Irreducible Generic Gelfand–Tsetlin Modules of $\mathfrak{gl}(n)^*$

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Abstract. We provide a classification and explicit bases of tableaux of all irreducible generic
Gelfand–Tsetlin modules for the Lie algebra $\mathfrak{gl}(n)$.

Key words: Gelfand–Tsetlin modules; Gelfand–Tsetlin basis; tableaux realization

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1 Introduction

Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra. The category of weight modules
of $\mathfrak{g}$ is interesting on its own on the one hand, and it contains some fundamental subcategories
like the category $\mathcal{O}$, categories of parabolically induced modules, Harish-Chandra modules on
the other. A weight $\mathfrak{g}$-module is a module which is a direct sum of simple $\mathfrak{h}$-modules, where $\mathfrak{h}$
is a fixed Cartan subalgebra of $\mathfrak{g}$. The classification of the simple weight modules is a very
hard problem which is solved only for $\mathfrak{g} = \mathfrak{sl}(2)$. However, the classification of the simple
objects is known for various subcategories of weight modules, including those with finite weight
multiplicities [5, 17].

The classification of the simple weight $\mathfrak{sl}(2)$-modules involves two parameters that correspond
to eigenvalues of the generators of a maximal commutative subalgebra of $U(\mathfrak{sl}(2))$, the Gelfand–
Tsetlin subalgebra. Such subalgebra can be defined for any $\mathfrak{sl}(n)$ and has a joint spectrum on
every finite-dimensional module. This observation leads naturally to the definition of a Gelfand–
Tsetlin module: a module that is the direct sum of its common generalized eigenspaces with
respect to the Gelfand–Tsetlin subalgebra $\Gamma$. Such modules were introduced in [2, 3, 4]. Note
that an irreducible Gelfand–Tsetlin modules does not need to be $\Gamma$-diagonalizable [6].

Gelfand–Tsetlin subalgebras and modules appear in various contexts. Such subalgebras were
considered in [22] in connection with subalgebras of maximal Gelfand–Kirillov dimension in the
universal enveloping algebra of a simple Lie algebra. Furthermore, Gelfand–Tsetlin subalgebras
are related to: general hypergeometric functions on the complex Lie group $\text{GL}(n)$ [13, 14];
solutions of the Euler equation [22]; and problems in classical mechanics in general [15, 16].

One natural question is to attempt the classification of all irreducible Gelfand–Tsetlin modules
of $\mathfrak{sl}(n)$. An explicit construction of all irreducible Gelfand–Tsetlin modules for the case
$n = 3$ was recently obtained in [10]. Various partial results for $\mathfrak{sl}(3)$ were previously obtained
in [1, 6, 7, 8, 9].

A generic Gelfand–Tsetlin module is a module spanned by tableaux with noninteger differences of entries in each row (see Definition 5.1). The present paper provides a classification of all irreducible generic Gelfand–Tsetlin modules of $\mathfrak{sl}(n)$ extending the result in [21] for $n = 3$.

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For simplicity we work with \( \mathfrak{gl}(n) \) instead of \( \mathfrak{sl}(n) \). We also obtain an explicit construction of all irreducible generic modules providing a Gelfand–Tsetlin type basis.

The organization of the paper is as follows. In Section 3 we introduce some basic definitions and preparatory results on Gelfand–Tsetlin modules. In Section 4 we list the Gelfand–Tsetlin formulas and use them to recall the classical result of Gelfand and Tsetlin for finite-dimensional \( \mathfrak{gl}(n) \)-modules. In Section 5 we introduce the notion of generic Gelfand–Tsetlin module and recall the classification of irreducible generic Gelfand–Tsetlin modules of \( \mathfrak{gl}(3) \). The main theorem in the paper, the classification of irreducible generic Gelfand–Tsetlin \( \mathfrak{gl}(n) \)-modules, is included in Section 6. In the last section we compute the number of irreducible Gelfand–Tsetlin modules in the so-called generic blocks.

2 Notation and conventions

Throughout the paper we fix an integer \( n \geq 2 \). The ground field will be \( \mathbb{C} \). For \( a \in \mathbb{Z} \), we write \( \mathbb{Z}_{\geq a} \) for the set of all integers \( m \) such that \( m \geq a \). Similarly, we define \( \mathbb{Z}_{< a} \), etc.

By \( \mathfrak{gl}(n) \) we denote the general linear Lie algebra consisting of all \( n \times n \) complex matrices, and by \( \{ E_{ij} \mid 1 \leq i, j \leq n \} \), the standard basis of \( \mathfrak{gl}(n) \) of elementary matrices. We fix the standard Cartan subalgebra \( \mathfrak{h} \), the standard triangular decomposition and the corresponding basis of simple roots of \( \mathfrak{gl}(n) \). The weights of \( \mathfrak{gl}(n) \) will be written as \( n \)-tuples \( (\lambda_1, \ldots, \lambda_n) \).

For a Lie algebra \( \mathfrak{a} \) by \( \mathcal{U}(\mathfrak{a}) \) we denote the universal enveloping algebra of \( \mathfrak{a} \). Throughout the paper \( \mathcal{U} = \mathcal{U}(\mathfrak{gl}(n)) \).

We will write the vectors in \( \mathbb{C}^{n(n+1)/2} \) in the following form:

\[
L = (l_{ij}) = (l_{ii}, \ldots, l_{nn}, \ldots, l_{21}, l_{22}, l_{11}).
\]

For \( 1 \leq j \leq i \leq n \), \( \delta^{ij} \in \mathbb{Z}_{\geq 0}^{n(n+1)/2} \) is defined by \( (\delta^{ij})_{ij} = 1 \) and all other \( (\delta^{ij})_{k\ell} \) are zero.

For \( i > 0 \) by \( S_i \) we denote the \( i \)th symmetric group. Throughout the paper we set \( G := S_n \times \cdots \times S_1 \).

3 Gelfand–Tsetlin modules

Recall that \( U = \mathcal{U}(\mathfrak{gl}(n)) \). Let for \( m \leq n \), \( \mathfrak{gl}_m \) be the Lie subalgebra of \( \mathfrak{gl}(n) \) spanned by \( \{ E_{ij} \mid i, j = 1, \ldots, m \} \). We have the following chain

\[
\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n.
\]

It induces the chain \( U_1 \subset U_2 \subset \cdots \subset U_n \) for the universal enveloping algebras \( U_m = \mathcal{U}(\mathfrak{gl}_m) \), \( 1 \leq m \leq n \). Let \( Z_m \) be the center of \( U_m \). The subalgebra of \( U \) generated by \( \{ Z_m \mid m = 1, \ldots, n \} \) will be called the (standard) Gelfand–Tsetlin subalgebra of \( U \) and will be denoted by \( \Gamma \).

Definition 3.1. A finitely generated \( U \)-module \( M \) is called a Gelfand–Tsetlin module (with respect to \( \Gamma \)) if

\[
M = \bigoplus_{m \in \text{Specm} \Gamma} M(m),
\]

where \( M(m) = \{ v \in M \mid m^k v = 0 \text{ for some } k \geq 0 \} \).

For each \( m \in \text{Specm} \Gamma \) we have associated a character \( \chi_m : \Gamma \to \mathbb{C} \) via \( \Gamma \). In the same way, for each non-zero character \( \chi : \Gamma \to \mathbb{C} \) we have that \( \text{Ker}(\chi) \) is a maximal ideal of \( \Gamma \). So, we have
a natural identification between characters of $\Gamma$ and elements of Specm $\Gamma$. Using characters we can define Gelfand–Tsetlin modules. A $U$-module $M$ is called Gelfand–Tsetlin module (with respect to $\Gamma$) if

$$M = \bigoplus_{\chi \in \Gamma^*} M(\chi),$$

where $M(\chi) = \{ v \in M : \forall g \in \Gamma, \exists k \in \mathbb{Z}_{>0} \text{ such that } (g - \chi(g))^kv = 0 \}$. The Gelfand–Tsetlin support of $M$ is the set $\text{Supp}_{GT}(M) := \{ \chi \in \Gamma^* : M(\chi) \neq 0 \}$.

**Lemma 3.2.** Any submodule of a Gelfand–Tsetlin module over $\mathfrak{gl}(n)$ is a Gelfand–Tsetlin module.

**Proof.** The proof is standard, but for a sake of completeness, we provide the important details. Let $M$ be a Gelfand–Tsetlin $\mathfrak{gl}(n)$-module and $N$ any submodule of $M$. We will prove that, if $\{\chi_1, \ldots, \chi_k\}$ is a set of distinct Gelfand–Tsetlin characters in $\text{Supp}_{GT}(M)$ such that $\sum_{i=1}^k v_i \in N$ with $v_i \in M(\chi_i)$, then $v_i \in N$ for all $i = 1, \ldots, k$.

Without loss of generality we assume that $k = 2$. Since $\chi_1 \neq \chi_2$, there exist $g \in \Gamma$ and $r \leq s$ in $\mathbb{Z}_{>0}$ such that $\chi_1(g) \neq \chi_2(g)$, $(g - \chi_1(g))^r(v_1) = 0$ and $(g - \chi_2(g))^s(v_2) = 0$. Let $a := \chi_1(g)$ and $b := \chi_2(g)$. Then, if $w = v_1 + v_2$ we have $(g - b)^sw = (g - b)^sv_1 \in N$. Let $y := (g - b)^sv_1$. We have that $y \in N$ on one hand and

$$y = ((g - a) + (a - b))^sv_1 = \sum_{k=0}^{r-1} \binom{s}{k} (a - b)^{s-k}(g - a)^kv_1 \in N$$

on the other. As $\binom{s}{k} (a - b)^{s-k} \neq 0$ for any $k$, using that $(g - a)^{r-1}y \in N$, we obtain $(g - a)^{r-1}v_1 \in N$. Reasoning in the same way, from $(g - a)^{r-1}y \in N$, and $(g - a)^{r-1}v_1, \ldots, (g - a)^{r-1}v_1 \in N$ we obtain $x^{r-i}v_1 \in N$. Hence $v_1 \in N$ and consequently, $v_2 \in N$. \hfill \blacksquare

One can choose the following generators of $\Gamma$: $\{c_{mk} | 1 \leq k \leq m \leq n\}$, where

$$c_{mk} = \sum_{(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k} E_{i_1\ell_1}E_{i_2\ell_2} \cdots E_{i_k\ell_k}. \quad (3.1)$$

Let $\Lambda$ be the polynomial algebra in the variables $\{\lambda_{ij} | 1 \leq j < i \leq n\}$. The action of the symmetric group $S_i$ on $\{\lambda_{ij} | 1 \leq j < i \leq n\}$ induces the action of the $G = S_n \times \cdots \times S_1$ on $\Lambda$. There is a natural embedding $\iota : \Gamma \rightarrow \Lambda$ given by $\iota(c_{mk}) = \gamma_{mk}(\lambda)$, where

$$\gamma_{mk}(\lambda) = \sum_{i=1}^{m} (\lambda_{mi} + m - 1)^k \prod_{j \neq i} \left( 1 - \frac{1}{\lambda_{mi} - \lambda_{mj}} \right). \quad (3.2)$$

Hence, $\Gamma$ can be identified with $G$-invariant polynomials in $\Lambda$.

**Remark 3.3.** In what follows, we will identify the set Specm $\Lambda$ of maximal ideals of $\Lambda$ with the set $\mathbb{C}^{\text{max}(\Lambda)}$. Then we have a surjective map $\pi : \text{Specm } \Lambda \rightarrow \text{Specm } \Gamma$. Moreover, since $\Lambda$ is integral over $\Gamma$, there are finitely many maximal ideals of $\Lambda$ that map to a fixed maximal ideal of $\Gamma$. The different maximal ideals of $\Lambda$ are obtained from each other under permutations in the group $G$.

If $\pi(\ell) = \mathfrak{m}$ for some $\ell \in \text{Specm } \Lambda$, then we write $\ell = \ell_{\mathfrak{m}}$ and say that $\ell_{\mathfrak{m}}$ is lying over $\mathfrak{m}$.
4 Finite-dimensional modules of $\mathfrak{gl}(n)$

In this section we recall a classical result of Gelfand and Tsetlin which provides an explicit basis for every irreducible finite-dimensional $\mathfrak{gl}(n)$-module.

**Definition 4.1.** For a vector $L = (l_{ij})$ in $\mathbb{C}^{\frac{n(n+1)}{2}}$, by $T(L)$ we will denote the following array with entries $\{l_{ij} : 1 \leq j \leq i \leq n\}$

\[
\begin{array}{cccccc}
 & l_{n1} & l_{n2} & \cdots & \cdots & l_{n,n-1} & l_{nn} \\
\hline
l_{n-1,1} & & & \cdots & \cdots & & \\
\hline
\vdots & \vdots & \ddots & & & \vdots & \\
\hline
l_{21} & l_{22} & & & & & \\
\hline
l_{11} & & & & & & \\
\end{array}
\]

Such an array will be called a **Gelfand–Tsetlin tableau** of height $n$. A Gelfand–Tsetlin tableau of height $n$ is called **standard** if $l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq k \leq n-1$.

Note that, for sake of convenience, the second condition above is slightly different from the original condition in [12].

**Theorem 4.2 ([12]).** Let $L(\lambda)$ be the finite-dimensional irreducible module over $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then there exist a basis of $L(\lambda)$ consisting of all standard tableaux $T(L) = T(l_{ij})$ with fixed top row $l_{nj} = \lambda_j - j + 1$. Moreover, the action of the generators of $\mathfrak{gl}(n)$ on $L(\lambda)$ is given by the Gelfand–Tsetlin formulas:

\[
\begin{align*}
E_{k,k+1}(T(L)) &= -\sum_{i=1}^{k} \left( \prod_{j=1}^{k+1} (l_{ki} - l_{k+1,j}) \right) \frac{1}{\prod_{j \neq i} (l_{ki} - l_{kj})} T(L + \delta^{ki}), \\
E_{k+1,k}(T(L)) &= \sum_{i=1}^{k} \left( \prod_{j=1}^{k-1} (l_{ki} - l_{k-1,j}) \right) \frac{1}{\prod_{j \neq i} (l_{ki} - l_{kj})} T(L - \delta^{ki}), \\
E_{kk}(T(L)) &= \left( k - 1 + \sum_{i=1}^{k} l_{ki} - \sum_{i=1}^{k-1} l_{k-1,i} \right) T(L),
\end{align*}
\]

if the new tableau $T(L \pm \delta^{ki})$ is not standard, then the corresponding summand of $E_{k,k+1}(T(L))$ or $E_{k+1,k}(T(L))$ is zero by definition. Furthermore, for $s \leq r$,

\[
c_{rs}(T(L)) = \gamma_{rs}(l)T(L),
\]

where $\{c_{rs}\}$ are the generators of $\Gamma$ defined in (3.1) and $\gamma_{rs}$ are defined in (3.2) (see [23]).

The formulas above are called **Gelfand–Tsetlin formulas** for $\mathfrak{gl}(n)$. These formulas were extended to the case of $U_q(\mathfrak{gl}(n))$ in [19].
5 Generic Gelfand–Tsetlin modules of $\mathfrak{gl}(n)$

Theorem 4.2 gives an explicit realization of any irreducible finite-dimensional $\mathfrak{gl}(n)$-module. Using the Gelfand–Tsetlin formulas, Drozd, Futorny and Ovsienko defined the class of infinite-dimensional generic modules for $\mathfrak{gl}(n)$ in [2].

Definition 5.1. A Gelfand–Tsetlin tableau $T(L)$ (equivalently, $L \in \mathbb{C}^{\frac{n(n+1)}{2}}$) is called generic if $l_{ki} - l_{kj} \notin \mathbb{Z}$ for all $1 \leq i \neq j \leq k \leq n - 1$. A character $\chi$ and $n = \text{Ker} \chi$ are called generic if $\ell_n$ is generic for one choice (hence for all choices) of $\ell_n$ lying over $n$. A Gelfand–Tsetlin module $M$ will be called a generic Gelfand–Tsetlin module if every $n$ in $\text{Supp}_{\text{GT}}(M)$ is generic.

Theorem 5.2 ([2, Section 2.3] and [18, Theorem 2]). Let $T(L) = T(l_{ij})$ be a generic Gelfand–Tsetlin tableau of height $n$. Denote by $B(T(L))$ the set of all Gelfand–Tsetlin tableaux $T(R) = T(r_{ij})$ satisfying $r_{nj} = l_{nj}$, $r_{ij} - l_{ij} \in \mathbb{Z}$ for $1 \leq j \leq i \leq n - 1$.

(i) The vector space $V(T(L)) = \text{span} B(T(L))$ has a structure of a $\mathfrak{gl}(n)$-module with action of the generators of $\mathfrak{gl}(n)$ given by the Gelfand–Tsetlin formulas (4.1).

(ii) The action of the generators of $\Gamma$ on the basis elements of $V(T(L))$ is given by (4.2).

(iii) The $\mathfrak{gl}(n)$-module $V(T(L))$ is a Gelfand–Tsetlin module all of whose Gelfand–Tsetlin multiplicities are 1.

Remark 5.3. The basis of the module in the previous theorem is

$$B(T(L)) = \{ T(L + z) : z \in \mathbb{Z}^{\frac{n(n+1)}{2}} \text{ and } z_{n1} = \cdots = z_{nn} = 0 \}.$$  

By a slight abuse of notation we will identify elements in $\mathbb{Z}^{\frac{n(n-1)}{2}}$ with elements $z \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ such that $z_{n1} = \cdots = z_{nn} = 0$. This will allow us to write $T(L + z)$ for $z \in \mathbb{Z}^{\frac{n(n+1)}{2}}$.

Remark 5.4. In what follows, we will apply Lemma 3.2 and use that the elements of $\Gamma$ separate the tableaux in the submodules of $V(T(L))$ in the following sense. Let $N$ be a $\mathfrak{gl}(n)$-submodule of $V(T(L))$, $g \in \mathfrak{gl}(n)$, and $T(R)$ be a tableau in $N$. Then, if $g \cdot T(R) = \sum_i c_i T(R_i)$ for some distinct tableaux $T(R_i)$ in $B(T(L))$ and nonzero $c_i \in \mathbb{C}$, we have $T(R_i) \in N$ for all $i$.

Theorem 5.5. If $n \in \text{Specm} \Gamma$ is generic, then there exists a unique irreducible Gelfand–Tsetlin module $N$ such that $N(n) \neq 0$.

Proof. Let $X_n = U/U_n$. We know that $X_n = U/U_n$ is a Gelfand–Tsetlin module. Furthermore, any irreducible Gelfand–Tsetlin module $M$ with $M(n) \neq 0$ is a homomorphic image of $X_n$, and $X_n(n)$ maps onto $M(n)$. Since both spaces $X_n(n)$ and $M(n)$ are $\Gamma$-modules then the projection $X_n(n) \to M(n)$ is a homomorphism of $\Gamma$-modules (see also [11, Corollary 5.3]). Taking into account that $\dim X_n(n) \leq n$, we conclude that $X_n$ has a unique maximal submodule (which does not intersect $X_n(n)$) and hence there exist a unique irreducible module $N$ with $N(n) \neq 0$.

Definition 5.6. If $T(R)$ is a generic tableau and $r \in \text{Specm} \Gamma$ corresponds to $R$ then, the unique module $N$ such that $N(r) \neq 0$ is called the irreducible Gelfand–Tsetlin module containing $T(R)$, or simply, the irreducible module containing $T(R)$.

Our goal is to describe explicitly the irreducible Gelfand–Tsetlin module containing $T(R)$ for every generic tableau $T(R)$. Below we recall how this is achieved in the case $n = 3$ in [20]. One should note that the methods used in [20] involve direct computations based on a case-by-case consideration, while in the present paper we provide an invariant proof. Also, we reformulate the result in [20] in terms of $T(L + z)$.
For any tableau $T(R) \in \{T(L+z) : z \in \mathbb{Z}^3\}$ and any $1 < p \leq 3$, $1 \leq s \leq p$, and $1 \leq u \leq p - 1$, define

$$\Omega^+(T(R)) := \{(p, s, u) : r_{p,s} - r_{p-1,u} \in \mathbb{Z}_{\geq 0}\}.$$

**Theorem 5.7** ([20]). If $T(L)$ is a generic Gelfand–Tsetlin tableau of height 3, then the following is a basis for the irreducible $\mathfrak{gl}(3)$-module containing $T(L)$:

$$\mathcal{I}(T(L)) := \{T(L+z) : z \in \mathbb{Z}^3 \text{ and } \Omega^+(T(L)) = \Omega^+(T(L+z))\}.$$  

The action of $\mathfrak{gl}(3)$ on this irreducible module is given by the Gelfand–Tsetlin formulas.

**Example 5.8.** Consider $a, b, c \in \mathbb{C}$ such that $\{a - b, a - c, b - c\} \cap \mathbb{Z} = \emptyset$, $L = (a, b, c|a, b+1|a)$ and

![Tableau](image)

then $\Omega^+(T(L)) = \{(3, 1, 1), (2, 1, 1)\}$. So, by Theorem 5.7, the irreducible module containing $T(L)$ has basis

$$\mathcal{I}(T(L)) = \{T(L+(m,n,k)) : (m,n,k) \in \mathbb{Z}^3, m \leq 0, k \leq m, \text{ and } n > -1\}.$$

### 6 Classification of irreducible generic Gelfand–Tsetlin $\mathfrak{gl}(n)$-modules

In this section we prove the main result in the paper, i.e. the generalization of Theorem 5.7 for $\mathfrak{gl}(n)$. For convenience we introduce and recall some notation.

**Notation 6.1.** Let $T(L) = T(l_{ij})$ be a fixed tableau of height $n$.

(i) $\mathcal{B}(T(L)) := \{T(L+z) : z \in \mathbb{Z}^{n(n-1)/2}\}$.

(ii) $V(T(L)) := \text{span} \mathcal{B}(T(L))$.

(iii) For any $T(R) = T(r_{ij}) \in \mathcal{B}(T(L))$ and for any $1 < p \leq n$, $1 \leq s \leq p$ and $1 \leq u \leq p - 1$ we define:

(a) $\omega_{p,s,u}(T(R)) := r_{p,s} - r_{p-1,u}$;

(b) $\Omega(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \mathbb{Z}\}$;

(c) $\Omega^+(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \mathbb{Z}_{\geq 0}\}$;

(d) $\mathcal{N}(T(R)) := \{T(Q) \in \mathcal{B}(T(L)) : \Omega^+(T(R)) \subseteq \Omega^+(T(Q))\}$;

(e) $W(T(R)) := \text{span} \mathcal{N}(T(R))$;

(f) $U \cdot T(R)$: the $\mathfrak{gl}(n)$-submodule of $V(T(L))$ generated by $T(R)$.

#### 6.1 Basis for the module generated by a single tableau

In order to find an explicit basis of every irreducible generic module, we first find a basis of $U \cdot T(R)$ for any tableau $T(R)$ in $\mathcal{B}(T(L))$.

**Proposition 6.2.** For any $T(R) \in \mathcal{B}(T(L))$, the Gelfand–Tsetlin formulas endow $W(T(R))$ with a $\mathfrak{gl}(n)$-module structure.
Lemma 6.3.

Let

\[ E_{k,k+1}(T(Q)) = -\sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k+1} (q_{ki} - q_{k+1,j})}{\prod_{j \neq i}^{k} (q_{ki} - q_{kj})} \right) T(Q + \delta^{ki}) \].

If \( E_{k,k+1}(T(Q)) \notin W(T(R)) \), then there exist \( k \) and \( i \) such that \( T(Q) \in N(T(R)) \) but \( T(Q + \delta^{ki}) \notin N(T(R)) \). That implies

\[ \Omega^{+}(T(R)) \subseteq \Omega^{+}(T(Q)) \text{ and } \Omega^{+}(T(R)) \not\subseteq \Omega^{+}(T(Q + \delta^{ki})). \]

Hence, there exists \( (p, s, u) \in \Omega^{+}(T(R)) \) such that \( \omega_{p,s,u}(T(Q)) \in \mathbb{Z}_{\geq 0} \) and \( \omega_{p,s,u}(T(Q + \delta^{ki})) \notin \mathbb{Z}_{\geq 0} \). The latter holds only in two cases:

\[ (p, s, u) \in \{(k, i, u), (k + 1, s, i) : 1 \leq u \leq k - 1; 1 \leq s \leq k + 1\}. \]

Note that if neither of these two cases hold, we have \( \omega_{p,s,u}(T(Q + \delta^{ki})) = \omega_{p,s,u}(T(Q)) \). We consider now each of the two cases separately.

(i) Suppose \( (p, s, u) = (k, i, u) \). Then \( \omega_{k,i,u}(T(Q)) = q_{ki} - q_{k-1,u} \in \mathbb{Z}_{\geq 0} \) and \( \omega_{k,i,u}(T(Q + \delta^{ki})) = (q_{ki} + 1) - q_{k-1,u} \notin \mathbb{Z}_{\geq 0} \), which is impossible.

(ii) Suppose \( (p, s, u) = (k + 1, s, i) \). Then

\[ \omega_{k+1,s,i}(T(Q)) = q_{k+1,s} - q_{ki} \in \mathbb{Z}_{\geq 0} \]

and

\[ \omega_{k+1,s,i}(T(Q + \delta^{ki})) = q_{k+1,s} - (q_{ki} + 1) \notin \mathbb{Z}_{\geq 0}. \]

Hence \( q_{k+1,s} - q_{ki} = 0 \) and then the coefficient of \( T(Q + \delta^{ki}) \) in the decomposition of

\[ E_{k,k+1}(T(Q)) = -\frac{\prod_{j=1}^{k+1} (q_{ki} - q_{k+1,j})}{\prod_{j \neq i}^{k} (q_{ki} - q_{kj})} = 0. \]

Therefore, the tableaux that appear with nonzero coefficients in \( E_{k,k+1}(T(Q)) \) are elements of \( N(T(R)) \). Hence, \( E_{k,k+1}(T(Q)) \in W(T(R)) \). The proof that \( E_{k+1,k}(T(Q)) \in W(T(R)) \) is analogous to the one of \( E_{k,k+1}(T(Q)) \in W(T(R)) \). The case \( g = E_{kk} \) is trivial because \( E_{kk} \) acts as a multiplication by a scalar on \( T(Q) \) and \( T(Q) \in N(T(R)) \subseteq W(T(R)) \).

Given any tableau \( T(R) \), there are three modules containing \( T(R) \): \( V(T(L)) \), \( W(T(R)) \) and \( U : T(R) \). We will show that \( W(T(R)) = U : T(R) \). For this we need the following lemmas.

Lemma 6.3. Let \( T(L) \) be a generic tableau. If \( 0 \neq z \in \mathbb{Z}_{\frac{n(n-1)}{2}}^{+} \) is such that \( \Omega^{+}(T(L)) \subseteq \Omega^{+}(T(L + z)) \) then, there exist \( i, j \) such that \( z_{ij} \neq 0 \) and

\[ \Omega^{+}(T(L)) \subseteq \Omega^{+}(T(L + z_{ij}\delta^{ij})) \subseteq \Omega^{+}(T(L + z)). \]

Proof. We will use the following definition in the proof of the lemma.
Definition 6.4. Given a generic tableau $T(R) \in \mathcal{B}(T(L))$, a chain in $T(R)$ of length $\ell$ starting in row $d$ is a subset of the entries of $T(R)$, $C = \{r_{d-i, s(d-i)}\}_{i=0,\ldots,\ell}$, where $1 \leq s(d-i) \leq d - i$ are such that $r_{d-i, s(d-i)} - r_{d-i-1, s(d-i-1)} \in \mathbb{Z}$ for any $i = 0, \ldots, \ell - 1$ (i.e. $\{(d-i, s(d-i)), s(d-i-1)\}_{i=0,\ldots,\ell} \subseteq \Omega(T(R))$). The chain is called maximal if

(i) $(d + 1, i, s(d)) \notin \Omega(T(R))$ for any $1 \leq i \leq d + 1$,
(ii) $(d - \ell, s(d-\ell), j) \notin \Omega(T(R))$ for any $1 \leq j \leq d - \ell - 1$.

For every $T(R)$ in $\mathcal{B}(T(L))$ we have that $\Omega^+(T(R)) = \bigsqcup_{1 \leq c \leq n} \Omega^+_c(T(R))$, where $\Omega^+_c(T(R)) := \{(p, s, u) \in \Omega^+(T(R)) : p = c\}$. In particular, (6.1) holds if and only if

$$\Omega^+_c(T(L)) \subseteq \Omega^+_c(T(L + z_{ij} \delta^{ij})) \subseteq \Omega^+_c(T(L + z)) \quad (6.2)$$

for any $1 \leq c \leq n$. For $c \notin \{i, i + 1\}$ we have $\Omega^+_c(T(L)) = \Omega^+_c(T(L + z_{ij} \delta^{ij}))$. So, in order to verify (6.2), it is enough to consider the cases $c = i, i + 1$.

Let consider $k, l$ such that $z_{kl} \neq 0$. Set for convenience $Q := L + z$. There exists a maximal chain $C$ in $T(Q)$ of length $\ell$, starting in row $d$ such that $q_{kl} \in C$. Suppose that $C = \{q_{[i]}\}_{i=0,\ldots,\ell}$ where $[i] := (d - i, s(d-1))$. If $\ell = 0$, then $C = \{q_{kl}\}$ and (6.1) is obvious for $z_{ij} = z_{kl}$.

Let $a$ and $b$ be the minimum and maximum of $\{i : z_{i[i]} \neq 0\}$, respectively. We have

$$\Omega^+_{d-a+1}(T(L + z_{[a]} \delta^{[a]})) = \Omega^+_{d-a+1}(T(L + z)),
\Omega^+_{d-b}(T(L + z_{[b]} \delta^{[b]})) = \Omega^+_{d-b}(T(L + z)). \quad (6.3)$$

Therefore (6.2) holds for the pairs $c = d - a + 1, z_{ij} = z_{[a]}$ and $c = d - b, z_{ij} = z_{[b]}$, respectively. Now, let $a \leq m \leq b$ and consider the 4 cases depending on what the signs of $z_{[a]}$ and $z_{[a+1]}$ are.

(i) $z_{[m]} > 0$ and $z_{[m+1]} \leq 0$. In this case (6.2) holds for $c = d - m$ and $z_{ij} = z_{[m]}$. In particular, if $z_{[a]} > 0$ and $z_{[a+1]} \leq 0$, using the first equation in (6.3), we conclude that (6.1) holds for $z_{ij} = z_{[a]}$.

(ii) $z_{[m]} < 0$ and $z_{[m-1]} \geq 0$. In this case (6.2) holds for $c = d - m + 1$ and $z_{ij} = z_{[m-1]}$. In particular, if $z_{[b]} < 0$ and $z_{[b-1]} \geq 0$, using the second equation in (6.3) we conclude that (6.1) holds for $z_{ij} = z_{[b]}$.

(iii) $z_{[m]} > 0$ and $z_{[m+1]} > 0$. In this case (6.2) holds for $c = d - m$ and

$$z_{ij} = \begin{cases} z_{[m]} & \text{if } l_{[m]} - l_{[m+1]} \in \mathbb{Z}_{\geq 0}, \\
z_{[m+1]} & \text{if } l_{[m+1]} - l_{[m]} \in \mathbb{Z}_{> 0}. \end{cases}$$

(iv) $z_{[m]} < 0$ and $z_{[m-1]} < 0$. In this case (6.2) holds for $c = d - m + 1$ and

$$z_{ij} = \begin{cases} z_{[m]} & \text{if } l_{[m-1]} - l_{[m]} \in \mathbb{Z}_{\geq 0}, \\
z_{[m-1]} & \text{if } l_{[m]} - l_{[m-1]} \in \mathbb{Z}_{> 0}. \end{cases}$$

Now combining (i)–(iv) we reduce the proof to the following two cases:

(a) $z_{[a]} > 0, z_{[a+1]} > 0, \ldots, z_{[b]} > 0$ and for any $t = 1, \ldots, b - a$, (6.2) holds for $c = d - a + t + 1$ and $z_{ij} = z_{[a+t]}$. In particular, (6.2) holds for $c = d - b + 1$ and $z_{ij} = z_{[b]}$. So, by the first equation in (6.3) we have that (6.1) holds for $z_{ij} = z_{[b]}$.

(b) $z_{[b]} < 0, z_{[b-1]} < 0, \ldots, z_{[a]} < 0$ and for any $t = 1, \ldots, b - a$, (6.2) holds for $c = d - (b - t)$ and $z_{ij} = z_{[b-t]}$. In particular, (6.2) holds for $c = d - a$ and $z_{ij} = z_{[a]}$. So, by the first equation in (6.3) we have that (6.1) holds for $z_{ij} = z_{[a]}$. \[\Box\]
Assume that a linear combination of tableaux. For any $p \geq 1$ we write $T(R) \preceq_{(1)} T(Q)$ if there exist $g \in \mathfrak{gl}(n)$ such that $T(Q)$ appears with nonzero coefficient in the decomposition of $g \cdot T(R)$ into a linear combination of tableaux. For any $p \geq 1$ we write $T(R) \preceq_{(p)} T(Q)$ if there exist tableaux $T(L^{(1)}), \ldots, T(L^{(p)})$, such that

$$T(R) = T(L^{(0)}) \preceq_{(1)} T(L^{(1)}) \preceq_{(1)} \cdots \preceq_{(1)} T(L^{(p)}) = T(Q).$$

As an immediate consequence of the definition of $\preceq_{(p)}$ we have the following.

**Lemma 6.6.** If $T(Q), T(Q^{(0)}), T(Q^{(1)})$ and $T(Q^{(2)})$ are tableaux in $\mathcal{B}(T(L))$ then:

(i) $T(Q^{(0)}) \preceq_{(p)} T(Q^{(1)})$ and $T(Q^{(1)}) \preceq_{(q)} T(Q^{(2)})$ imply $T(Q^{(0)}) \preceq_{(p+q)} T(Q^{(2)})$;

(ii) $T(Q) \preceq_{(1)} T(Q)$.

**Corollary 6.7.** If $T(R), T(Q) \in \mathcal{B}(T(L))$ are generic Gelfand–Tsetlin tableaux such that $T(R) \preceq_{(p)} T(Q)$ for some $p \in \mathbb{Z}_{\geq 0}$, then $T(Q) \in U \cdot T(R)$.

**Proof.** By Lemma 5.4 and the definition of the relation $\preceq_{(1)}$, we first verify that $T(R) \preceq_{(1)} T(Q)$ implies $T(Q) \in U \cdot T(R)$. Now, using Lemma 6.6(i), if $T(R) \preceq_{(p)} T(Q)$ for some $p$ then $T(Q) \in U \cdot T(R)$.

The next theorem provides a convenient basis for the submodule of $V(T(L))$ generated by a fixed tableau. Recall the definition of $\mathcal{N}(T(R))$ in Notation 6.1(iii)(d).

**Theorem 6.8.** For any tableau $T(R) \in \mathcal{B}(T(L))$, $U \cdot T(R) = W(T(R))$. In particular, $\mathcal{N}(T(R))$ forms a basis of $U \cdot T(R)$, and the action of $\mathfrak{gl}(n)$ on $U \cdot T(R)$ is given by the Gelfand–Tsetlin formulas.

**Proof.** By Proposition 6.2, $U \cdot T(R) \subseteq W(T(R))$. To prove that $W(T(R)) \subseteq U \cdot T(R)$ we will show that $T(Q) \in U \cdot T(R)$ for any $T(Q) \in \mathcal{N}(T(R))$. By Corollary 6.7, it is enough to prove that $T(R) \preceq_{(p)} T(Q)$ for some positive integer $p$.

Suppose that $T(Q) = T(R + z) \in \mathcal{N}(T(R))$ for some $z \in \mathbb{Z}^{n(n-1) \times 2}$. Let $t$ be the number of non-zero components of $z$. We will prove that $T(R) \preceq_{(p)} T(Q)$ using induction on $t$.

Let us first consider the case $t = 1$ (the case $t = 0$ is trivial, since then $T(Q) = T(R)$) and $z_{ij} > 0$. We will first prove that $T(R + l \delta_{ij}) \preceq_{(1)} T(R + (l + 1) \delta_{ij})$ for any $0 \leq l \leq z_{ij} - 1$. This will imply

$$T(R) \preceq_{(1)} T(R + \delta_{ij}) \preceq_{(1)} T(R + 2 \delta_{ij}) \preceq_{(1)} \cdots \preceq_{(1)} T(R + z_{ij} \delta_{ij}) = T(Q),$$

and then $T(R) \preceq_{(z_{ij})} T(Q)$. To prove that $T(R + l \delta_{ij}) \preceq_{(1)} T(R + (l + 1) \delta_{ij})$ we show that the coefficient of $T(R + (l + 1) \delta_{ij})$ in the decomposition of $E_{i,i+1}(T(R + l \delta_{ij}))$ is not zero. In fact, by the Gelfand–Tsetlin formulas, that coefficient is

$$a_l := \frac{\prod_{k=1}^{i+1}(r_{ij} - r_{i+1,k} + l)}{\prod_{k \neq j}(r_{ij} - r_{ik} + l)}.$$

Assume that $a_l = 0$. Then $r_{ij} - r_{i+1,k} + l = 0$ for some $k$, which implies $\omega_{i+1,k,j}(T(R)) = r_{i+1,k} - r_{ij} = l \in \mathbb{Z}_{\geq 0}$. But, since $T(Q) \in \mathcal{N}(T(R))$, we have

$$l - z_{ij} = r_{i+1,k} - r_{ij} - z_{ij} = \omega_{i+1,k,j}(T(Q)) \in \mathbb{Z}_{\geq 0}.$$

Therefore we have $0 \leq l \leq z_{ij} - 1$ and $z_{ij} \leq l$, which is a contradiction. Hence, $T(R) \preceq_{(z_{ij})} T(Q)$. 

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Let now \( t = 1 \) and \( z_{ij} < 0 \). Using the same arguments as in the case \( z_{ij} > 0 \), we prove that 
\( T(R) \preceq (z_{ij}) T(Q) \) using \( |z_{ij}| \) applications of \( E_{i+1,t} \). This completes the proof for \( t = 1 \).

Assume now that for any \( w \in \mathbb{Z}^\frac{n(n-1)}{2} \) with at most \( t \) nonzero components, and such that 
\( \Omega^+(T(R)) \subseteq \Omega^+(T(R + w)) \), we have \( T(R) \preceq (p) T(R + w) \) for some \( p \). Let us consider \( z \) with \( t + 1 \) nonzero components. Since \( \Omega^+(T(R)) \subseteq \Omega^+(T(R + z)) \), by Lemma 6.3, there exist \( i, j \) such that 

\[
\Omega^+(T(R)) \subseteq \Omega^+(T(R + z_{ij} \delta^{ij})) \subseteq \Omega^+(T(R + z)).
\]

Using the induction hypothesis for the pairs of tableaux \((T(R), T(R + z_{ij} \delta^{ij}))\) and \((T(R + z_{ij} \delta^{ij}), T(R + z))\), there exist \( p, q \in \mathbb{Z}_{\geq 0} \) such that

\[
T(R) \preceq (p) T(R + z_{ij} \delta^{ij}) \quad \text{and} \quad T(R + z_{ij} \delta^{ij}) \preceq (q) T(R + z).
\]

Thus, by Lemma 6.6(i), \( T(R) \preceq (p + q) T(R + z) \). \( \Box \)

**Proposition 6.9.** Let \( T(R) \) and \( T(Q) \) be in \( \mathcal{B}(T(L)) \). Then \( U \cdot T(R) = U \cdot T(Q) \) if and only if \( \Omega^+(T(Q)) = \Omega^+(T(R)) \).

**Proof.** Using Theorem 6.8 and the definitions of \( W(T(R)), W(T(Q)), \Omega^+(T(R)), \) and \( \Omega^+(T(Q)) \), we can prove a stronger statement: \( U \cdot T(R) \subseteq U \cdot T(Q) \) if and only if \( \Omega^+(T(Q)) \subseteq \Omega^+(T(R)) \). \( \Box \)

**Corollary 6.10.** \( U \cdot T(R) = V(T(L)) \) whenever \( \Omega^+(T(R)) = \emptyset \).

**Definition 6.11.** We will write \( T(Q) \sim_{\Omega^+} T(R) \) if \( \Omega^+(T(R)) = \Omega^+(T(Q)) \).

**Proposition 6.12.** Every submodule of \( V(T(L)) \) is finitely generated.

**Proof.** Let \( N \) be any submodule of \( V(T(L)) \) and \( \Phi \) the set of all tableaux \( T(R) \) in \( N \) such that \( \Omega^+(T(P)) \subseteq \Omega^+(T(R)) \) implies \( \Omega^+(T(P)) = \Omega^+(T(R)) \). By Theorem 6.8, \( N = \sum_{T(R) \in \Phi} U \cdot T(R) \) and by Proposition 6.9, we can write \( N = \bigoplus_{T(R) \in \tilde{\Phi}} U \cdot T(R) \), where \( \tilde{\Phi} \) is a set of distinct representatives of \( \Phi \sim_{\Omega^+} \) (hence \( \Omega^+(T(R)) \neq \Omega^+(T(Q)) \) for any \( T(R), T(Q) \) in \( \tilde{\Phi} \)). Now, since \( \Omega(T(L)) \) is a finite set, then \( \tilde{\Phi} \) is finite. \( \Box \)

### 6.2 Basis for irreducible modules containing a given tableau

By Theorem 6.8, the module generated by a tableau \( T(R) \) has basis \( N(T(R)) \). For the purpose of the next theorem let us introduce the following equivalence on \( \mathbb{C}_{n(n+1)}^{\frac{n(n+1)}{2}} \).

**Definition 6.13.** We write \( z \sim w \) for \( z, w \in \mathbb{C}_{n(n+1)}^{\frac{n(n+1)}{2}} \) if and only if one of the two cases hold.

(i) \( z - w \in \mathbb{Z}_{\frac{n(n-1)}{2}} \) and \( z \sim_{\Omega^+} w \).

(ii) \( z \in Gw \).

Now we are ready to formulate and prove the main theorem in the paper.

**Theorem 6.14.** The irreducible module containing \( T(R) \) has a basis of tableaux 

\[
\mathcal{I}(T(R)) = \{ T(Q) \in \mathcal{B}(T(R)) : \Omega^+(T(Q)) = \Omega^+(T(R)) \}.
\]

The action of \( \mathfrak{gl}(n) \) on this irreducible module is given by the Gelfand–Tsetlin formulas (4.1). Therefore the set of irreducible generic Gelfand–Tsetlin modules is in one-to-one correspondence with \( \mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}} / \sim \), where \( \mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}} \) stands for the set of generic vectors in \( \mathbb{C}_{n(n+1)}^{\frac{n(n+1)}{2}} \).
**Proof.** For each tableau $T(R)$, we have an explicit construction of the module containing $T(R)$ (recall Definition 5.6):

$$M(T(R)) := U \cdot T(R)/\left( \sum U \cdot T(Q) \right),$$

where the sum is taken over tableaux $T(Q)$ such that $T(Q) \in U \cdot T(R)$ and $U \cdot T(Q)$ is a proper submodule of $U \cdot T(R)$.

The module $M(T(R))$ is simple. Indeed, this follows from the fact that for any nonzero tableau $T(S)$ in $M(T(R))$ we have $U \cdot T(S) = U \cdot T(R)$ and, hence, $T(S)$ generates $M(T(R))$.

By Theorem 6.8 and Proposition 6.9, a basis for a proper submodule $U \cdot T(Q)$ of $U \cdot T(R)$ is $\{ T(S) : \Omega^+(T(R)) \subseteq \Omega^+(T(Q)) \subseteq \Omega^+(T(S)) \}$ so, a basis for the module $\sum U \cdot T(Q)$ is $\{ T(S) : \Omega^+(T(R)) \subseteq \Omega^+(T(S)) \}$.

Therefore, $T(T(R))$ is a basis for $M(T(R))$.

To show that $C_{\text{gen}}(n+1) / \sim$ parameterizes the set of all irreducible generic Gelfand–Tsetlin modules we use Theorem 5.5 and the fact that $\ell, \ell' \in \text{Specm } \Lambda$ lie over the same $m$ in $\text{Specm } \Gamma$ if and only if $\ell \in G\ell'$ (see Remark 3.3). □

### 7 Number of irreducible modules in generic blocks

**Definition 7.1.** For any generic tableau $T(L)$, the block associated with $T(L)$ is the set of all Gelfand–Tsetlin $\mathfrak{g}l(n)$-modules with Gelfand–Tsetlin support contained in $\text{Supp}_{\text{GT}}(V(T(L)))$.

Theorem 6.14 describe explicit bases of the irreducible modules in the block associated with $V(T(L))$. In this section we will use this description to compute the number of nonisomorphic irreducible modules in this block.

**Definition 7.2.** For any $T(R) = T(r_{ij}) \in \mathcal{B}(T(L))$, $1 < p \leq n$ and $1 \leq u \leq p - 1$, define \( d_{pu}(T(R)) \) to be the number of distinct elements in

$$\{ r_{ps} : (p, s, u) \in \Omega(T(R)) \}.$$

**Remark 7.3.** For any generic tableau $T(R) = T(r_{ij}) \in \mathcal{B}(T(L))$ of height $n$ we have:

(i) $d_{pu}(T(L)) = d_{pu}(T(R))$ for any $1 < p \leq n$, $1 \leq u \leq p - 1$;

(ii) if $p \neq n$, then $d_{pu}(T(R)) \leq 1$ for any $1 \leq u \leq p - 1$.

**Example 7.4.** Suppose $a, b, c \in \mathbb{C}$ are such that $\{ a - b, a - c, b - c \} \cap \mathbb{Z} = \emptyset$. If $R = (a, a - 1, b|a, b|c)$, then

$$T(R):= \begin{array}{ccc}
\hline
a & a - 1 & b \\
\hline
a & b \\
c 
\hline
\end{array}$$

$$d_{31}(T(R)) = 2, d_{32}(T(R)) = 1, d_{21}(T(R)) = 0 \text{ and } d_{22}(T(R)) = 0.$$

**Remark 7.5.** For each tableau $T(R)$ we have an one-to-one correspondence between the set $\{0, 1, \ldots, d_{pu}(T(L))\}$ and the subset $\{0, i_1, \ldots, i_{d_{pu}(T(L))}\}$ of $\{0, 1, \ldots, p\}$ defined as follows: $i_1 = 1$ and $i_k = \min \{ x : r_{px} \notin \{r_{p_1x}, \ldots, r_{p_{k-1}x}\} \}$. 
Theorem 7.6. For any generic tableau $T(L)$, the number of irreducible modules in the block associated with $T(L)$ is
\[
\prod_{1 \leq u \leq \rho-1 < n} (d_{pu}(T(L)) + 1).
\]
In particular, $V(T(L))$ is irreducible if and only if $d_{pu}(T(L)) = 0$ for any $p$ and $u$, or equivalently, if and only if $\Omega(T(L)) = \emptyset$.

Proof. By Theorem 6.14, the irreducible modules are in one-to-one correspondence with the subsets of $\Omega(T(L))$ of the form $\Omega^+(T(L + z))$. For any $T(R) \in B(T(L))$, we can decompose $\Omega(T(R))$ into a disjoint union $\Omega(T(R)) = \bigsqcup_{p,u} \Omega_{pu}(T(R))$, where
\[
\Omega_{p,u}(T(R)) = \{(p, 1, u), (p, 2, u), \ldots, (p, p, u)\} \cap \Omega(T(R)).
\]
Now, if $\Omega^+_{p,u}(T(R)) := \Omega_{p,u} \cap \Omega^+(T(R))$, one can write $\Omega^+(T(R)) = \bigsqcup_{p,u} \Omega^+_{pu}(T(R))$. For $p$, $u$ fixed, let us denote by $s_{p,u}$ the number of different subsets of the form $\Omega^+_{p,u}(T(R))$. So, the number of different subsets of the form $\Omega^+(T(R))$ is $\prod_{p,u} s_{p,u}$. Let $\{T(R^{(i)})\}_{i=1}^{s_{pu}}$ be a set of tableaux such that $\{\Omega^+_{p,u}(T(R^{(i)}))\}_{i=1}^{s_{pu}}$ is the set of all distinct sets of the form $\Omega^+_{p,u}(T(R))$. We have a one-to-one correspondence between $\{T(R^{(i)})\}_{i=1}^{s_{pu}}$ and the set $\{0, i_1, \ldots, i_{s_{pu}}(T(L))\}$ constructed in Remark 7.5. More explicitly, this correspondence is defined my the map:
\[
T(R^{(i)}) \mapsto \begin{cases} 
\min\{j : (p, j, u) \in \Omega^+(T(R^{(i)}))\}, & \text{if } \Omega^+_{p,u}(T(R^{(i)})) \neq \emptyset, \\
0, & \text{if } \Omega^+_{p,u}(T(R^{(i)})) = \emptyset.
\end{cases}
\]
Therefore, $s_{pu} = d_{pu}(T(L)) + 1$. \hfill \blacksquare

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