The Farey Graph

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This is joint work with David Singerman and Keith Wicks, subsequently published in [2]. The modular group

$$\Gamma = PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) / \{ \pm I \}$$

acts on the upper half-plane $$U = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$$ and on the rational projective line $$\hat{\mathbb{Q}} = \mathbb{Q} \cup \{ \infty \}$$ as a group of Möbius transformations

$$z \mapsto \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbb{Z}, \, ad - bc = 1).$$

Its action on $$\hat{\mathbb{Q}}$$ is transitive but imprimitive: for each positive integer $$N \neq 2, 5$$ there is a $$\Gamma$$-invariant equivalence relation on $$\hat{\mathbb{Q}}$$ with $$N$$ equivalence classes. We study the action of $$\Gamma$$ on $$\hat{\mathbb{Q}}$$ by using suborbital graphs (introduced in 1967 by Sims [3] for finite permutation groups). These are $$\Gamma$$-invariant directed graphs with vertex-set $$\hat{\mathbb{Q}}$$, their edge-sets being the orbits of $$\Gamma$$ on the cartesian square $$\hat{\mathbb{Q}}^2$$. Apart from the trivial case, corresponding to the diagonal orbit, there is one suborbital graph $$\mathcal{G}_{u,n}$$ for each integer $$n \geq 1$$ and for each of the $$\phi(n)$$ units $$u \mod (n)$$: its edge-set is the orbit containing the pair $$(\infty, u/n)$$. Reversing edges induces a pairing of suborbital graphs, in which $$\mathcal{G}_{u,n}$$ is paired with $$\mathcal{G}_{v,n}$$ where $$uv \equiv -1 \mod (n)$$; thus $$\mathcal{G}_{u,n}$$ is self-paired (and can be represented as an undirected graph) if and only if $$u^2 \equiv -1 \mod (n)$$.

The simplest example is the Farey graph $$\mathcal{F} = \mathcal{G}_{1,1}$$: the vertex $$\infty$$ is joined to the integers, while two rational numbers $$r/s$$ and $$x/y$$ (in reduced form) are adjacent in $$\mathcal{F}$$ if and only if $$ry - sx = \pm 1$$, or equivalently if they are consecutive terms in some Farey sequence $$F_m$$ (consisting of the rationals $$x/y$$ with $$|y| \leq m$$, arranged in increasing order). If we draw the edges of $$\mathcal{F}$$ as hyperbolic geodesics in $$\mathcal{U}$$ (euclidean semicircles and half-lines), they do not cross, so we have an embedding of $$\mathcal{F}$$; the faces are hyperbolic triangles, giving a triangulation $$\mathcal{T}$$ of $$\mathcal{U}$$ with ‘ideal vertices’ on the boundary. Both $$\mathcal{F}$$ and $$\mathcal{T}$$ have automorphism group $$PGL(2, \mathbb{Z})$$, which contains $$\Gamma$$ as its orientation-preserving subgroup of index 2. The triangulation $$\mathcal{T}$$ acts as a universal object for triangular maps, each of which is isomorphic to a quotient $$\mathcal{T}/M$$ for some subgroup $$M$$ of $$PGL(2, \mathbb{Z})$$. It follows from Belyi’s Theorem [1] that the Riemann surfaces defined as algebraic curves over the field $$\overline{\mathbb{Q}}$$ of algebraic numbers are those obtained in this way from compact orientable triangular maps, that is, they are the compactifications of the surfaces $$\mathcal{U}/M$$ where $$M$$ has finite index in $$\Gamma$$. 

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The Farey graph $G_{1,1}$ is connected, but if $n > 1$ then $G_{u,n}$ is a disjoint union of

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$$

subgraphs (where $p$ ranges over the distinct primes dividing $n$): their vertex-sets are the equivalence classes in $\mathbb{Q}$ where we define $r/s \equiv x/y$ if and only if $ry - sx \equiv 0 \mod (n)$. For a given $n$ these subgraphs are permuted transitively by $\Gamma$, so they are all isomorphic to the subgraph $\mathcal{F}_{u,n}$ containing $\infty$, consisting of the vertices $x/y$ with $y \equiv 0 \mod(n)$. This subgraph is connected if and only if $n \leq 4$. Each $\mathcal{F}_{u,n}$ is embedded in $U$ to give a tessellation $T_{u,n}$: for instance $T_{1,2}$ is the universal map [4], in the sense that every map is isomorphic to a quotient of $T_{1,2}$ by some group of automorphisms.

$G_{u,n}$ contains directed triangles if and only if $u^2 \pm u + 1 \equiv 0 \mod(n)$, a typical example being $\infty \rightarrow u/n \rightarrow (u \pm 1)/n \rightarrow \infty$; however, only $G_{1,1} = \mathcal{F}$ contains anti-directed triangles, such as $\infty \rightarrow 1 \leftarrow 2 \rightarrow \infty$. For $n > 1$, $G_{u,n}$ is a forest if it is self-paired or if $n$ is even. We conjecture that $G_{u,n}$ is a forest if and only if it contains no triangles, that is, if and only if $u^2 \pm u + 1 \not\equiv 0 \mod(n)$. (This conjecture has subsequently been proved by Mehmet Akbas.)


