can be identified with the centraliser of G in S^E, that is, the group of all permutations which commute with G.

(Historical note: A slight modification of these ideas allows one to describe any oriented map, whether bipartite or not, by a pair of permutations [6]. Although generally regarded as a modern development, this use of permutations can be traced back at least as far as Hamilton's construction of what we now call Hamiltonian cycles in the icosahedron [4].)

Every bipartite map $\mathcal{B}$ is a quotient of the *universal bipartite map* $\hat{\mathcal{B}}$, drawn on the upper half-plane $\mathcal{U} = \{ z \in \mathbf{C} \mid \mathrm{Im}(z) > 0 \}$; the vertices of $\hat{\mathcal{B}}$ are the rational numbers r/s (in reduced form) with s odd, coloured black or white as r is even or odd, and the edges are the hyperbolic geodesics (euclidean semi-circles) joining vertices r/s and x/y with $ry - sx = \pm 1$. The automorphisms of $\hat{\mathcal{B}}$ are the Möbius transformations

$$z \mapsto \frac{az+b}{cz+d} \qquad (a,b,c,d \in \mathbf{Z},\ ad-bc=1) \tag{$*$}$$

such that $a \equiv d \equiv 1$ and $b \equiv c \equiv 0 \bmod (2)$; these form a normal subgroup $\Gamma(2)$ of index 6 in the modular group $\Gamma = PSL_2(\mathbf{Z})$ of all transformations $(*)$, called the *principal congruence subgroup* of level 2 in Γ. Now $\Gamma(2)$ is a free group of rank 2, generated by the transformations

$$T_0 : z \mapsto \frac{z}{-2z+1} \qquad \text{and} \qquad T_1 : z \mapsto \frac{z-2}{2z-3}\,,$$

so there is an epimorphism $\theta : \Gamma(2) \to G$, $T_i \mapsto g_i$, which gives a transitive permutation representation of $\Gamma(2)$ on E. If B denotes the subgroup $\theta^{-1}(G_e)$ of $\Gamma(2)$ fixing an edge e of $\mathcal{B}$ then B acts as a group of automorphisms of $\hat{\mathcal{B}}$, and one can show that $\mathcal{B} \cong \hat{\mathcal{B}}/B$. The underlying surface of $\hat{\mathcal{B}}/B$ is now a compact Riemann surface $\overline{\mathcal{U}/B} = (\mathcal{U} \cup \mathbf{Q} \cup \{\infty\})/B$, formed by compactifying $\mathcal{U}/B$ with finitely many points, corresponding to the orbits of B on the extended rationals $\mathbf{Q} \cup \{\infty\}$. One can regard $\hat{\mathcal{B}}/B$ as a rigid, conformal model of $\mathcal{B}$, with a complex structure induced from that of $\mathcal{U}$: for instance, the edges are geodesics, the angles between edges around any vertex are equal, and the automorphisms of $\mathcal{B}$ are conformal isometries of the Riemann surface.

Riemann showed that a Riemann surface X is compact if and only if it is isomorphic to the Riemann surface of an algebraic curve $f(x,y) = 0$ for some polynomial $f(x,y) \in \mathbf{C}[x,y]$. Computationally and theoretically, the most satisfactory polynomials are those with coefficients in the field $\overline{\mathbf{Q}}$ of algebraic numbers; results of Belyĭ [1] and Weil [9] imply that the Riemann surfaces corresponding to such polynomials $f(x,y) \in \overline{\mathbf{Q}}[x,y]$ are those obtained from bipartite maps by the above method. More precisely, Belyĭ showed that a compact Riemann surface X is defined over $\overline{\mathbf{Q}}$ if and only if there is a Belyĭ function β from X to the Riemann sphere $\Sigma = \mathbf{C} \cup \{\infty\}$, that is, a meromorphic function on X which is unbranched over $\Sigma \setminus \{0,1,\infty\}$. In these circumstances, X is the underlying surface of the bipartite map $\mathcal{B} = \beta^{-1}(\mathcal{B}_1)$, where $\mathcal{B}_1$ is the trivial bipartite map $\hat{\mathcal{B}}/\Gamma(2)$ on Σ with a black vertex at 0, a white vertex at 1, and a single edge along the unit interval $[0,1]$. If each edge of $\mathcal{B}$ is identified with the sheet of the covering $\beta : X \to \Sigma$ which contains it, then the monodromy group G of $\mathcal{B}$ coincides with the monodromy group of β, regarded as a group of permutations of the sheets; in particular, the elements $g_0, g_1, g_\infty \in G$ describe how the sheets are permuted by lifting small loops in Σ around the branch-points $0, 1$ and ∞. Similarly, the automorphism group of $\mathcal{B}$ is identified with the group of covering transformations of β.

Since $\overline{\mathbf{Q}}$ is the union of the Galois (finite normal) extensions $K \geq \mathbf{Q}$ in $\mathbf{C}$, it follows that the *absolute Galois group* $\mathbf{G} = \mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ of $\overline{\mathbf{Q}}$ over $\mathbf{Q}$ is the projective limit of the finite Galois groups Gal $(K/\mathbf{Q})$ of these algebraic number fields; as such, it is an uncountable profinite group, which embodies the whole of classical Galois theory over $\mathbf{Q}$. This group $\mathbf{G}$ is of fundamental importance in several areas of mathematics: for instance, the representation theory of $\mathbf{G}$ played a crucial role in Wiles's proof of Fermat's Last Theorem [10], and the Inverse Galois Problem (Hilbert's still unproved conjecture that every finite group is a Galois group over $\mathbf{Q}$) is equivalent to showing that every finite group is an epimorphic image of $\mathbf{G}$. Fortunately, Belyĭ's Theorem provides us with an explicit realisation of $\mathbf{G}$ in terms of bipartite maps, which is beginning to add to our rather meagre knowledge of this complicated group.

In [3], Grothendieck showed that the natural action of $\mathbf{G}$ on polynomials over $\overline{\mathbf{Q}}$ induces an action of $\mathbf{G}$ on bipartite maps (and on other similar combinatorial objects, generally known as *dessins d'enfants*), through the above correspondence between maps and polynomials. Although $\mathbf{G}$ preserves such properties of a map as its genus, the numbers and valencies of its black and white vertices, its monodromy group and its automorphism group, this action of $\mathbf{G}$ is nevertheless faithful, in the sense that each non-identity element of $\mathbf{G}$ sends some bipartite map to a non-isomorphic bipartite map. Moreover, this action remains faithful even when restricted to such simple objects as plane trees (maps on the sphere with one face). One therefore has a combinatorial approach to Galois theory, which is attracting interest from a wide spectrum of mathematicians and theoretical physicists (for whom maps are an effective discrete approximation to the compact Riemann surfaces which play a major role in quantum gravity).

References

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