On Free Field Realizations of $W(2, 2)$-Modules

Dražen ADAMOVIĆ† and Gordan RADOBOLJA‡

† Department of Mathematics, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia
E-mail: adamovic@math.hr
URL: https://web.math.pmf.unizg.hr/~adamovic/
‡ Faculty of Science, University of Split, Rudera Boškovića 33, 21 000 Split, Croatia
E-mail: gordan@pmfst.hr

Received June 09, 2016, in final form December 03, 2016; Published online December 06, 2016
http://dx.doi.org/10.3842/SIGMA.2016.113

Abstract. The aim of the paper is to study modules for the twisted Heisenberg–Virasoro algebra $\mathcal{H}$ at level zero as modules for the $W(2, 2)$-algebra by using construction from [J. Pure Appl. Algebra 219 (2015), 4322–4342, arXiv:1405.1707]. We prove that the irreducible highest weight $\mathcal{H}$-module is irreducible as $W(2, 2)$-module if and only if it has a typical highest weight. Finally, we construct a screening operator acting on the Heisenberg–Virasoro vertex algebra whose kernel is exactly $W(2, 2)$ vertex algebra.

Key words: Heisenberg–Virasoro Lie algebra; vertex algebra; $W(2, 2)$ algebra; screening-operators

2010 Mathematics Subject Classification: 17B69; 17B67; 17B68; 81R10

1 Introduction

Lie algebra $W(2, 2)$ was first introduced by W. Zhang and C. Dong in [20] as part of a classification of certain simple vertex operator algebras. Its representation theory has been studied in [14, 15, 18, 19] and several other papers. Although $W(2, 2)$ is an extension of the Virasoro algebra, its representation theory is very different. This is most notable with highest weight representations. It was shown in [19] that some Verma modules contain a cosingular vector.

Highest weight representation theory of the twisted Heisenberg–Virasoro Lie algebra has also been studied recently. Representations with nontrivial action of $C_I$ have been developed in [6]. Representations at level zero, i.e., with trivial action of $C_I$ were studied in [8] due to their importance in some constructions over the toroidal Lie algebras (see [7, 9]). In this case, a free field realization of highest weight modules along with the fusion rules for a suitable category of modules were obtained in [4].

Irreducible highest weight modules of highest weights $(0, 0)$ over these algebras carry the structure of simple vertex operator algebras. Let us denote these vertex operator algebras as $L^{W(2, 2)}(c_L, c_W)$ and $L^{H}(c_L, c_{L,I})$. It was proved in [4] that simple vertex operator algebra $L^{W(2, 2)}(c_L, c_W)$ embeds into Heisenberg–Virasoro vertex operator algebra $L^{H}(c_L, c_{L,I})$ so that $c_W = -24c_{L,I}^2$. As a result each highest weight module over $\mathcal{H}$ is also a $W(2, 2)$-module.

In this paper we shall completely describe the structure of the irreducible highest weight $\mathcal{H}$-modules as $W(2, 2)$-modules. We show that in generic case the resulting $W(2, 2)$-module is irreducible. However, in case of a module of highest weight such that associated Verma module over $W(2, 2)$ contains cosingular vectors (we shall call this kind of weight atypical), irreducible $\mathcal{H}$-module is reducible over $W(2, 2)$. We shall denote the irreducible highest weight $\mathcal{H}$-module
Our construction uses an extension \( L^\mathcal{H}(c_L, 0, c_L, I, h, h_I) \) shortly as \( L^\mathcal{H}(h, h_I) \). We also use the following notation\(^1\)

\[
h_{p,r} = \frac{(1 - p^2) c_L - 2}{24} + p(p - 1) + \frac{1 - r}{2}
\]

for \( p, r \in \mathbb{Z}_{>0} \). Define

\[
\mathcal{A} \mathcal{T}_\mathcal{H}(c_L, c_L, I) = \{(h, (1 \pm p)c_L) | p, r \in \mathbb{Z}_{>0}\}.
\]

We call a weight \((h, h_I)\) atypical for \( \mathcal{H} \) (resp. typical) if \((h, h_I) \in \mathcal{A} \mathcal{T}_\mathcal{H}(c_L, c_L, I)\) (resp. \((h, h_I) \notin \mathcal{A} \mathcal{T}_\mathcal{H}(c_L, c_L, I)\)). We shall refer to a highest weight module over \( \mathcal{H} \) as (a)typical if its highest weight is (a)typical for \( \mathcal{H} \).

The next theorem gives a main result of the paper.

**Theorem 1.1.** Assume that \( c_L, I \neq 0 \).

1. \( L^\mathcal{H}(h, h_I) \) is irreducible as a \( W(2, 2) \)-module if and only if

   \[
   (h, h_I) \notin \mathcal{A} \mathcal{T}_\mathcal{H}(c_L, c_L, I).
   \]

2. If \((h, h_I) \in \mathcal{A} \mathcal{T}_\mathcal{H}(c_L, c_L, I)\) then \( L^\mathcal{H}(h, h_I) \) is a non-split extension of two irreducible highest weight \( W(2, 2) \)-modules.

We recall some aspects of representation theories of infinite-dimensional Lie algebras \( \mathcal{H} \) and \( W(2, 2) \) in Section 2. The main results on the branching rules will be proved in Section 3. From the free field realization in [4] follows that irreducible \( \mathcal{H} \)-modules are pairweise contragredient. For half of these modules, proofs rely on a \( W(2, 2) \)-homomorphism between Verma modules over \( W(2, 2) \) and \( \mathcal{H} \) which is induced by a homomorphism of vertex operator algebras. The rest is then proved elegantly by passing to contragredients. We also prove a very interesting result that the Verma module for \( \mathcal{H} \) with typical highest weight is an infinite direct sum of irreducible \( W(2, 2) \)-modules (cf. Theorem 3.7). This result presents a \( W(2, 2) \)-analogue of certain Feigin–Fuchs modules for the Virasoro algebra (cf. Remark 3.8).

From the results in the paper, we see that the vertex algebra \( L^{W(2,2)}(c_L, c_W) \) has many properties similar to the \( W \)-algebras appearing in logarithmic conformal field theory (LCFT):

- \( L^{W(2,2)}(c_L, c_W) \) admits a free field realization inside of the Heisenberg–Virasoro vertex algebra \( L^\mathcal{H}(c_L, c_L, I) \).
- Typical modules are realized as irreducible modules for \( L^\mathcal{H}(c_L, c_L, I) \).
- In the atypical case, irreducible \( L^\mathcal{H}(c_L, c_L, I) \)-modules as \( L^{W(2,2)}(c_L, c_W) \)-modules have semi–simple rank two.

The singlet vertex algebra \( M(1) \) has similar properties. \( M(1) \) is realized as kernel of a screening operator inside the Heisenberg vertex algebra \( M(1) \) (cf. [1]). In Section 4 we construct the screening operator

\[
S_1: L^\mathcal{H}(c_L, c_L, I) \rightarrow L^\mathcal{H}(1, 0),
\]

which commutes with the action of \( W(2, 2) \)-algebra such that

\[
\text{Ker}_{L^\mathcal{H}(c_L, c_L, I)} S_1 \cong L^{W(2,2)}(c_L, c_W).
\]

Our construction uses an extension \( \mathcal{V}_\text{ext} \) of the vertex algebra \( L^\mathcal{H}(c_L, c_L, I) \) by a non-weight module for the Heisenberg–Virasoro vertex algebra. In our forthcoming paper [5], we shall present an explicit realization of \( \mathcal{V}_\text{ext} \) and apply this construction to the study of intertwining operators and logarithmic modules.

\(^1\)We emphasise a term \( \frac{c_L - 2}{24} \) for its importance in a free field realization of \( \mathcal{H} \) (see [4] for details).
2 Lie algebra $W(2, 2)$ and the twisted Heisenberg–Virasoro

Lie algebra at level zero

$W(2, 2)$ is a Lie algebra with basis $\{L(n), W(n), C_L, C_W : n \in \mathbb{Z}\}$ over $\mathbb{C}$, and a Lie bracket

$$[L(n), L(m)] = (n - m) L(n + m) + \delta_{n, -m} \frac{n^3 - n}{12} C_L,$$

$$[L(n), W(m)] = (n - m) W(n + m) + \delta_{n, -m} \frac{n^3 - n}{12} C_W,$$

$$[W(n), W(m)] = [\cdot, C_L] = [\cdot, C_W] = 0.$$

Highest weight representation theory over $W(2, 2)$ was studied in [14, 19]. However, representations treated in these papers have equal central charges $C_L = C_W$. These results have recently been generalised to $C_L \neq C_W$ in [15]. Here we state the most important results. Verma module with central charge $(c_L, c_W)$ and highest weight $(h, h_W)$ is denoted by $V_W^{(2, 2)}(c_L, c_W, h, h_W)$, its highest weight vector by $v_{h, h_W}$ and irreducible quotient module by $L_W^{(2, 2)}(c_L, c_W, h, h_W)$.

Recall the definition of a cosingular vector. Homogeneous vector $v \in M$ is called cosingular (or subsingular) if it is not singular in $M$ and if there is a proper submodule $N \subset M$ such that $v + N$ is a singular vector in $M/N$.

**Theorem 2.1** ([15, 19]). Let $c_W \neq 0$.

(i) Verma module $V_W^{(2, 2)}(c_L, c_W, h, h_W)$ is reducible if and only if $h_W = \frac{1-p^2}{24} c_W$ for some $p \in \mathbb{Z}_{>0}$. In that case, there exists a singular vector $u_p' \in \mathbb{C}[W(-1), \ldots, W(-p)]v_{h, h_W}$ such that $U(W(2, 2))u_p' \cong V_W^{(2, 2)}(c_L, c_W, h + p, h_W)$.

(ii) A quotient module

$$V_W^{(2, 2)}(c_L, c_W, h, h_W)/U(W(2, 2))u_p' =: \tilde{L}_W^{(2, 2)}(c_L, c_W, h_p, r, h_W)$$

is reducible if and only if $h = h_p, r$ for some $r \in \mathbb{Z}_{>0}$. In that case, there is a cosingular vector $u_p \in V_W^{(2, 2)}(c_L, c_W, h, h_W)_{h = h_p, r}$ such that $u_p := u_p + U(W(2, 2))u_p'$ is a singular vector in $\tilde{L}_W^{(2, 2)}(c_L, c_W, h_p, r, h_W)$ which generates a submodule isomorphic to $L_W^{(2, 2)}(c_L, c_W, h_p, r, h_W)$. The short sequence

$$0 \rightarrow L_W^{(2, 2)}(c_L, c_W, h_p, r, h_W) \rightarrow \tilde{L}_W^{(2, 2)}(c_L, c_W, h_p, r, h_W) \rightarrow 0,$$

where the highest weight vector in $L_W^{(2, 2)}(c_L, c_W, h_p, r, h_W)$ maps to $u_p$ is exact. Define

$$\mathcal{AT}_{W(2, 2)}(c_L, c_W) = \left\{ \left( h_p, \frac{1-p^2}{24} c_W \right) \mid p, r \in \mathbb{Z}_{>0} \right\}.$$

**Remark 2.2.** We will refer to the (modules of) highest weights $(h, h_W) \in \mathcal{AT}_{W(2, 2)}(c_L, c_W)$ as *atypical* for $W(2, 2)$, and otherwise as *typical*. Again, we refer to a highest weight $W(2, 2)$-module as (a)typical depending on its highest weight. So a Verma module over $W(2, 2)$ contains a nontrivial cosingular vector if and only if it is atypical.

**Proposition 2.3.** Let $h_W = \frac{1-p^2}{24} c_W$, $p \in \mathbb{Z}_{>0}$.

\[2\text{This module is denoted by } L' \text{ in [15, 19]. We change notation to } \tilde{L} \text{ due to use of superscript } W(2, 2).\]
(i) Let \((h_{p,r}, h_W), r \in \mathbb{Z}_{>0}\) be an atypical weight and \(k \in \mathbb{Z}\). Then \((h_{p,r} + kp, h_W)\) is atypical if and only if \(k < \frac{r}{2}\).

(ii) Atypical Verma module \(V^{W(2,2)}(h_{p,r}, h_W)\) contains exactly \(\left\lfloor \frac{r+1}{2} \right\rfloor\) cosingular vectors. The weights of these vectors are \(h_{p,r} + (r - i)p = h_{p,r+2i}, i = 0, \ldots, \left\lfloor \frac{r+1}{2} \right\rfloor\).

**Proof.** (i) Directly from Theorem 2.1 since \(h_{p,r} + kp = h_{p,r-2k}\).

(ii) Follows from (i) since \(V^{W(2,2)}(h_{p,r}, h_W)\) contains an infinite chain of submodules isomorphic to Verma modules of highest weights \(h_{p,r} + ip = h_{p,r-2i}, i > 0\). Applying Theorem 2.1 to each of these submodules we obtain cosingular vectors of weights

\[ h_{p,r-2i} + (r - 2i)p = h_{p,r} + (r - i)p = h_{p,r+2i} \]
as long as \(r - 2i > 0\).

**Remark 2.4.** Standard PBW basis for \(V^{W(2,2)}(c_L, c_W, h, h_W)\) consists of vectors

\[ W(-m_s) \cdots W(-m_1)L(-n_t) \cdots L(-n_1)\text{v}_{h,h_W} \]
such that \(m_s \geq \cdots \geq m_1 \geq 1, n_t \geq \cdots \geq n_1 \geq 1\). The only nonzero component of \(u_{r,p}\) belonging to \(\mathbb{C}[L(-1), L(-2), \ldots]v\) is \(L(-p)^r\text{v}_{h,h_W}\) [19].

Define \(P_2(n) = \sum_{i=0}^{n} P(n - i)P(i)\) where \(P\) is a partition function with \(P(0) = 1\). We have the following character formulas [19]

\[ \text{char} V^{W(2,2)}(c_L, c_W, h, h_W) = q^h \sum_{n \geq 0} P_2(n)q^n = q^h \prod_{k \geq 1} (1 - q^k)^{-2}, \]

for all \(h, h_W \in \mathbb{C}\). If \(h_W = \frac{1-p^2}{2}c_W\), then

\[ \text{char} \bar{L}^{W(2,2)}(c_L, c_W, h, h_W) = q^h (1 - q^p) \sum_{n \geq 0} P_2(n)q^n = q^h (1 - q^p) \prod_{k \geq 1} (1 - q^k)^{-2}. \]

If \((h, h_W)\) is typical for \(W(2,2)\), then this is the character of an irreducible highest weight module. Finally, the character of atypical irreducible module is

\[ \text{char} \bar{L}^{W(2,2)}(c_L, c_W, h_{p,r}, h_W) = q^{h_{p,r}} (1 - q^p) \prod_{n \geq 0} P_2(n)q^n \]

\[ = q^{h_{p,r}} (1 - q^p) \prod_{k \geq 1} (1 - q^k)^{-2}. \]

The twisted Heisenberg–Virasoro algebra \(\mathcal{H}\) is the universal central extension of the Lie algebra of differential operators on a circle of order at most one. It is the infinite-dimensional complex Lie algebra with a basis

\[ \{L(n), I(n) : n \in \mathbb{Z}\} \cup \{C_L, C_{LI}, C_I\} \]

and commutation relations

\[ [L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m} \frac{n^3-n}{12} C_L, \]

\[ [L(n), I(m)] = -mI(n+m) - \delta_{n,-m}(n^2+n) C_{LI}, \]

\[ [I(n), I(m)] = n\delta_{n,-m} C_I, \quad [\mathcal{H}, C_L] = [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0. \]
The Lie algebra $\mathcal{H}$ admits the following triangular decomposition

$$
\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^0 \oplus \mathcal{H}^+,
$$

where

$$
\mathcal{H}^\pm = \text{span}_\mathbb{C}\{I(\pm n), L(\pm n) | n \in \mathbb{Z}_{>0}\}, \quad \mathcal{H}^0 = \text{span}_\mathbb{C}\{I(0), L(0), C_L, C_{L,I}, C_I\}.
$$

Although they seem to be two similar extensions of the Virasoro algebra, representation theories of $W(2,2)$ and $\mathcal{H}$ are different. The main reason for that lies in the fact that $I(0)$ is a central element, while $W(0)$ is not. However, applying free field realization, we shall see that highest weight modules over the two algebras are related.

Denote by $V^\mathcal{H}(c_L, c_{L,I}, h, h_I)$ the Verma module and by $v_{h,h_I}$ its highest weight vector. $C_L, C_I, C_{L,I}, L(0)$ and $I(0)$ act on $v_{h,h_I}$ by scalars $c_L, c_I, c_{L,I}, h$ and $h_I$, respectively. Then $(c_L, c_{L,I})$ is called a central charge, and $(h, h_I)$ a highest weight. In this paper we consider central charges $(c_L, 0, c_{L,I})$ such that $c_{L,I} \neq 0$.

**Theorem 2.5** ([8]). Let $c_{L,I} \neq 0$. Verma module $V^\mathcal{H}(c_L, 0, c_{L,I}, h, h_I)$ is reducible if and only if $h_I = (1 \pm p)c_{L,I}$ for some $p \in \mathbb{Z}_{>0}$. In that case, there is a singular vector $v_p^\pm$ of weight $p$, which generates a maximal submodule in $V^\mathcal{H}(c_L, 0, c_{L,I}, h, h_I)$ isomorphic to $V^\mathcal{H}(c_L, 0, c_{L,I}, h + p, h_I)$.

**Remark 2.6.** In case $h_I = (1 + p)c_{L,I}$ an explicit formula for a singular vector $v_p^\pm$ is obtained using Schur polynomials in $I(-1), \ldots, I(-p)$. See [4] for details. Assume that $x \in U(W(2,2))_-$ is such that $xv_{h,h_I} \in V^\mathcal{H}(c_L, 0, c_{L,I}, h, h_I)$ lies in a maximal submodule. Then $x$ does not have a nontrivial additive component (in PBW basis) that belongs to $\mathbb{C}[L(-1), L(-2), \ldots]$ [8].

There is an infinite chain of Verma submodules generated by singular vectors $v_k^{\pm}, k \in \mathbb{Z}_{>0}$, with all the subquotients being irreducible. Note that there is no mention of $\tilde{L}\mathcal{H}$ since there are no cosingular vectors in $V^\mathcal{H}$.

The following character formulas were obtained in [8]:

$$
\text{char } V^\mathcal{H}(c_L, 0, c_{L,I}, h, h_I) = q^h \sum_{n \geq 0} P_2(n) q^n = q^h \prod_{k \geq 1} (1 - q^k)^{-2},
$$

$$
\text{char } L^\mathcal{H}(c_L, 0, c_{L,I}, h, h_I) = q^h (1 - q^h) \sum_{n \geq 0} P_2(n) q^n = q^h (1 - q^h) \prod_{k \geq 1} (1 - q^k)^{-2}.
$$

**Remark 2.7.** Throughout the rest of the paper we work with highest weight modules over the Lie algebras $W(2,2)$ and $\mathcal{H}$ so we always denote algebra in superscript. In order to avoid too cumbersome notation, we omit central charges. Therefore, we write $V^{\mathcal{H}}(h, h_I)$ for Verma module over $\mathcal{H}$, $V^{W(2,2)}(h, h_W)$ for Verma module over $W(2,2)$ and so on. We always assume that $c_W$ and $c_{L,I}$ are nonzero. Moreover, if we work with several modules over both algebras, $c_L$ is equal for all modules.

We shall write $(x)_{W(2,2)}$ for a cyclic submodule $U(W(2,2))x$ and $(x)_{\mathcal{H}}$ for $U(\mathcal{H})x$. Finally, $\cong_{W(2,2)}$ denotes an isomorphism of $W(2,2)$-modules.

### 3 Irreducible highest weight modules

In this section we present main results of the paper which completely describe the structure of (irreducible) highest weight modules for $\mathcal{H}$ as $W(2,2)$-modules. The main tool is the homomorphism between $W(2,2)$ and the Heisenberg–Virasoro vertex algebras from [4].

$L^{W(2,2)}(c_L, c_W, 0, 0)$ is a simple universal vertex algebra associated to Lie algebra $W(2,2)$ (cf. [19, 20]) which we denote by $L^{W(2,2)}(c_L, c_W)$. It is generated by fields

$$
L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \quad W(z) = Y(W, z) = \sum_{n \in \mathbb{Z}} W(n) z^{-n-2},
$$
where \( \omega = L(-2)1 \) and \( W = W(-2)1 \). Each highest weight \( W(2,2) \)-module is also a module over a vertex operator algebra \( L^{W(2,2)}(c_L, c_W) \).

Likewise (see [7]) \( L^H(c_L, 0, c_{L,I}, 0, 0) \) is a simple Heisenberg–Virasoro vertex operator algebra, which we denote by \( L^H(c_L, c_{L,I}) \). This algebra is generated by the fields
\[
L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n - 2}, \quad I(z) = Y(I, z) = \sum_{n \in \mathbb{Z}} I(n) z^{-n - 1},
\]
where \( \omega = L(-2)1 \) and \( I = I(-1)1 \). Moreover, highest weight \( H \)-modules are modules over a vertex operator algebra \( L^H(c_L, c_{L,I}) \).

It was shown in [4] that there is a monomorphism of vertex operator algebras
\[
\Psi: L^{W(2,2)}(c_L, c_W) \to L^H(c_L, c_{L,I}),
\]
where \( c_W = -24c_{L,I}^2 \). By means of \( \Psi \), each highest weight module over \( H \) becomes an \( L^{W(2,2)}(c_L, c_W) \)-module and therefore a module over \( W(2,2) \). In particular, \( \Psi \) induces a non-trivial \( W(2,2) \)-homomorphism (which we shall denote by the same letter)
\[
\Psi: V^{W(2,2)}(c_L, c_W, h, h_W) \to V^H(c_L, 0, c_{L,I}, h, h_I),
\]
where \( c_W = -24c_{L,I}^2 \) and \( h_W = h_I(h_I - 2c_{L,I}) \). \( \Psi \) maps the highest weight vector \( v_{h,h_W} \) to the highest weight vector \( v_{h,h_I} \) and the action of \( W(-n) \) on \( V^H(c_L, 0, c_{L,I}, h, h_I) \) is given by
\[
W(-n) \equiv 2c_{L,I}(n - 1)I(-n) + \sum_{i \in \mathbb{Z}} I(-i)I(-n + i),
\]
\[
W(-n) \equiv 2c_{L,I} \left( n - 1 + \frac{h_I}{c_{L,I}} \right) I(-n) + \sum_{i \neq 0, n} I(-i)I(-n + i).
\]

Note that \( h_W = \frac{1-p^2}{24} c_W \) if and only if \( h_I = (1 \pm p)c_{L,I} \), so either both of these Verma modules are irreducible, or they are reducible with singular vectors at equal levels. Moreover, \( (h, h_W) \in AT_{W(2,2)}(c_L, c_W) \) if and only if \( (h, h_I) \in AT_H(c_L, c_{L,I}) \).

Throughout the rest of this section we assume that \( c_W = -24c_{L,I}^2 \).

**Lemma 3.1** ([4, Lemma 7.2]). Suppose that \( h_I \neq (1 - p)c_{L,I} \) for all \( p \in \mathbb{Z}_{>0} \). Then \( \Psi \) is an isomorphism of \( W(2,2) \)-modules. In particular, if \( h_I \neq (1 \pm p)c_{L,I} \) for \( p \in \mathbb{Z}_{>0} \), then
\[
L^H(h, h_I) \cong W^{(2,2)}(h, h_W),
\]
where \( h_W = h_I(h_I - 2c_{L,I}) \).

**Lemma 3.2.** Suppose that \( x \in V^H(h, h_I) \) is \( H \)-singular. Then \( x \) is \( W(2,2) \)-singular as well. In particular, if \( y \) is a homogeneous vector such that \( x = \Psi(y) \), then \( y \) is either singular or cosingular vector in \( V^{W(2,2)}(h, h_W) \).

**Proof.** Let \( x \in V^H(h, h_I) \) be a \( H \)-singular vector, i.e., \( L(k)x = I(k)x = 0 \) for all \( k \in \mathbb{Z}_{>0} \). From (3.2) we have
\[
W(n)x = -2c_{L,I}(n + 1)I(n)x + \sum_{i \in \mathbb{Z}} I(-i)I(n + i)x,
\]
so \( W(n)x = 0 \) for all \( n \in \mathbb{Z}_{>0} \). Therefore, \( x \) is \( W(2,2) \)-singular. If \( x = \Psi(y) \), then \( L(k)y, W(k)y \in \ker \Psi \) for \( k > 0 \). Therefore \( y + \ker \Psi \) is a singular vector in \( V^{W(2,2)}(h, h_W)/\ker \Psi \). \( \blacksquare \)
Theorem 3.3. Let \( p \in \mathbb{Z}_{>0} \).

(i) If \((h, (1 + p)c_{L, I})\) is typical for \( \mathcal{H} \) (equivalently if \((h, \frac{1-p^2}{24} c_W)\) is typical for \(W(2,2)\)) then
\[
L^H(h, (1 + p)c_{L, I}) \cong L^W_{W(2,2)}(h, \frac{1-p^2}{24} c_W).
\]
(3.3)

(ii) If \((h_{p, r}, (1 + p)c_{L, I}) \in \mathcal{AT}(c_{L}, c_{L, I})\) (equivalently if \((h_{p, r}, \frac{1-p^2}{24} c_W) \in \mathcal{AT}_{W(2,2)}(c_{L}, c_W)\)) then
\[
L^H(h_{p, r}, (1 + p)c_{L, I}) \cong L^W_{W(2,2)}(h_{p, r}, \frac{1-p^2}{24} c_W)
\]
and the short sequence of \(W(2,2)\)-modules
\[
0 \rightarrow L^W_{W(2,2)}(h_{p, r} + rp, \frac{1-p^2}{24} c_W) \rightarrow L^H(h_{p, r}, (1 + p)c_{L, I})
\]
\[
\rightarrow L^W_{W(2,2)}(h_{p, r}, \frac{1-p^2}{24} c_W) \rightarrow 0
\]
is exact.

Proof. By Lemma 3.1, \( \Psi \) is an isomorphism of Verma modules and thus by Lemma 3.2 it maps a \(W(2,2)\)-singular vector \( u_p^\ast \) to an \( \mathcal{H} \)-singular vector \( \Psi(u_p^\ast) \). If \( h \neq h_{p, r} \), both of these vectors generate maximal submodules in respective Verma modules so (3.3) follows.

Now suppose that \( h = h_{p, r} \). We need to show that a cosingular vector \( u_{rp} \) is not mapped into a maximal submodule of \( V^H(h_{p, r}, h_I) \). But \( u_{rp} \) has \( L(-p)^i v \) as an additive component (see Remark 2.4), and by construction (3.1), \( \Psi(u_{rp}) \) also must have this additive component. However, \( \Psi(u_{rp}) \) can not lie in a maximal \( \mathcal{H} \)-submodule of \( V^H(h, h_I) \) (see Remark 2.6). This means that isomorphism \( \Psi \) of Verma modules induces a \( W(2,2) \)-isomorphism of \( \tilde{L}^W_{W(2,2)}(h, h_W) \) and \( L^H(h, h_I) \) for all \( h \in \mathbb{C} \). Exactness of (3.4) is just an application of (2.1).

Remark 3.4. Note that the image \( \Psi(u_{rp}) \) of a \(W(2,2)\)-cosingular vector is neither \( \mathcal{H} \)-singular, nor \( \mathcal{H} \)-cosingular in \( V^H(h_{p, r}, (1 + p)c_{L, I}) \). For example, \( L(-1)v_{0,0} \) in \( V^H(0, 2c_{L, I}) \) is \(W(2,2)\)-cosingular, but not \( \mathcal{H} \)-singular since \( I(1)L(-1)v_{0,0} = 2c_{L, I}v_{0,0} \).

If \( h_I = (1 - p)c_{L, I} \), then \( \Psi \) is not an isomorphism. We shall present a \(W(2,2)\)-structure of Verma module later. In order to examine irreducible \(W(2,2)\)-modules we apply the properties of contragredient modules.

Let us recall the definition of contragredient module (see [12]). Assume that \((M, Y_M)\) is a graded module over a vertex operator algebra \( V \) such that \( M = \bigoplus_{n=0}^\infty M(n) \), \( \dim M(n) < \infty \) and suppose that there is \( \gamma \in \mathbb{C} \) such that \( L(0)M(n) \equiv (\gamma + n) M(n) \). The contragredient module \((M^*, Y_{M^*})\) is defined as follows. For every \( n \in \mathbb{Z}_{>0} \) let \( M(n)^* \) be the dual vector space and let \( M^* = \bigoplus_{n=0}^\infty M(n)^* \) be a restricted dual of \( M \). Consider the natural pairing \( \langle \cdot, \cdot \rangle : M^* \otimes M \rightarrow \mathbb{C} \). Define the linear map \( Y_{M^*} : V \rightarrow \text{End} M^*[z, z^{-1}] \) such that
\[
\langle Y_{M^*}(v, z)m^\prime, m \rangle = \langle m^\prime, Y_M(e^{zL(1)}(-z^{-2})L(0)v, z^{-1})m \rangle
\]
(3.5)
for each \( v \in V, m \in M, m^\prime \in M^* \). Then \((M^*, Y_{M^*})\) is a \( V \)-module.

In particular, choosing \( v = \omega = L_{-2}1 \) in (3.5) one gets
\[
\langle L(n)m^\prime, m \rangle = \langle m^\prime, L(-n)m \rangle.
\]
Simple calculation with \( I \in L^H(c_{L}, c_{L, I}) \) and \( W \in L^W_{W(2,2)}(c_{L}, c_W) \) shows that
\[
\langle I(n)m^\prime, m \rangle = \langle m^\prime, -I(-n)m + \delta_{n, 0}2c_{L, I} \rangle, \quad \langle W(n)m^\prime, m \rangle = \langle m^\prime, W(-n)m \rangle.
\]
Therefore we get the following result (the first and third relations were given in [4]):
Lemma 3.5.

\[ L^H(h, h_I)^* \cong L^H(h, -h_I + 2c_{L,I}), \quad L^{W(2,2)}(h, h_W)^* \cong L^{W(2,2)}(h, h_W). \]

In particular,

\[ L^H(h, (1 \pm p)c_{L,I})^* \cong L^H(h, (1 \mp p)c_{L,I}). \]

Directly from Theorem 3.3 and Lemma 3.5 follows

Corollary 3.6. Let \( p \in \mathbb{Z}_{>0} \).

(i) If \( (h, (1 - p)c_{L,I}) \) is typical for \( \mathcal{H} \) (equivalently if \( (h, \frac{1-p^2}{24}c_W) \) is typical for \( W(2,2) \)) then

\[ L^H(h, (1 - p)c_{L,I}) \cong W^{W(2,2)}(h, \frac{1-p^2}{24}c_W). \]

(ii) If \( (h, (1 - p)c_{L,I}) \in \mathcal{A}T^H(c_{L,I}, c_{L,I}) \) (equivalently if \( (h, \frac{1-p^2}{24}c_W) \in \mathcal{A}T^{W(2,2)}(c_{L,I}, c_W) \)) then

\[ L^H(h, (1 - p)c_{L,I}) \cong W^{W(2,2)}(h, \frac{1-p^2}{24}c_W)^*. \]

and the short sequence of \( W(2,2) \)-modules

\[ 0 \to W^{W(2,2)}(h, \frac{1-p^2}{24}c_W) \to L^H(h, (1 - p)c_{L,I}) \to W^{W(2,2)}(h, \frac{1-p^2}{24}c_W) \to 0 \]

is exact.

From Lemma 3.1, Theorem 3.3 and Corollary 3.6 follow assertions of Theorem 1.1.

Finally, we show that Verma module over \( \mathcal{H} \) is an infinite direct sum of irreducible \( W(2,2) \)-modules. Recall that \( V^H(h, (1 - p)c_{L,I}) \) has a series of singular vectors \( v_{-ip}^-, i \in \mathbb{Z}_{\geq 0} \) (for \( i = 0 \), we set \( v_0^- = v_{h,h_I}^- \)) which generate a descending chain of Verma submodules over \( \mathcal{H} \):

\[ \langle v_{h,h_I}^- \rangle_{\mathcal{H}} \supseteq V^H(h, h_I) \]

\[ \langle v_{-ip}^- \rangle_{\mathcal{H}} \cong V^H(h + p, h_I) \]

\[ \langle v_{-ip}^- \rangle_{\mathcal{H}} \cong V^H(h + ip, h_I) \]

\[ \langle v_{-ip}^- \rangle_{\mathcal{H}} \cong V^H(h + (i + 1)p, h_I) \]

Therefore one may identify \( V^H(h + ip, h_I) \) with a submodule of \( V^H(h, h_I) \) and a singular vector \( v_{-ip}^- \in V^H(h, h_I) \) with the highest weight vector \( v_{h+ip,h_I}^- \in V^H(h + ip, h_I) \). We will prove that in a typical case each of those vectors generates an irreducible \( W(2,2) \)-submodule.
**Theorem 3.7.** Let \( p \in \mathbb{Z}_{\geq 0} \). Suppose that \((h, (1 - p)c_{L,I}) \notin \mathcal{A}(c_L, c_{L,I})\). Then we have the following isomorphism of \(W(2,2)\)-modules

\[
V^H(h, (1 - p)c_{L,I}) \cong_{W(2,2)} \bigoplus_{i \geq 0} L^{W(2,2)}(h + ip, \frac{1 - p^2}{24} c_W).
\]

**Proof.** First we notice that the vertex algebra homomorphism \( \Psi : L^{W(2,2)}(c_W, c_L) \rightarrow L^H(c_W, c_L) \), for every \( i \in \mathbb{Z}_{\geq 0} \) induces the following non-trivial homomorphism of \(W(2,2)\)-modules:

\[
\Psi^{(i)} : V^{W(2,2)}(h + ip, \frac{1 - p^2}{24} c_W) \rightarrow \langle v^{-}_i \rangle_{W(2,2)} \subset V^H(h + ip, (1 - p)c_{L,I}),
\]

which maps the highest weight vector of \( V^{W(2,2)}(h + ip, \frac{1 - p^2}{24} c_W) \) to \( v^{-}_i \). Since \((h, \frac{1 - p^2}{24} c_W)\) is typical it follows from Proposition 2.3(i) that \((h + ip, \frac{1 - p^2}{24} c_W)\) are typical for all \( i \in \mathbb{Z}_{\geq 0} \) as well.

Let \( h_W = \frac{1 - p^2}{24} c_W \). Consider the homomorphism \( \Psi^{(i)} : V^{W(2,2)}(h + ip, h_W) \rightarrow V^H(h + ip, h_I) \) above. Applying (3.2), we get

\[
\Psi^{(i)}(W(-p)v_{h+ip,h_W}) = \sum_{i=1}^{p-1} I(-i)I(i-p)v_{h+ip,h_I},
\]

so \((W(-p)v_{h+ip,h_W}) \notin \text{Im}\ \Psi^{(i)}\). Since the Verma modules \( V^{W(2,2)}(h + ip, h_W) \) and \( V^H(h + ip, h_I) \) have equal characters, it follows that \( \text{Ker}\ \Psi^{(i)} \) contains a singular vector in \( V^{W(2,2)}(h + ip, h_W) \) of conformal weight \( h + (i+1)p \). Since the weight \((h + ip, h_W)\) is typical, the maximal submodule in \( V^{W(2,2)}(h + ip, h_W) \) is generated by this singular vector so we conclude that \( \text{Ker}\ \Psi^{(i)} \) is the maximal submodule in \( V^{W(2,2)}(h + ip, h_W) \). Therefore

\[
\text{Im}\ \Psi^{(i)} = \langle v_{h+ip,h_I} \rangle_{W(2,2)} \cong L^{W(2,2)}(h + ip, h_W).
\]

In this way we get a series of \(W(2,2)\)-monomorphisms

\[
L^{W(2,2)}(h + ip, h_W) \hookrightarrow V^H(h, (1 - p)c_{L,I}), \quad i \in \mathbb{Z}_{\geq 0}
\]

mapping \( v_{h+ip,h_W} \) to a singular vector \( v^{-}_i \). Let \( v^{-}_j \) be an \( \mathcal{H} \)-singular vector in \( V^H(h + ip, (1 - p)c_{L,I}) \) of weight \( h + jp \), for \( j > i \). By Lemma 3.2, \( v^{-}_j \) is singular for \( W(2,2) \) and therefore \( v^{-}_j \notin \langle v_{h+ip,h_I} \rangle_{W(2,2)} \) for \( j > i \). We conclude that the images of morphisms (3.6) have trivial pairwise intersections (since these images are non-isomorphic irreducible modules), so their sum is direct. The assertion follows from the observation that the character of this sum is

\[
\sum_{i=0}^{\infty} q^{h + ip}(1 - q^p) \prod_{k \geq 1} (1 - q^k)^{-2} = q^h \prod_{k \geq 1} (1 - q^k)^{-2} = \text{char}\ V^H(h, (1 - p)c_{L,I}).
\]

**Remark 3.8.** It is interesting to notice that our Theorem 3.7 shows that \( V^H(h, h_I) \) can be considered as a \( W(2,2) \)-analogue of certain Feigin–Fuchs modules for the Virasoro algebra which are also direct sums of infinitely many irreducible modules (cf. [11], [2, Theorem 5.1]).

From the previous theorem follows

\[
V^H(h, h_I) \cong_{W(2,2)} \langle v_{h,h_I} \rangle_{W(2,2)} = L^{W(2,2)}(h, h_W).
\]
\begin{align*}
\langle v_p^- \rangle_{W(2,2)} & \cong L^{W(2,2)}(h + p, h_W) \\
\oplus & \\
\vdots & \\
\oplus & \\
\langle v_{ip}^- \rangle_{W(2,2)} & \cong L^{W(2,2)}(h + ip, h_W)
\end{align*}

In atypical case however, the $W(2,2)$-submodules generated by $H$-singular vectors are nested as follows. Consider $V^H(h_{p,r}, h_{1})$ where $(h_{p,r}, h_{1}) \in AT_H(c_L, c_L, l)$. Then $\Psi^0$ maps a cosingular vector $u_{ip} \in V^{W(2,2)}(h_{p,r}, h_W)$ to a singular vector $v_{rp}^-$. In other words we have

\[ \langle v_{ip}^- \rangle_{W(2,2)} \subseteq \langle v_{rp}^- \rangle_{W(2,2)} \cong L^{W(2,2)}(h_{p,r}, h_W). \]

Using the same argument in view of Proposition 2.3 we see that

\[ \langle v_{(r-i)p}^- \rangle_{W(2,2)} \subseteq \langle v_{ip}^- \rangle_{W(2,2)} \cong L^{W(2,2)}(h_{p,r} + ip, h_W), \quad i = 1, \ldots, \left\lfloor \frac{r-1}{2} \right\rfloor. \]

Therefore,

\[ \langle v_{hp,r,h_{1}} \rangle_{W(2,2)}/\langle v_{ip}^- \rangle_{W(2,2)} \cong L^{W(2,2)}(h_{p,r}, h_W), \]

\[ \langle v_{ip}^- \rangle_{W(2,2)}/\langle v_{(r-i)p}^- \rangle_{W(2,2)} \cong L^{W(2,2)}(h_{p,r} + ip, h_W), \quad i < \frac{r-1}{2}, \]

\[ \langle v_{ip}^- \rangle_{W(2,2)} \cong L^{W(2,2)}(h_{p,r} + ip, h_W), \quad i \geq \frac{r-1}{2}. \]

In this case, $(p-i)v_{hp,r,h_{1}}$ are $W(2,2)$-cosingular vectors in $V^H(h_{p,r}, h_{1})$.

**Example 3.9.** Consider $p = 1$ case. Singular vector in $V^H(h, 0)$ is $u'_1 = (L(-1) + \frac{h}{c_L, l}I(-1))v_{h,0}$, and $u'_1$ generates a copy of $V^H(h + 1, 0)$.

$r = 1$: $\Psi: V^{W(2,2)}(0,0) \to V^H(0,0)$ maps a singular vector $u'_1 = W(-1)v_{0,0}$ to 0 and a cosingular vector $u_1 = L(-1)v_{0,0}$ to $H$-singular vector $v_1^- = L(-1)v_{0,0}$. We get the short exact sequence of $W(2,2)$-modules

\[ 0 \to L^{W(2,2)}(0,0) \to L^H(0,0) \to L^{W(2,2)}(1,0) \to 0, \]

which is an expansion of (3.1) considered as a homomorphism of $W(2,2)$-modules. The rightmost module is generated by a projective image of $I(-1)v_{0,0}$. Therefore, $L^H(c_L, c_L, l)$ is generated over $W(2,2)$ by $v_{0,0}$ and $I(-1)v_{0,0}$.

$r \in \mathbb{Z}_{>0}$: In general, a cosingular vector $u_{rp} \in V^{W(2,2)}(\frac{1-r}{2},0)$ maps to a singular vector $v_r^- \in V^H(\frac{1-r}{2},0)$ of weight $\frac{1-r}{2}$.

\[ v_r^- = \prod_{i=0}^{r-1} \left( L(-1) + \frac{1-r + 2i}{2c_L, l}I(-1) \right) v_{\frac{1-r}{2},0}. \]

### 4 Screening operators and $W(2,2)$-algebra

We think that the vertex algebra $L^{W(2,2)}(c_L, c_W)$ is a very interesting example of non-rational vertex algebra, which admits similar fusion ring of representations as some $W$-algebras appearing in LCFT (cf. [1, 2, 10, 13]). Since $W$-algebras appearing in LCFT are realized as kernels of
screening operators acting on certain modules for Heisenberg vertex algebras, it is natural to ask if \( L^{W(2,2)}(c_L, c_W) \) admits similar realization. In [4] we embedded the \( W(2,2) \)-algebra as a subalgebra of the Heisenberg–Virasoro vertex algebra. In this section we shall construct a screening operator \( S_1 \) such that the kernel of this operator is exactly \( L^{W(2,2)}(c_L, c_W) \).

Let us first construct a non-semisimple extension of the vertex algebra \( L^H(c_L, c_{L,I}) \). Recall that the Lie algebra \( H \) admits the triangular decomposition (2.2). Let \( E = \text{span}_C\{v^0, v^1\} \) be 2-dimensional \( H^{\geq 0} = H^0 \oplus H^+ \)-module such that \( H^+ \) acts trivially on \( E \) and

\[
\begin{align*}
L(0) v^i &= v^i, & i = 0, 1, \\
I(0) v^1 &= v^0, & I(0) v^0 = 0, \\
C_L v^i &= c_L v^i, & C_L, I v^i = c_{L,I} v^i, \\
C_I v^i &= 0, & i = 1, 2.
\end{align*}
\]

Consider now induced \( H \)-module

\[
\tilde{E} = U(H) \otimes_{U(H^{\geq 0})} E.
\]

By construction, \( \tilde{E} \) is a non-split self-extension of the Verma module \( V^H(1,0) \):

\[
0 \to V^H(1,0) \to \tilde{E} \to V^H(1,0) \to 0.
\]

Moreover, \( \tilde{E} \) is a restricted module for \( H \) and therefore it is a module over vertex operator algebra \( L^H(c_L, c_{L,I}) \). Since

\[
\tilde{E} \cong E \otimes U(H^-)
\]

as a vector space, the operator \( L(0) \) defines a \( \mathbb{Z}_{\geq 0} \)-gradation on \( \tilde{E} \).

Note that \((L(-1) + I(-1)/c_{L,I})v_0\) is a singular vector in \( \tilde{E} \) and it generates the proper submodule. Finally we define the quotient module

\[
\mathcal{U} = \frac{\tilde{E}}{U(H_0)(L(-1) + I(-1)/c_{L,I})v_0}.
\]

**Proposition 4.1.** \( \mathcal{U} \) is a \( \mathbb{Z}_{\geq 0} \)-graded module for the vertex operator algebra \( L^H(c_L, c_{L,I}) \):

\[
\mathcal{U} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathcal{U}(m), \quad L(0) \mathcal{U}(m) \equiv (m + 1) \text{Id}.
\]

The lowest component \( \mathcal{U}(0) \cong E \). Moreover, \( \mathcal{U} \) is a non-split extension of the Verma module \( V^H(1,0) \) by the simple highest weight module \( L^H(1,0) \):

\[
0 \to L^H(1,0) \to \mathcal{U} \to V^H(1,0) \to 0.
\]

**Proof.** By construction \( \mathcal{U} \) is a graded quotient of a \( \mathbb{Z}_{\geq 0} \)-graded \( L^H(c_L, c_{L,I}) \)-module \( \tilde{E} \). The lowest component is \( \mathcal{U}(0) \cong E \). Submodule \( U(H_0).v^0 \) is isomorphic to \( L^H(1,0) \), and the projective image of \( v^1 \) generates the Verma module \( V^H(1,0) \) since \( I(0)v^1 = v^0 \). For the same reason, this exact sequence does not split. 

Now we consider \( L^H(c_L, c_{L,I}) \)-module

\[
\mathcal{V}^{\text{ext}} := L^H(c_L, c_{L,I}) \oplus \mathcal{U}.
\]

By using [16, Theorem 4.8.1] (see also [3, 17]) we have that \( \mathcal{V}^{\text{ext}} \) has the structure of a vertex operator algebra with vertex operator map \( Y_{\text{ext}} \) defined as follows:

\[
Y_{\text{ext}}(a_1 + w_1, z)(a_2 + w_2) = Y(a_1, z)(a_2 + w_2) + e^{-L(-1)} Y(a_2, -z)w_1,
\]

where \( a_1, a_2 \in L^H(c_L, c_{L,I}), w_1, w_2 \in \mathcal{U} \).
Take now $v^i \in E \subset \mathcal{U}$, $i = 0, 1$ as above and define

$$S_i(z) = Y_{\text{ext}}(v^i, z) = \sum_{n \in \mathbb{Z}} S_i(n) z^{-n-1}.$$ 

By construction

$$S_1(z) \in \text{End} \left( L^\mathcal{H}(c_L, c_L, 1), L^\mathcal{H}(1, 0) \right)(z).$$

**Proposition 4.2.** For all $n, m \in \mathbb{Z}$ we have:

$$[L(n), S_i(m)] = -m S_i(n + m), \quad i = 0, 1,$$

$$[W(n), S_0(m)] = 0, \quad [W(n), S_1(m)] = 2mc_L S_0(n + m).$$

In particular, $S_0(0)$ and $S_1(0)$ are screening operators. Moreover,

$$S_1 = S_1(0): \ L^\mathcal{H}(c_L, c_L, 1) \to L^\mathcal{H}(1, 0)$$

is nontrivial and $S_1(0) \mathcal{I}(-1) 1 = -v_0$.

**Proof.** Since $L(k) v^i = \delta_{k,0} v^i$ for $k \geq 0$, commutator formula gives that

$$[L(n), S_i(m)] = -m S_i(n + m).$$

Next we calculate $[W(n), S_1(m)]$. We have

$$W(-1)v^1 = 2I(-1)v^0 = -2c_L, I(-1)v^0,$$

$$W(0)v^1 = -2c_L, I v^0, \quad W(n)v^1 = 0, \quad n \geq 0.$$ 

This implies that

$$[W(n), S_1(m)] = 2c_L, I m S_0(n + m).$$

Since $W(n)v^0 = 0$ for $n \geq -1$ we get

$$[W(n), S_0(m)] = 0.$$ 

Therefore we have proved that $S_i(0), i = 0, 1$ are screening operators. Next we have

$$S_1(0) \mathcal{I}(-1) 1 = \text{Res}_z Y_{\text{ext}}(v^1, z) \mathcal{I}(-1) = \text{Res}_z e^{zL(-1)} Y(1) \mathcal{I}(-1) 1, -z v^1 = -v_0.$$ 

The proof follows.

**Theorem 4.3.** $S_1$ is a derivation of the vertex algebra $\mathcal{V}_{\text{ext}}$ and we have

$$\text{Ker}_{L^\mathcal{H}(c_L, c_L, 1)} S_1 \cong L^{W(2,2)}(c_L, c_W).$$

**Proof.** By construction $S_1 = \text{Res}_z Y_{\text{ext}}(v^1, z)$, so $S_1$ is a derivation so $\mathcal{W} = \text{Ker}_{L^\mathcal{H}(c_L, c_L, 1)} S_1$ is a vertex subalgebra of $L^\mathcal{H}(c_L, c_L, 1)$. Since

$$S_1 L(-2) 1 = S_1 W(-2) 1 = 0$$

we have that $L^{W(2,2)}(c_L, c_W) \subset \mathcal{W}$. Since $S_1 \mathcal{I}(-1) 1 \neq 0$, we have that $\mathcal{I}(-1) 1$ does not belong to $\mathcal{W}$. By using the fact that $L^\mathcal{H}(c_L, c_L, 1)$ is as $W(2, 2)$-module generated by singular vector $1$ and cosingular vector $\mathcal{I}(-1) 1$ (see Example 3.9) we get that $\mathcal{W} = L^{W(2,2)}(c_L, c_W)$. The proof follows.

**Remark 4.4.** Of course, every $\mathcal{V}_{\text{ext}}$-module becomes a $W(2, 2)$-module with screening operator $S_1$. Similar statement holds for intertwining operators. Constructions of such modules and intertwining operators require different techniques which we will present in our forthcoming paper [5].
Acknowledgements

The authors are partially supported by the Croatian Science Foundation under the project 2634 and by the Croatian Scientific Centre of Excellence QuantixLie.

References


