Center of Twisted Graded Hecke Algebras for Homocyclic Groups

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Abstract. We determine explicitly the center of the twisted graded Hecke algebras associated to homocyclic groups. Our results are a generalization of formulas by M. Douglas and B. Fiol in [J. High Energy Phys. 2005 (2005), no. 9, 053, 22 pages].

Key words: twisted graded Hecke algebra; homocyclic group

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1 Main results

The notion of twisted graded Hecke algebras was introduced by S. Witherspoon in [10]; they are variants of the graded Hecke algebras of V. Drinfel’d [4] and G. Lusztig [6] (see also [7]) and twisted symplectic reflection algebras of T. Chmutova [2]. To a finite dimensional complex vector space $V$, a finite subgroup $G$ of $GL(V)$, and a 2-cocycle $\alpha$ of $G$, the associated twisted graded Hecke algebra $H$ is, by definition, a Poincaré–Birkhoff–Witt deformation of the crossed-product algebra $SV\#_{\alpha}G$, where $SV$ denotes the symmetric algebra of $V$. The center of $SV\#_{\alpha}G$ is $(SV)^G$, and it is a natural question to determine the center of $H$. In the non-twisted case, the center of the graded Hecke algebra associated to a finite real reflection group was determined by G. Lusztig in [5, Theorem 6.5]. In this paper, we determine the center of $H$ for the twisted graded Hecke algebra in [10, Example 2.16], where $V = \mathbb{C}^n$ and $G$ is isomorphic to a homocyclic group $(\mathbb{Z}/\ell\mathbb{Z})^{n-1}$. (By a homocyclic group, we mean a direct product of cyclic groups of the same order.) In this example, the algebra $H$ is finitely generated as a module over its center; the center of $H$ therefore plays an important role in the representation theory of $H$. We show that the center of $H$ is generated by $n+1$ elements subject to one relation, which we determine explicitly. Our results are a generalization of formulas by M. Douglas and B. Fiol who considered the special case when $n = 3$ in their paper [3] on $\mathbb{C}^3/(\mathbb{Z}/\ell\mathbb{Z})^2$ orbifolds with discrete torsion.

We state our main results in this section and give the proofs in Section 2. We shall work over $\mathbb{C}$. Let $n$ be an integer $\geq 3$, and $\ell$ an integer $\geq 2$. Let $V = \mathbb{C}^n$ and let $x_1, \ldots, x_n$ be the standard basis of $V$. Let $G$ be the subgroup of $SL_n(\mathbb{C})$ consisting of all diagonal matrices $g$ satisfying $g^\ell = 1$. Let $\zeta$ be a primitive $\ell$-th root of unity.

Notation 1.1. All subscripts are taken modulo $n$. For example, $x_{n+1} = x_1$.

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For $i = 1, \ldots, n$, let $g_i$ be the element of $G$ such that
\[
g_i(x_j) = \begin{cases} 
\zeta x_j, & \text{if } j = i, \\
\zeta^{-1} x_j, & \text{if } j = i + 1, \\
x_j, & \text{else.}
\end{cases}
\]

Observe that $g_n = g_1^{-1} \cdots g_{n-1}^{-1}$. We have an isomorphism $(\mathbb{Z}/\ell \mathbb{Z})^{n-1} \sim \sim G$ defined by sending $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ to $g_1, \ldots, g_{n-1}$, respectively.

Define the 2-cocycle $\alpha : G \times G \rightarrow \mathbb{C}^\times$ of $G$ by
\[
\alpha(g^i_1 \cdots g^i_{n-1}, g^j_1 \cdots g^j_{n-1}) = \zeta^{-i_1 i_2 - i_3 \cdots - i_{n-2} i_{n-1}}.
\]

If $E$ is an algebra, an action of $G$ on $E$ is a homomorphism $G \rightarrow \text{Aut}(E)$. Recall that for any algebra $E$ and an action of $G$ on $E$, one has the crossed product algebra $E \#_\alpha G$. As a vector space, $E \#_\alpha G$ is $E \otimes \mathbb{C}G$; the product is defined by
\[
(r \otimes g)(s \otimes h) = \alpha(g,h) r(g \cdot s) \otimes gh
\]
for all $r, s \in E$ and $g, h \in G$. If $g, h \in G$, then we shall denote their product in $E \#_\alpha G$ by $g * h$; thus,
\[
g * h = \alpha(g,h)gh.
\]

One has, for any $i, j \in \{1, \ldots, n\}$ with $|i - j| \notin \{1, n - 1\}$,
\[
g_{i+1} * g_i = \zeta g_i * g_{i+1}, \quad g_i * g_j = g_j * g_i.
\]

Let $t = (t_1, \ldots, t_n) \in \mathbb{C}^n$, and write $TV$ for the tensor algebra of $V$. Following [10, Example 2.16], we make the following definition.

**Definition 1.2.** Let $H$ be the associative algebra defined as the quotient of $TV \#_\alpha G$ by the relations:
\[
x_ix_{i+1} - x_{i+1}x_i = t_ig_i, \quad x_ix_j - x_jx_i = 0
\]
for all $i, j \in \{1, \ldots, n\}$ with $|i - j| \notin \{1, n - 1\}$.

**Remark 1.3.** By [10, Theorem 2.10] and [10, Example 2.16], the algebra $H$ in Definition 1.2 is a twisted graded Hecke algebra for $G$. (However, when $n > 3$ and $\ell = 2$, this is not the most general twisted graded Hecke algebra for $G$; see [10, Example 2.16] and [9, Example 5.1].)

Let $\mathbb{C}[y_1^\pm, \ldots, y_n^\pm]$ be the algebra of Laurent polynomials in the variables $y_1, \ldots, y_n$. The group $G$ acts on $\mathbb{C}[y_1^\pm, \ldots, y_n^\pm]$ by
\[
g_i y_1^{p_1} \cdots y_n^{p_n} = \zeta^{p_n-p_{n+1}} y_1^{p_1} \cdots y_n^{p_n}
\]
for all $i \in \{1, \ldots, n-1\}$ and $p_1, \ldots, p_n \in \mathbb{Z}$.

**Proposition 1.4.** There is an injective homomorphism
\[
\Theta : H \rightarrow \mathbb{C}[y_1^\pm, \ldots, y_n^\pm] \#_\alpha G
\]
such that
\[
\Theta(x_i) = y_i - \left( \frac{\zeta t_i}{\zeta - 1} \right) y_{i+1}^{-1} g_i, \quad i = 1, \ldots, n - 1
\]
(1.1)
\[
\Theta(g_i) = g_i
\]
(1.2)
for all $i \in \{1, \ldots, n\}$.
Let
\[ I = \{ \{i_1 < \cdots < i_k\} \mid k \geq 0; \; i_1, \ldots, i_k \in \{1, \ldots, n\} \}, \]
\[ J = \{ \{i_1 < \cdots < i_k\} \in I \mid |i_r - i_s| \not\in \{1, n-1\} \text{ for all } r, s \}. \]

Define the elements \( \delta, \varepsilon_1, \ldots, \varepsilon_n \) of \( \mathbb{Z}^n \) by
\[ \delta = (1, 1, \ldots, 1), \]
\[ \varepsilon_1 = (1, 1, 0, \ldots, 0), \]
\[ \varepsilon_2 = (0, 1, 1, 0, \ldots), \]
\[ \ldots, \]
\[ \varepsilon_n = (1, 0, \ldots, 0, 1). \]

**Notation 1.5.** For any variables \( \omega_1, \ldots, \omega_n \) and \( p = (p_1, \ldots, p_n) \in \mathbb{Z}^n \), we denote by \( \omega^p \) the expression \( \omega_1^{p_1} \cdots \omega_n^{p_n} \).

We set
\[ \tau_i = \frac{t_i}{\zeta - 1} \quad \text{for} \quad i = 1, \ldots, n - 1, \quad \tau_n = \frac{\zeta t_n}{\zeta - 1}. \]

Define the element \( w \in \mathcal{H} \) by
\[ w = \sum_{\{i_1 < \cdots < i_k\} \in J} \tau_{i_1} \cdots \tau_{i_k} x^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} g_{i_1} \cdots g_{i_k}. \]

**Example 1.6.** If \( n = 3 \), then
\[ w = x_1 x_2 x_3 + \tau_1 x_3 g_1 + \tau_2 x_1 g_2 + \tau_3 x_2 g_3 = x_1 x_2 x_3 + \frac{1}{\zeta - 1} (t_1 x_3 g_1 + t_2 x_1 g_2 + \zeta t_3 x_2 g_3). \]

In particular, if \( n = 3 \) and \( \ell = 2 \), the formula for \( w \) is in [1, Lemma 7.1].

**Theorem 1.7.** The center of \( \mathcal{H} \) is generated as an algebra by \( x_1^\ell, \ldots, x_n^\ell \), and \( w \).

Let \( Z \) be the center of \( \mathcal{H} \). For \( r = 0, \ldots, \lfloor \ell/2 \rfloor \), set
\[ \nu_r = (-1)^r \frac{\ell}{\ell - r} \binom{\ell - r}{r}, \]
and set
\[ \widetilde{\tau}_i = \tau_i^\ell \quad \text{for} \quad i = 1, \ldots, n - 1, \quad \widetilde{\tau}_n = (-1)^{n-1} \tau_n^\ell. \]

We define a polynomial \( F \) in the \( n + 1 \) variables \( a_1, \ldots, a_n \) and \( b \) by
\[ F = \sum_{\{i_1 < \cdots < i_k\} \in J} \widetilde{\tau}_{i_1} \cdots \widetilde{\tau}_{i_k} a^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} - \sum_{r=0}^{\lfloor \ell/2 \rfloor} (-1)^r \nu_r \zeta^{r(n-2)} \nu_r (\tau_1 \cdots \tau_n)^r b^{\ell - 2r}. \quad (1.3) \]

**Corollary 1.8.** The assignment
\[ a_i \mapsto x_i^\ell \quad \text{for} \quad i = 1, \ldots, n, \quad b \mapsto w \quad (1.4) \]
defines an isomorphism
\[ \mathbb{C}[a_1, \ldots, a_n, b]/(F) \xrightarrow{\sim} Z. \quad (1.5) \]

In the undeformed case, when \( t_1 = \cdots = t_n = 0 \), the polynomial \( F \) is equal to \( a_1 \cdots a_n - b^\ell \).
2 Proof of main results

Proof of Proposition 1.4. For $i = 1, \ldots, n - 1$, we define $\Theta(x_i)$, $\Theta(x_n)$, and $\Theta(g_i)$ by (1.1) and (1.2). It follows from a straightforward verification that $\Theta$ is a well-defined homomorphism.

It remains to see that $\Theta$ is injective. Observe that $H$ is spanned by the monomials $x^p y^q$ for $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $q \in G$, where $p_1, \ldots, p_n \geq 0$. We call $p_1 + \cdots + p_n$ the total degree of the monomial $x^p y^q$. The image of $x^p y^q$ under $\Theta$ is the sum of $y^p q$ with terms of strictly smaller total degrees. Therefore, if $\alpha \in H$ is nonzero, we can write it as a sum $\alpha_0 + \alpha_1 + \cdots$, where $\alpha_k$ is a linear combination of monomials $x^p y^q$ with total degree $k$. If $k$ is the maximal integer with $\alpha_k$ nonzero, then $\Theta(\alpha_k)$ is nonzero, and hence $\Theta(\alpha)$ is also nonzero.

Remark 2.1. It follows from Proposition 1.4 that the monomials $x_1^{p_1} \cdots x_n^{p_n}$ for non-negative integers $p_1, \ldots, p_n$ and $g \in G$ form a basis for $H$ (called the PBW basis of $H$). This was first proved in [10, Example 2.16] using [10, Theorem 2.10].

We have an increasing filtration on $H$ defined by setting $\deg(x_i) = 1$ and $\deg(g) = 0$ for all $i \in \{1, \ldots, n\}$, $g \in G$.

The proof of (2.3) in the following lemma is the key calculation in this paper.

Lemma 2.2.

(i) One has:

\[
\Theta(x_i^\ell) = y_i^\ell - \zeta^\ell_i y_{i+1}^\ell, \\
\Theta(x_n^\ell) = y_n^\ell - (-1)^{(n-1)} \zeta_n^\ell y_1^\ell,
\]

for all $i \in \{1, \ldots, n-1\}$.

(ii) One has:

\[
\Theta(u) = y_1 \cdots y_n + (-1)^{n} \zeta^{n-2} \tau_1 \cdots \tau_n y_1^{-1} \cdots y_n^{-1}.
\]

Proof. (i) To prove (2.1), we need to show that

\[
(y_i - \zeta \tau_i y_{i+1}^{-1}g_i) \cdots (y_i - \zeta \tau_i y_{i+1}^{-1}g_i) = y_i^\ell - \tau_i^\ell y_{i+1}^\ell.
\]

Since $g_i y_i = \zeta y_i g_i$ and $g_i y_{i+1}^{-1} = \zeta y_{i+1}^{-1} g_i$, the product on the left hand side of (2.4) is a linear combination of $y_i^k y_{i+1}^{-1} g_i^{\ell-k}$ for $k = 0, 1, \ldots, \ell$. Moreover, the coefficient of $y_i^k y_{i+1}^{-1} g_i^{\ell-k}$ in this linear combination is the same as the coefficient of $u^k$ when we expand the product

\[
(u - \zeta\tau_i)(u - \zeta^{-1}\tau_i) \cdots (u - \zeta\tau_i)
\]

in the polynomial ring $\mathbb{C}[u]$. Since the polynomial in (2.5) is equal to $u^\ell - \tau_i^\ell$, the identity (2.1) follows. The proof of (2.2) is similar except that

\[
\prod_{i=1}^{n} g_i = (-1)^{(n-1)}.
\]

(ii) For any $h_* = \{h_1 < \cdots < h_j\} \in I$, we let

\[
\begin{align*}
h'_* &= \{h_r \in h_* \mid h_s - h_r \in \{1, 1-n\} \text{ for some } s\}, \\
\chi(h_*) &= |\{h_r \in h'_* \mid h_r \neq n\}| - |\{h_r \in h'_* \mid h_r = n\}|,
\end{align*}
\]
\[ E(h_*) = \zeta(x(h_*)) \tau_1 \cdots \tau_n y_{\delta}^{-\varepsilon_{h_1}} \cdots \varepsilon_{h_j} g_{h_1} \cdots g_{h_j}. \]

Now suppose \( i_* = \{ i_1, \ldots, i_k \} \in J \). Let \( D \) be the subset of \( \{ 1, \ldots, n \} \) consisting of all \( d \) such that \( d \neq i_r, i_r + 1 \) (mod \( n \)) for all \( r \). We denote by \( d_1 < \cdots < d_p \) the elements of \( D \). Then

\[
\Theta(\tau_1 \cdots \tau_k x^{\delta_{\varepsilon_{i_1}} \cdots \varepsilon_{i_k}} g_{i_1} \cdots g_{i_k}) = \tau_1 \cdots \tau_k \left( y_{d_1} - \frac{\zeta t_{d_1}}{\zeta - 1} y_{d_1}^{-1} \right) \cdots \left( y_{d_p} - \frac{\zeta t_{d_p}}{\zeta - 1} y_{d_p}^{-1} \right) g_{i_1} \cdots g_{i_k}
\]

where, for \( r = 1, \ldots, p \),

\[
Y_{d_r}(S) = \begin{cases} 
   y_{d_r}, & \text{if } d_r \notin S, \\
   -\zeta(\zeta - 1)^{-1} t_{d_r} y_{d_r}^{-1} g_{d_r}, & \text{if } d_r \in S.
\end{cases}
\]

Setting \( h_* = i_* \cup S \), we obtain \(^1\)

\[
\Theta(\tau_1 \cdots \tau_k x^{\delta_{\varepsilon_{i_1}} \cdots \varepsilon_{i_k}} g_{i_1} \cdots g_{i_k}) = \sum_{h_* \in I \setminus i_* \subset h_*'} (-1)^{|h_*| - |i_*|} E(h_*).
\]

Hence,

\[
\Theta(w) = \sum_{\{i_1 < \cdots < i_k\} \in J} \Theta(\tau_1 \cdots \tau_k x^{\delta_{\varepsilon_{i_1}} \cdots \varepsilon_{i_k}} g_{i_1} \cdots g_{i_k}) = \sum_{i_* \in J} \sum_{h_* \in I \setminus i_* \subset h_*'} (-1)^{|h_*| - |i_*|} E(h_*)) = \sum_{h_* \in I} \left( E(h_*) \sum_{i_* \subset h_* \subset h_*'} (-1)^{|h_*| - |i_*|} \right).
\]

If \( |h_*| = n \), then \( h_*' = h_* \). If \( |h_*| \notin \{0, n\} \), then \( h_*' \neq h_* \). Therefore,

\[
E(h_*) \sum_{i_* \subset h_* \subset h_*'} (-1)^{|h_*| - |i_*|} = \begin{cases} 
   y_1 \cdots y_n, & \text{if } |h_*| = 0, \\
   (-1)^n \zeta^{-2} \sum \tau_1 \cdots \tau_n y_1^{-1} \cdots y_n^{-1} & \text{if } |h_*| = n, \\
   0 & \text{else}.
\end{cases}
\]

**Proof of Theorem 1.7.** It is easy to see that the center of \( SV \#_G \) is the algebra of \( G \)-invariant elements \( (SV)^G \) of \( SV \), and moreover, the algebra \( (SV)^G \) is generated by \( x_i^\ell \) \((i = 1, \ldots, n)\) and \( x_1 \cdots x_n \).

Using Lemma 2.2, we see that

\[
\Theta(x_i^\ell) \text{ for } i = 1, \ldots, n, \quad \text{and} \quad \Theta(w)
\]

are in the center of \( \mathbb{C}[y_1^\pm, \ldots, y_n^\pm] \#_G \). Since the homomorphism \( \Theta \) is injective, the elements \( x_i^\ell \) \((i = 1, \ldots, n)\) and \( w \) are in the center of \( H \). Since the principal symbols of \( x_i^\ell, \ldots, x_n^\ell \) and \( w \) in \( SV \#_G \) are, respectively, \( x_i^\ell, \ldots, x_n^\ell \) and \( x_1 \cdots x_n \), the theorem follows from a standard argument. \( \blacksquare \)

\(^1\)Note that if \( d_r \in S \) but \( d_r + 1 \in D - S \), then the term \( g_{d_r} \) in \( Y_{d_r}(S) \) appears on the left of the term \( y_{d_r+1} \) of \( Y_{d_r+1}(S) \) and one has \( g_d y_{d_r+1} = \zeta^{-1} y_{d_r+1} g_d \). However, if \( n \in S \) but \( 1 \in D - S \), then the term \( g_n \) in \( Y_n(S) \) already appears to the right of the term \( y_1 \) of \( Y_1(S) \). This is the reason why the definition of \( \tau_n \) differs from the corresponding definitions of \( \tau_1, \ldots, \tau_{n-1} \) by a factor of \( \zeta \).
Proof of Corollary 1.8. Let $\tilde{a}_1 = \Theta(x_1^0), \ldots, \tilde{a}_n = \Theta(x_n^0)$, and $\tilde{b} = \Theta(w)$. By Lemma 2.2,

\[ \tilde{a}_i = y_i^\ell \tau_i y_{i+1}^{-\ell} \quad \text{for} \quad i = 1, \ldots, n, \]
\[ \tilde{b} = y_1 \cdots y_n + (-1)^n \zeta n^{-2} \tau_1 \cdots \tau_n y_1^{-1} \cdots y_n^{-1}. \]

By a calculation completely similar to the proof of (2.3), one has

\[ \sum_{\{i_1 < \cdots < i_k\} \in J} \tilde{a}_{i_1} \cdots \tilde{a}_{i_k} \tilde{a}^{\delta_{i_1} \cdots \delta_{i_k}} = (y_1 \cdots y_n)^\ell + (-1)^n (\tau_1 \cdots \tau_n)^\ell (y_1 \cdots y_n)^{-\ell}. \quad (2.6) \]

We claim that we also have

\[ \sum_{r=0}^{[\ell/2]} (-1)^n \zeta^{(n-2)r} \nu_r (\tau_1 \cdots \tau_n)^{\ell-2r} = (y_1 \cdots y_n)^\ell + (-1)^n (\tau_1 \cdots \tau_n)^\ell (y_1 \cdots y_n)^{-\ell}. \quad (2.7) \]

To see this, recall that the Chebyshev polynomials of the first kind are defined recursively by

\[ T_0(\xi) = 1, \quad T_1(\xi) = \xi, \quad \text{and} \]
\[ T_m(\xi) = 2\xi T_{m-1}(\xi) - T_{m-2}(\xi) \quad \text{for} \quad m = 2, 3, \ldots. \]

It is well known (and can be easily proved by induction) that

\[ 2T_\ell \left( \frac{\xi}{2} \right) = \sum_{r=0}^{[\ell/2]} \nu_r \xi^{\ell-2r}, \quad (2.8) \]
\[ 2T_\ell \left( \frac{\xi + \xi^{-1}}{2} \right) = \xi^\ell + \xi^{-\ell}. \quad (2.9) \]

By (2.8) and (2.9), one has the identity

\[ \xi^\ell + \xi^{-\ell} = \sum_{r=0}^{[\ell/2]} \nu_r (\xi + \xi^{-1})^{\ell-2r}, \]

and hence the identity

\[ \xi^\ell + \varrho^{2\ell} \xi^{-\ell} = \sum_{r=0}^{[\ell/2]} \nu_r \varrho^{2r} (\xi + \varrho^2 \xi^{-1})^{\ell-2r} \]

where $\xi$ and $\varrho$ are formal variables. By setting $\xi = y_1 \cdots y_n$ and choosing $\varrho$ to be a square-root of $(-1)^n \zeta n^{-2} \tau_1 \cdots \tau_n$, we obtain (2.7).

By Proposition 1.4, Theorem 1.7, and the equations (2.6) and (2.7), the assignment (1.4) defines a surjective homomorphism

\[ \Phi : \mathbb{C}[a_1, \ldots, a_n, b] \to \mathbb{Z} \]

such that $\Phi(F) = 0$. Suppose $D \in \mathbb{C}[a_1, \ldots, a_n, b]$ and $\Phi(D) = 0$. We can write

\[ D = \sum_{r=0}^{\ell-1} D_r(a_1, \ldots, a_n) b^r + R, \]

where $D_r(a_1, \ldots, a_n) \in \mathbb{C}[a_1, \ldots, a_n]$ for $r = 0, \ldots, \ell - 1$, and $R \in (F)$. Thus,

\[ \sum_{r=0}^{\ell-1} D_r(x_1^\ell, \ldots, x_n^\ell) w^r = 0. \quad (2.10) \]
We claim that $D_r(a_1, \ldots, a_n) = 0$ for all $r$. Suppose not; then let $m$ be the maximal integer such that $D_m(a_1, \ldots, a_n) \neq 0$. Let $x_1^l p_1 \cdots x_n^l p_n$ be a monomial in $D_m(x_1^l, \ldots, x_n^l)$ with nonzero coefficient. Since $0 \leq m < \ell$, when we write the left hand side of (2.10) in terms of the PBW basis, the coefficient of $x_1^l p_1 + m \cdots x_n^l p_n + m$ is nonzero, a contradiction. Hence, the kernel of $\Phi$ is $(F)$. This proves (1.5). ■

**Remark 2.3.** When $n = 3$, the algebra $H$ is Morita equivalent to a deformed Sklyanin algebra $S_{\text{def}}$ defined by C. Walton in [8, Definition IV.2]. More precisely, if $n = 3$ and

$$e = \frac{1}{\ell} \sum_{r=0}^{\ell-1} g_1^r,$$

one has $HeH = H$ and $eHe \cong S_{\text{def}}$ where the parameters for $S_{\text{def}}$ (following the notations in [8, Definition IV.2]) are $a = 1$, $b = \zeta$, $c = d_i = 0$, and $e_i = -\zeta t_i$ for $i = 1, 2, 3$. This follows from the observation that, for $n = 3$, setting $\phi_i = x_i g_{i+1}$, one has $\phi_i \phi_{i+1} - \zeta \phi_{i+1} \phi_i = \zeta t_i$ for all $i$. The algebra $S_{\text{def}}$ (with above parameters) was first studied by M. Douglas and B. Fiol, see [3, (3.10)]. Our formulas (1.1)–(1.2) are a generalization of [3, (4.6)], and our equation (1.3) is a generalization of [3, (4.7)]. The formulas in (2.1)–(2.3) are generalizations of [3, (4.8)].

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**References**


