Generalised Chern–Simons Theory 
and $G_2$-Instantons over Associative Fibrations

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Abstract. Adjusting conventional Chern–Simons theory to $G_2$-manifolds, one describes
$G_2$-instantons on bundles over a certain class of 7-dimensional flat tori which fiber non-
trivially over $T^4$, by a pullback argument. Moreover, if $c_2 \neq 0$, any (generic) deformation of
the $G_2$-structure away from such a fibred structure causes all instantons to vanish. A brief
investigation in the general context of (conformally compatible) associative fibrations $f : 
Y^7 \to X^4$ relates $G_2$-instantons on pullback bundles $f^*E \to Y$ and self-dual connections on
the bundle $E \to X$ over the base, a fact which may be of independent interest.

Key words: Chern–Simons; Yang–Mills; $G_2$-manifolds; associative fibrations

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1 Introduction

This article fits in the context of gauge theory in higher dimensions, following the seminal
works of S. Donaldson & R. Thomas, G. Tian and others [4, 16]. The common thread to such
generalisations is the presence of a closed differential form on the base manifold $Y$, inducing
an analogous notion of anti-self-dual connections, or instantons, on bundles over $Y$. In the
case at hand, $G_2$-manifolds are 7-dimensional Riemannian manifolds with holonomy in the Lie
group $G_2$, which implies the existence of precisely such a structure. This allows one to make
sense of $G_2$-instantons as the energy-minimising gauge classes of connections, solutions to the
Corresponding Yang–Mills equation.

Heuristically, $G_2$-instantons are somewhat analogous to flat connections in dimension 3.
Given a bundle over a compact 3-manifold, with space of connections $\mathcal{A}$ and gauge group $\mathcal{G}$, the
Chern–Simons functional is a multi-valued real function on the quotient $B = \mathcal{A}/\mathcal{G}$, with integer
periods, whose critical points are precisely the flat connections [3, § 2.5]. Similar theories can
be formulated in higher dimensions in the presence of a suitable closed differential form [4, 15];
e.g. on a $G_2$-manifold $(Y, \varphi)$, the coassociative 4-form $\ast \varphi$ allows for the definition of a functional
of Chern–Simons type$^1$. Its ‘gradient’, the Chern–Simons 1-form, vanishes precisely at the $G_2$-
instantons, hence it detects the solutions to the Yang–Mills equation. These gauge-theoretic
preliminaries are covered in Section 2.

On the other hand, one may understand $G_2$-manifolds as a particular case of the rich theory
of calibrated geometries [6], for which the $G_2$-structure $\varphi$ is a calibration 3-form. Then a 3-
dimensional submanifold $P$ is said to be associative if it is calibrated by $\varphi$, i.e., if $\varphi|_P =
d\text{Vol}|_P$. The deformation theory of associative submanifolds is known to be obstructed [9], so
their occurrence in families, e.g. fibering over a 4-manifold, is nongeneric and somewhat exotic.
Nonetheless, we may consider theoretically, at first, the existence of instantons over associative

$^1$In fact only the condition $d\ast \varphi = 0$ is required, so the discussion extends to cases in which the $G_2$-structure $\varphi$
is not necessarily torsion-free.
fibrations \( f: Y^7 \to X^4 \). Given a bundle \( E \to X \), a connection \( A \) on its pullback \( E \) is locally of the form

\[
A = A_{t}(x) + \sum_{i=1}^{3} \sigma_{i}(x,t)dt^{i},
\]

where \( \{A_{t}\} \) is a family of connections on \( E \) parametrised by the associative fibers \( P_{x} := f^{-1}(x) \) and \( \sigma_{i} \in \Omega^{0}(Y, f^{*}g_{E}) \). In Section 3.1 I prove the following relation between \( G_{2} \)-instantons and families of self-dual connections over the base:

**Theorem 1.** Let \( f: Y \to X \) define an associative fibration and \( E \to Y \) be the pullback from an indecomposable vector bundle \( E \to X \).

(i) If a connection \( A \) on \( E \) is a \( G_{2} \)-instanton, then \( \{A_{t}\} \) is a family of self-dual connections on \( E \), satisfying

\[
\frac{\partial A_{t}}{\partial t^{i}} = d_{A_{t}}\sigma_{i}, \quad i = 1, 2, 3.
\]

(ii) If, moreover, the family \( A_{t} \equiv A_{t_{0}} \) is constant, then \( A = f^{*}A_{t_{0}} \) is a pullback.

NB: We denote henceforth by \( \mathcal{M}^{4}_{+} \) the moduli space of SD connections on the base and by \( \mathcal{M}^{7}_{\varphi} \) the moduli space of \( G_{2} \)-instantons relative to \( G_{2} \)-structure \( \varphi \).

Finally, over the remaining of Section 3, these ideas are applied to a concrete example of certain \( T^{3} \)-fibrations over \( T^{4} \), topologically equivalent to the 7-torus, which I will call \( G_{2} \)-torus fibrations [11]. Deforming the metric (i.e. the lattice) on \( T^{4} \) induces a change on the fibration map and hence on the \( G_{2} \)-structure, and one can use Chern–Simons formalism to see how this affects the moduli of \( G_{2} \)-instantons:

**Theorem 2.** Let \( f: \mathbb{T} \to T^{4} \) be a \( G_{2} \)-torus fibration, \( E \to \mathbb{T} \) be the pullback of an indecomposable vector bundle \( E \to T^{4} \) and \( \varphi \) denote the \( G_{2} \)-structure of \( \mathbb{T} \); then

(i) every SD connection on \( E \) lifts to a \( G_{2} \)-instanton on \( E \), i.e.,

\[
f^{*}\mathcal{M}^{4}_{+} \subset \mathcal{M}^{7}_{\varphi};
\]

(ii) if, moreover, \( c_{2}(E) \neq 0 \), then any perturbation \( \varphi + \phi \) away from the class of fibred structures causes the moduli space of \( G_{2} \)-instantons to vanish, i.e.,

\[
\mathcal{M}^{7}_{\varphi + \phi} = \varnothing.
\]

The construction of \( G_{2} \)-instantons is a recent and active research area. Indeed Theorem 2 yields nontrivial, albeit nongeneric, examples of \( G_{2} \)-instanton moduli, whenever a complex vector bundle \( E \to T^{4} \) admits SD connections. The interested reader will find other examples in works of Walpuski, Clarke and the author [2, 11, 12, 13, 17]. In the high-energy physics community, solutions to a very similar problem in the context of \( G_{2} \)-structures with torsion have been found eg. for cylinders over nearly-Kähler homogeneous spaces [5] and more generally for cones over nontrivial manifolds admitting real Killing spinors [7].

Finally, a paper just published by Wang [18] makes significant progress towards a Donaldson theory over higher-dimensional foliations, which seems to encompass our \( G_{2} \)-torus fibration as a special, codimension 4 tight foliation, whose leaf space is the smooth 4-manifold \( X \). It is inspiring to speculate whether an invariant of the corresponding foliated moduli space can be explicitly computed for some suitable bundle \( E \to \mathbb{T} \), or indeed if that space coincides with our definition of \( \mathcal{M}^{7} \).
2 Gauge theory over $G_2$-manifolds

I will concisely recall the essentials of gauge theory on $G_2$-manifolds, while referring the interested reader to a more detailed exposition in [12].

Let $Y$ be an oriented smooth 7-manifold; a $G_2$-structure is a smooth 3-form $\varphi \in \Omega^3(Y)$ such that, at every point $p \in Y$, one has $\varphi_p = f_p^*(\varphi_0)$ for some frame $f_p : T_p Y \to \mathbb{R}^7$ and (adopting the conventions of [14])

$$\varphi_0 = e^{567} + \omega_1 \wedge e^5 + \omega_2 \wedge e^6 + \omega_3 \wedge e^7$$

with

$$\omega_1 = e^{12} - e^{34}, \quad \omega_2 = e^{13} - e^{42}, \quad \text{and} \quad \omega_3 = e^{14} - e^{23}.$$ 

Moreover, $\varphi$ determines a Riemannian metric $g(\varphi)$ induced by the pointwise inner-product

$$\langle u, v \rangle_{e^{1...7}} := \frac{1}{6}(u \wedge \varphi_0) \wedge (v \wedge \varphi_0) \wedge \varphi_0,$$ 

under which $^\ast \varphi_0$ is given pointwise by

$$^\ast \varphi_0 = e^{1234} - \omega_1 \wedge e^{67} - \omega_2 \wedge e^{75} - \omega_3 \wedge e^{56}.$$ 

Such a pair $(Y, \varphi)$ is a $G_2$-manifold if $d\varphi = 0$ and $d^\ast \varphi = 0$.

2.1 The $G_2$-instanton equation

The $G_2$-structure allows for a 7-dimensional analogue of conventional Yang–Mills theory, yielding a notion of (anti-)self-duality for 2-forms. Under the usual identification between 2-forms and matrices, we have $g_2 \subset \mathfrak{so}(7) \simeq \Lambda^2$, so we denote $\Lambda^2_{\mathbb{R}} := g_2$ and $\Lambda^2_{\mathbb{R}}$ its orthogonal complement in $\Lambda^2$:

$$\Lambda^2 = \Lambda^2_{\mathbb{R}} \oplus \Lambda^2_{\mathbb{R}}.$$ 

It is easy to check that $\Lambda^2_{\mathbb{R}} = \langle e_1 \cdot \varphi_0, \ldots, e_7 \cdot \varphi_0 \rangle$, hence the orthogonal projection onto $\Lambda^2_{\mathbb{R}}$ in (4) is given by

$$L_{\ast \varphi_0} : \Lambda^2 \to \Lambda^6,$$

$$\eta \mapsto \eta \wedge ^\ast \varphi_0$$

in the sense that [1, p. 541]

$$L_{\ast \varphi_0} | _{\Lambda^2_{\mathbb{R}}} : \Lambda^2_{\mathbb{R}} \to \Lambda^6 \quad \text{and} \quad L_{\ast \varphi_0} | _{\Lambda^2_{\mathbb{R}}} = 0.$$ 

Furthermore, since (4) splits $\Lambda^2$ into irreducible representations of $G_2$, a little inspection on generators reveals that $(\Lambda^2_{\mathbb{R}})_{\mathbb{T}}$ is respectively the $-2$-eigenspace of the $G_2$-equivariant linear map

$$T_{\varphi_0} : \Lambda^2 \to \Lambda^2,$$

$$\eta \mapsto T_{\varphi_0} \eta := \ast (\eta \wedge \varphi_0).$$

Consider now a vector bundle $E \to Y$ over a compact $G_2$-manifold $(Y, \varphi)$; the curvature $F := F_A$ of some connection $A$ decomposes according to the splitting (4):

$$F_A = F_7 \oplus F_{14}, \quad F_i \in \Omega^2_i(\text{End } E), \quad i = 7, 14.$$
The $L^2$-norm of $F_A$ is the Yang–Mills functional:

$$\text{YM}(A) := \|F_A\|^2 = \|F_7\|^2 + \|F_{14}\|^2.$$  \hfill (5)

It is well-known that the values of $\text{YM}(A)$ can be related to a certain characteristic class of the bundle $E$, given (up to choice of orientation) by

$$\kappa(E) := -\int_Y \text{tr} (F_A^2) \wedge \varphi.$$ 

Using the property $d\varphi = 0$, a standard argument of Chern–Weil theory [10] shows that the de Rham class $[\text{tr}(F_A^2) \wedge \varphi]$ is independent of $A$, thus the integral is indeed a topological invariant. The eigenspace decomposition of $T_\varphi$ implies (up to a sign)

$$\kappa(E) = -2\|F_7\|^2 + \|F_{14}\|^2,$$

and combining with (5) we get

$$\text{YM}(A) = -\frac{1}{2}\kappa(E) + 3\|F_{14}\|^2 = \kappa(E) + 3\|F_7\|^2.$$ 

Hence $\text{YM}(A)$ attains its absolute minimum at a connection whose curvature lies either in $\Lambda^2_7$ or in $\Lambda^2_{14}$. Moreover, since $\text{YM} \geq 0$, the sign of $\kappa(E)$ obstructs the existence of one type or the other, so we fix $\kappa(E) \geq 0$ and define $G_2$-instantons as connections with $F \in \Lambda^2_{14}$, i.e., such that $\text{YM}(A) = \kappa(E)$. These are precisely the solutions of the $G_2$-instanton equation:

$$F_A \wedge \ast \varphi = 0$$ \hfill (6a)

or, equivalently,

$$F_A - \ast (F_A \wedge \varphi) = 0.$$ \hfill (6b)

If instead $\kappa(E) \leq 0$, we may still reverse orientation and consider $F \in \Lambda^2_{14}$, but then the above eigenvalues and energy bounds must be adjusted accordingly, which amounts to a change of the ($-$) sign in (6b).

### 2.2 Definition of the Chern–Simons functional $\vartheta$

Gauge theory in higher dimensions can be formulated in terms of the geometric structure of manifolds with exceptional holonomy [4]. In particular, instantons can be characterised as critical points of a Chern–Simons functional, hence zeroes of its gradient 1-form [3]. The explicit case of $G_2$-manifolds, which we now describe, was first examined in the author’s thesis [11].

Let $E \to Y$ be a vector bundle; the space $\mathcal{A}$ is an affine space modelled on $\Omega^1(g_E)$ so, fixing a reference connection $A_0 \in \mathcal{A}$,

$$\mathcal{A} = A_0 + \Omega^1(g_E)$$

and, accordingly, vectors at $A \in \mathcal{A}$ are 1-forms $a, b, \ldots \in T_A \mathcal{A} \simeq \Omega^1(g_E)$ and vector fields are maps $\alpha, \beta, \ldots : \mathcal{A} \to \Omega^1(g_E)$. In this notation we define the Chern–Simons functional by

$$\vartheta(A) := \frac{1}{2} \int_Y \text{tr} \left( d_{A_0} a \wedge a + \frac{2}{3} a \wedge a \wedge a \right) \wedge \ast \varphi,$$

fixing $\vartheta(A_0) = 0$. This function is obtained by integration of the Chern–Simons 1-form

$$\rho(\beta)_A = \rho_A(\beta_A) := \int_Y \text{tr}(F_A \wedge \beta_A) \wedge \ast \varphi.$$ \hfill (7)
We find $\vartheta$ explicitly by integrating $\rho$ over paths $A(t) = A_0 + ta$, from $A_0$ to any $A = A_0 + a$:

$$\vartheta(A) - \vartheta(A_0) = \int_0^1 \rho_{A(t)}(\dot{A}(t))dt = \int_0^1 \left( \int_Y \text{tr} \left( (F_{A_0} + td_{A_0}a + t^2 a \wedge a) \wedge a \right) \right) dt$$

$$= \frac{1}{2} \int_Y \text{tr} \left( d_{A_0}a \wedge a + \frac{2}{3} a \wedge a \wedge a \right) \wedge \ast \varphi + K,$$

where $K = K(A_0, a)$ is a constant and vanishes if $A_0$ is an instanton.

The co-closedness condition $d \ast \varphi = 0$ implies that the 1-form (7) is closed, so the procedure doesn’t depend on the path $A(t)$. Indeed, given tangent vectors $a, b \in \Omega^1(\mathfrak{g}_E)$ at $A$, the leading term in the expansion of $\rho$,

$$\rho_{A+a}(b) - \rho_{A}(b) = \int_Y \text{tr}(d_A a \wedge b) \wedge \ast \varphi + O(|b|^2),$$

is symmetric by Stokes’ theorem:

$$\int_Y \text{tr}(d_A a \wedge b - a \wedge d_A b) \wedge \ast \varphi = \int_Y \text{d}(\text{tr}(b \wedge a) \wedge \ast \varphi) = 0.$$

We conclude that

$$\rho_{A+a}(b) - \rho_{A}(b) = \rho_{A+a}(a) - \rho_{A}(a) + O(|b|^2)$$

and, comparing reciprocal Lie derivatives on parallel vector fields $\alpha \equiv a, \beta \equiv b$ near a point $A$, we have:

$$d\rho(\alpha, \beta)_A = (\mathcal{L}_b \rho)_A(a) - (\mathcal{L}_a \rho)_A(b) = \lim_{h \to 0} \frac{1}{h} \{ \rho_{A+hb}(a) - \rho_{A}(a) \} - (\rho_{A+ha}(b) - \rho_{A}(b)) \}

= \lim_{h \to 0} \frac{1}{h} \{ (\rho_{A+hb}(ha) - \rho_{A}(ha)) - (\rho_{A+ha}(hb) - \rho_{A}(hb)) \} = 0.$$

Since $\mathcal{A}$ is contractible, by the Poincaré lemma $\rho$ is the derivative of some function $\vartheta$. Again by Stokes, $\rho$ vanishes along $G$-orbits $\text{im}(d_A) \simeq T_A \{ G.A \}$. Thus $\rho$ descends to the quotient $\mathcal{B}$ and so does $\vartheta$, locally.

### 2.3 Periodicity of $\vartheta$

Consider the gauge action of $g \in G$ and some path $\{ A(t) \}_{t \in [0,1]} \subset \mathcal{A}$ connecting an instanton $A$ to $g.A$. The natural projection $p_1 : Y \times [0,1] \to Y$ induces a bundle

$$\begin{array}{ccc}
E_g & \overset{\tilde{p}_1}{\longrightarrow} & E \\
\downarrow & & \downarrow \\
Y \times [0,1] & \overset{p_1}{\longrightarrow} & Y
\end{array}$$

and, using $g$ to identify the fibres $(E_g)_0 \simeq (E_g)_1$, one may think of $E_g$ as a bundle over $Y \times S^1$. Moreover, in some local trivialisation, the path $A(t) = A_i(t)dx^i$ gives a connection $\mathbf{A} = A_0 dt + A_i dx^i$ on $E_g$:

$$(A_0)(t, \rho) = 0, \quad (A_i)(t, \rho) = A_i(t).$$

The corresponding curvature 2-form is $F_{\mathbf{A}} = (F_{\mathbf{A}})_0 dx^i \wedge dt + (F_{\mathbf{A}})_{jk} dx^j \wedge dx^k$, where

$$(F_{\mathbf{A}})_0 = \dot{A}_i, \quad (F_{\mathbf{A}})_{jk} = (F_A)_{jk}.$$
The periods of \( \vartheta \) are then of the form
\[
\vartheta(g,A) - \vartheta(A) = \int_0^1 \rho_{A(t)}(\dot{A}(t)) dt = \int_{Y \times [0,1]} \text{tr}(F_{A(t)} \wedge \dot{A}(t) dx^1) \wedge dt \wedge \ast \varphi \\
= \int_{Y \times S^1} \text{tr} F_A \wedge F_A \wedge \ast \varphi = \frac{1}{8\pi^2} \langle c_2(E_g) \sim \ast \varphi, Y \times S^1 \rangle.
\]

The Künneth formula for \( Y \times S^1 \) gives
\[
H^4(Y \times S^1, \mathbb{R}) = H^4(Y, \mathbb{R}) \oplus H^3(Y, \mathbb{R}) \otimes H^1(S^1, \mathbb{R})
\]
and obviously \( H^4(Y) \sim \ast \varphi = 0 \) so, denoting by \( c_2'(E_g) \) the component lying in \( H^3(Y) \) and by \( S_g := \left[ \frac{1}{8\pi^2} c_2'(E_g) \right]^{PD} \) its normalised Poincaré dual, we are left with
\[
\vartheta(g,A) - \vartheta(A) = \langle \ast \varphi, S_g \rangle.
\]
Consequently, the periods of \( \vartheta \) lie in the set
\[
\left\{ \int_{S_g} \ast \varphi \mid S_g \in H_4(Y, \mathbb{R}) \right\}.
\]

That may seem odd at first, because \( \ast \varphi \) is not, in general, an integral class and so the set of periods is dense. However, as long as our interest remains in the study of the moduli space \( \mathcal{M} = \text{Crit}(\rho) \) of \( G_2 \)-instantons, there is not much to worry, for the gradient \( \rho = d\vartheta \) is unambiguously defined on \( \mathcal{B} \).

3 Instantons over \( G_2 \)-torus fibrations

Instances of \( G_2 \)-manifolds fibred by associative submanifolds in the literature are relatively scarce, not least because their deformation theory is zero-index elliptic [9] and therefore any new examples will be somewhat exotic. A few trivial cases include the products \( T^7 = T^4 \times T^3 \) and \( K3 \times T^3 \) and also \( CY^3 \times S^1 \) given a family of curves in the Calabi–Yau [8, §10.8]. The example I will propose is unique in the sense that the total space is not a Riemannian product.

3.1 Instantons over associative fibrations

We consider pullback bundles over smooth associative fibrations, and relate \( G_2 \)-instantons to their gauge theory over the base; in particular we do not address the possibility of singular fibres.

**Definition 1.** A \( G_2 \)-manifold \( (Y^7, \varphi) \) is called an associative fibration over a compact oriented Riemannian four-manifold \( (X^4, \eta) \) if it is the total space of a Riemannian submersion \( f : Y \to X \) such that each fibre \( P_x := f^{-1}(x) \subset Y \) is a smooth associative submanifold.

Since each fibre \( P_x \) is 3-dimensional and orientable, its tangent bundle is differentiably trivial and we may choose global coordinates \( t = (t^1, t^2, t^3) \) induced respectively by a global coframe \( \{ e_5, e_6, e_7 \} := \{ dt^1, dt^2, dt^3 \} \). Thus near each \( y \in P_x \) we may complete the triplet into a local orthogonal coframe \( \{ e_1, \ldots, e_7 \} \) of \( T^*Y \) such that \( \varphi_y \) has the form (1), and the point \( y \) is unambiguously described by \( (x, t(y)) \).

**Lemma 1.** Let \( f : Y \to X \) define an associative fibration and \( E \to Y \) be the pullback from a vector bundle \( E \to X \); then a connection \( A \) on \( E \) is self-dual if, and only if, \( f^*A \) is a \( G_2 \)-instanton on \( E \).
**Proof.** Let $F := (F_{\cdot A})_y$ be the curvature 2-form at $y \in P_x$; then

$$ *\varphi(F \wedge \varphi)_{\text{loc}} = *\varphi(F \wedge (\varphi|_{P_x} + \omega_1 \wedge e^5 + \omega_2 \wedge e^6 + \omega_3 \wedge e^7)) = *\eta F + *\varphi(O(F^-) \wedge f^*d\text{Vol}_y), $$

where $O(F^-) := (F_{34} - F_{12})e^5 + (F_{42} - F_{13})e^6 + (F_{23} - F_{14})e^7$ vanishes precisely when $A$ is self-dual, i.e., when $F = *\eta F$ satisfies the $G_2$-instanton equation (6b).

We are now in position to prove Theorem 1. Let us examine the general form of a $G_2$-instanton on $E$. An arbitrary connection $A$ on $E$ is locally of the form

$$ A(y) = A_t(x) + \sum_{i=1}^3 \sigma_i(x,t)dt^i, $$

where $\{A_t\}_{t \in (P_x)}$ is a family of connections on $E$ and $\sigma_i \in \Omega^0(Y, f^*g_E)$. The curvature of $A$ is

$$ F_A = F_{A_t} + \sum_{i=1}^3 \left( d_{A_t}\sigma_i - \frac{\partial A_t}{\partial t^i} \right) \wedge dt^i + F_\sigma $$

with

$$ F_\sigma := \sum_{i,j=1}^3 \left( \frac{\partial \sigma_i}{\partial t^j} - \frac{\partial \sigma_j}{\partial t^i} + \frac{1}{2}[\sigma_i, \sigma_j] \right) dt^i \wedge dt^j. $$

Replacing $F_A$ into the $G_2$-instanton equation (6a) and using the expression (3) of $*\varphi$ in the natural frame $\{e_1, \ldots, e_7\}$, we have

$$ \left( F_{A_t} + \sum_{i=1}^3 \left( d_{A_t}\sigma_i - \frac{\partial A_t}{\partial t^i} \right) \wedge e^{4+i} + F_\sigma \right) \wedge (e^{1234} - \omega_1 \wedge e^{67} - \omega_2 \wedge e^{75} - \omega_3 \wedge e^{56}) = 0. $$

Using the following elementary properties

$$ F_{A_t} \wedge e^{1234} = 0, \quad F_{A_t} \wedge \omega_1 \wedge e^{67} = [(F_{A_t})_{34} - (F_{A_t})_{12}] (*e^5), $$

$$ F_{A_t} \wedge \omega_2 \wedge e^{75} = [(F_{A_t})_{42} - (F_{A_t})_{13}] (*e^6), \quad F_{A_t} \wedge \omega_3 \wedge e^{56} = [(F_{A_t})_{23} - (F_{A_t})_{14}] (*e^7), $$

$$ F_\sigma \wedge e^{4+i} \wedge e^{4+j} = 0, \quad F_\sigma \wedge e^{1234} = (F_\sigma)_{23} (*e^5) + (F_\sigma)_{31} (*e^6) + (F_\sigma)_{12} (*e^7), $$

and the fact that each $d_{A_t}\sigma_i$ and $\frac{\partial A_t}{\partial t^i}$ are locally 1-forms on the base, hence their wedge product with $e^{1234} = d\text{Vol}_y$ vanishes, the equation simplifies to

$$ \sum_{i=1}^3 \left( d_{A_t}\sigma_i - \frac{\partial A_t}{\partial t^i} \right) \wedge \omega_i = 0 \quad \text{and} \quad F_{A_t}^- - Q(F_\sigma) = 0, $$

where $Q$ is the linear map on 2-forms defined by

$$ Q(dt^i \wedge dt^j) = Q(e^{4+i} \wedge e^{4+j}) := \sum_{k=1}^3 e^{ijk} \omega_k. $$

On the other hand, if $A = A_t + \sum \sigma_i$ is a $G_2$-instanton, then it minimises the Yang–Mills functional (5). This implies

$$ \sum \left\| d_{A_t}\sigma_i - \frac{\partial A_t}{\partial t^i} \right\|^2 + \|F_\sigma\|^2 = 0, $$

where $\|\cdot\|$ is the natural frame $\{e_1, \ldots, e_7\}$. The functional $\mathcal{F}(A)$ is

$$ \mathcal{F}(A) := \int_M \frac{1}{2} \|F_A\|^2 - \int_M \sum_{k=1}^3 e^{ijk} \omega_k, $$

and the fact that each $d_{A_t}\sigma_i$ and $\frac{\partial A_t}{\partial t^i}$ are locally 1-forms on the base, hence their wedge product with $e^{1234} = d\text{Vol}_y$ vanishes, the equation simplifies to

$$ \sum_{i=1}^3 \left( \frac{\partial \sigma_i}{\partial t^j} - \frac{\partial \sigma_j}{\partial t^i} + \frac{1}{2}[\sigma_i, \sigma_j] \right) dt^i \wedge dt^j = 0. $$

The functional $\mathcal{F}(A)$ is a $G_2$-instanton equation (6b).
since otherwise the pullback component $A_t$ alone would violate the minimum energy:

$$YM(A_t) = \| F_{A_t} \|^2 < \| F_A \|^2 = YM(A).$$

In particular $F_A \equiv 0$ and so every $A_t$ must be SD. Finally, if the family $A_t \equiv A_{t_0}$ is constant, then $d_{A_{t_0}} \sigma_i = 0$ implies $\sigma \equiv 0$, since by assumption $E$ is indecomposable and therefore does not admit nontrivial parallel sections. This concludes the proof of Theorem 1.

**Remark 1.** If $\mathcal{M}_+^4$ is discrete, then by continuity the family $\{A_t\}$ is contained in a gauge orbit; if the family is constant, then $A$ is a pullback.

### 3.2 G\textsubscript{2}-torus fibrations

A 7-torus $T^7 = \mathbb{R}^7 / \Lambda$ naturally inherits the $G_2$-structure $\varphi$ from $\mathbb{R}^7$. Recall from Section 2.2 that a connection $A$ on some bundle over $T^7$ is a $G_2$-instanton if and only if it is a zero of the Chern–Simons 1-form (7):

$$\rho_A(b) = \int_{T^7} \text{tr}(F_A \wedge b) \wedge *\varphi.$$  \hspace{1cm} (8)

One asks what is the behaviour of the moduli space of $G_2$-instantons under perturbations $\varphi \to \varphi + \phi$ of the $G_2$-structure. More precisely, given suitable assumptions, one asks whether $(\varphi + \phi)$-instantons exist at all once we deform the lattice. As a working example, we consider the following class of flat $T^3$-fibred 7-tori:

**Definition 2.** A $G_2$-torus fibration structure is a triplet $(\eta, L, \alpha)$ in which:

- $\eta$ is a metric on $\mathbb{R}^4$;
- $L$ is a lattice on the subspace $\Lambda^2_+(\mathbb{R}^4, \eta)$ of $\eta$-self-dual 2-forms;
- $\alpha : \mathbb{R}^4 \to \Lambda^2_+(\mathbb{R}^4, \eta)$ is a linear map.

Given the above data, set $V = \mathbb{R}^4 \oplus \Lambda^2_+$ and form the torus $T = V / \tilde{L}$, with the lattice

$$\tilde{L} \doteq \{ (\mu, \nu + \alpha \mu) | \mu \in \mathbb{Z}^4, \nu \in L \} \subset V.$$  

Then $T$ inherits from $V$ the $G_2$-structure $\varphi$ which makes the generators of $\tilde{L}$ orthonormal with respect to the induced inner-product (2). It is straightforward to check that $T$ is an associative fibration as in Definition 1: denoting by $e^5, e^6, e^7$ the $(\nu + \alpha \mu)$-orthonormal basis of the fibre $\Lambda^2_+$, the flat $G_2$-structure (1) simplifies to $\varphi|_{\Lambda^2_+} = e^{507} = d \text{Vol}_\varphi|_{\Lambda^2_+}$; moreover the lattice $\tilde{L}$ on every tangent subspace normal to the fibre is just the lattice $\mu$ from the base, so the corresponding metrics are the same. Although $T$ fibres over the 4-torus $\mathbb{R}^4 / \mu$, the induced metric $g(\varphi)$ is not, in general, a Riemannian product.

Suppose the moduli space $\mathcal{M}_+^4$ of self-dual connections on $E \to T^4$ is nonempty; then we have trivial solutions to the $G_2$-instanton equation on the pullback $E \to T$ simply by lifting $\mathcal{M}_+^4$ as in Lemma 1, which proves the first part of Theorem 2:

**Corollary 1.** If $A$ is a self-dual connection on $E \to T^4$, then its pullback $f^* A$ by the fibration map $f : T \to T^4$ is a $G_2$-instanton on $E$.

For future reference, I denote the set of such $\varphi$-instantons obtained by lifts from $\mathcal{M}_+^4$ by

$$\mathcal{M}_+^4 := f^* \mathcal{M}_+^4 \subset \mathcal{B}^7.$$  \hspace{1cm} (9)

We know from 4-dimensional gauge theory that SD connections on a complex vector bundle $E \to T^4$ correspond to stable holomorphic structures on $E$, thus in such cases we have examples of $G_2$-instantons on bundles over $T$. 

3.3 Deformations of $\mathbb{T}$

Working on a bundle $E \to \mathbb{T}$ with compact structure group over a fixed $G_2$-torus fibration, let us ponder in generality about the behaviour of instantons under a deformation of the $G_2$-structure:

$$\varphi \to \varphi + \phi, \quad \ast \varphi \varphi \to \ast \varphi \varphi + \xi_\phi, \quad \xi_\phi := \ast \varphi + \phi - \ast \varphi \varphi \in \Omega^4(\mathbb{T}).$$

An arbitrary deformation $\phi$ does not in general preserve the fibred structure of $\mathbb{T}$:

**Proposition 1.** A deformation $\xi_\phi \in \Lambda^4(\mathbb{T})$ of the coassociative 4-form $\ast \varphi \varphi$ has four orthogonal components, with the following significance:

$$\begin{align*}
\Lambda^4(\mathbb{R}^4 \oplus \Lambda_+^2) &= \Lambda^4(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4) \oplus \Lambda^1(\Lambda_+^2) \oplus \Lambda^2(\mathbb{R}^4) \oplus \Lambda^3(\Lambda_+^2), \\
(I) &\text{ corresponds to a rescaling of the metric } \eta \text{ on } \mathbb{R}^4; \\
(II) &\text{ redefines the map } \alpha; \\
(III) &\text{ splits as } \text{Hom}(\Lambda_+^2, \Lambda_+^2) \oplus \text{Hom}(\Lambda_+^2, \Lambda_+^2), \text{ where the first factor modifies the lattice } L \text{ and the second one affects the conformal class of } \eta; \\
(IV) &\text{ parametrises deformations transverse to the fibred structures.}
\end{align*}$$

**Proof.** Let us examine the four cases.

(I) If $\xi_\phi \in \Lambda^4(\mathbb{R}^4) \simeq \mathbb{R}$, then it must be a multiple of $\ast \varphi \varphi |_{\mathbb{R}^4} = e^{1234} = d \text{Vol}_\eta$.

(II) Since $\Lambda^3(\mathbb{R}^4) \otimes \Lambda^1(\Lambda_+^2) \simeq \mathbb{R}^4 \otimes (\Lambda_+^2)^* \simeq \text{Hom}(\mathbb{R}^4, \Lambda_+^2)$, such deformations are precisely linear maps $\mathbb{R}^4 \to \Lambda_+^2$.

(III) Clearly $\Lambda^2(\mathbb{R}^4) = \Lambda_+^2 \oplus \Lambda_+^2$ and $\Lambda^2(\Lambda_+^2) \simeq (\Lambda_+^2)^*$, so the product decomposes as

$$(\Lambda_+^2 \otimes (\Lambda_+^2)^*) \oplus (\Lambda_+^2 \otimes (\Lambda_+^2)^*) \simeq \text{Hom}(\Lambda_+^2, \Lambda_+^2) \oplus \text{Hom}(\Lambda_+^2, \Lambda_+^2).$$

Now, on one hand, acting with an endomorphism on $\Lambda_+^2$ is equivalent to redefining the triplet $\{e^5, e^6, e^7\}$, hence the lattice $L \subset \Lambda_+^2$. On the other hand, since the orthogonal split $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_+^2$ is conformally invariant, a map $\Lambda_+^2 \to \Lambda_+^2$ redefines the orthogonal complement of $\Lambda_+^2$ and hence the conformal class.

(IV) Since $\Lambda^3(\Lambda_+^2) \simeq \mathbb{R}$, this component is just $\Lambda^1(\mathbb{R}^4)$, which is irreducible in the sense that $\mathbb{T}$ has no distinguished subspaces in $\mathbb{R}^4$. Then either every 7-torus is a $G_2$-fibration, which is obviously false, or these are precisely the deformations away from said structures. 

We will now describe what happens to the zeroes of (8) under the corresponding perturbation of the Chern–Simons 1-form:

$$\rho \to \rho \phi := \rho + r_{\phi}, \quad (r_{\phi})_A(b) = \int_{\mathbb{T}} \text{tr}(F_A \wedge b) \wedge \xi_\phi.$$

Clearly a $\varphi$-instanton $A$ is also a $(\varphi + \phi)$-instanton if and only if $(r_{\phi})_A \equiv 0$. There is little reason, however, to expect such a coincidence; as we will see, the topology of the bundle may constrain the existence of instantons under certain – indeed most – deformations.

Denoting henceforth by $\mathcal{A}$ the space of connections over the 7-manifold $\mathbb{T}$, let us briefly digress into the translation action of some vector $v \in \mathbb{T}$ on some $A \in \mathcal{A}$. The first order variation is given by the bundle-valued 1-form

$$(\beta_v)_A := v \lrcorner \text{d}F_A,$$

which we interpret as a vector in $T_A \mathcal{A}$. Notice first that in the direction $\beta_v$ the value of the Chern–Simons 1-form is independent of the base-point:
Lemma 2. The function $\rho(\beta_{\nu}) : A \to \mathbb{R}$ is constant.

Proof. The computation is straightforward:

$$
\rho(\beta_{\nu})_{A+ha} = \int_{T} \text{tr} F_{A+ha} \wedge v \cdot F_{A+ha} \wedge *\varphi = -\frac{1}{2} \int_{T} \text{tr} F_{A+ha} \wedge F_{A+ha} \wedge (v \cdot \varphi)
$$

$$
= -\frac{1}{2} \int_{T} (\text{tr} F_{A} \wedge d\chi + d\chi) \wedge (v \cdot \varphi) = -\frac{1}{2} \int_{T} \text{tr} F_{A} \wedge F_{A} \wedge (v \cdot \varphi) = \rho(\beta_{\nu})_{A},
$$

where $d\chi$ is the exact differential given by Chern–Weil theory and we use Stokes’ theorem and Cartan’s identity $d(v \cdot \varphi) = L_{v}(\varphi) = 0$, since $\varphi$ is constant on the flat torus. \hfill \Box

Similarly, evaluating $r_{\phi}$ on $\beta_{\nu}$ gives

$$
r_{\phi}(\beta_{\nu})_{A} = \int_{T} \text{tr}(F_{A} \wedge (\beta_{\nu})_{A}) \wedge \xi_{\phi} = -\frac{1}{2} \int_{T} \text{tr}(F_{A} \wedge F_{A}) \wedge (v \cdot \xi_{\phi}) = \langle c_{2}(E), S_{\phi}(v) \rangle,
$$

where $S_{\phi}(v) = -\frac{1}{2}[v \cdot \xi_{\phi}]^{PD}$, and this depends only on the topology of $E$, not on the point $A$.

Remark 2. Hence we may interpret $\phi$ as defining a linear functional

$$
N_{\phi} : \mathbb{R}^{7} \to \mathbb{R},
$$

$$
v \mapsto \langle c_{2}(E), S_{\phi}(v) \rangle,
$$

such that $N_{\phi} \neq 0$ implies no $\varphi$-instanton is still a $(\varphi + \phi)$-instanton. This is, however, a rather weak obstruction, since the map $\phi \mapsto N_{\phi}$ has kernel of dimension at least 28 and thus, in principle, leaves plenty of possibilities for instantons of perturbed $G_{2}$-structures.

Now consider specifically a translation vector on the base $v \in T^{4}$. Notice that for deformations $\phi$ of types (I), (II) or (III) the contraction of $\xi_{\phi}$ with such $v$ gives $S_{\phi}(v) = 0$, so $\phi$ only effectively contributes to the function $\rho(\beta_{\nu})$ when $\xi_{\phi} \in \Lambda^{1}(\mathbb{R}^{4})$, which means the perturbed torus is no longer a fibred structure (Proposition 1). Moreover, either the bundle $E$ is flat and $\beta_{\nu}$ vanishes identically, or $c_{2}(E) \neq 0$ and the following holds:

Lemma 3. If $c_{2}(E) \neq 0$ and $\phi$ is of type (IV), then there exists $v \in T^{4}$ such that $r_{\phi}(\beta_{\nu})$ is a non-zero constant.

Proof. Denoting $T^{3}$ the typical fibre of $f$ (and setting $\text{Vol}(T^{3}) = 1$), we may assume

$$
\xi_{\phi} = -2\varepsilon \wedge d\text{Vol}_{T^{3}}
$$

for some $0 \neq \varepsilon \in \Lambda^{1}(T^{4})$. One can always choose $v \in T^{4}$ such that $\varepsilon(v) \neq 0$, and consider $(\beta_{\nu})_{A} = v \cdot F_{A}$. Then

$$
r_{\phi}(\beta_{\nu})_{A} = -2 \int_{T^{3}} \text{tr}(F_{A} \wedge v \cdot F_{A}) \wedge \varepsilon \wedge d\text{Vol}_{T^{3}} = -2 \int_{T^{4}} \text{tr}(F_{A} \wedge v \cdot F_{A}) \wedge \varepsilon = \varepsilon(v) \cdot c_{2}(E),
$$

which is nonzero by assumption. \hfill \Box

So far we know from Corollary 1 that the set $\mathcal{M}_{+}^{4}$ of self-dual connections (modulo gauge) over $T^{4}$ lifts to instantons (cf. (9)) of the original $G_{2}$-structure $\varphi$ (i.e. to zeroes of $\rho$). However, for bundles with non-trivial $c_{2}$, this generic case degenerates precisely under deformations of type (IV):

Proposition 2. Let $E \to (T, \varphi)$ be the pullback of a stable $SU(n)$-bundle $E$ over $T^{4}$ with $c_{2}(E) \neq 0$; then $E$ admits no $(\varphi + \phi)$-instantons, for any perturbation $\phi$ away from a fibred structure (i.e. of type (IV) in Proposition 1).
**Proof.** Fix a lifted $\varphi$-instanton $A \in \widetilde{M}_4^\varphi$; for any $A + ha \in \mathcal{A}$, Lemma 2 gives $\rho_{A+ha}(\beta_v) = \rho_A(\beta_v) = 0$. Taking $v \in T^4$ as in Lemma 3 we have

$$\rho_\phi(\beta_v)_{A+ha} = r_\phi(\beta_v)_{A+ha} + \rho(\beta_v)_{A+ha} = \varepsilon(v) \cdot c_2(E) + \rho(\beta_v)_{A+ha} \neq 0,$$

hence $A + ha$ is not a $(\varphi + \phi)$-instanton. 

Combining Corollary 1 and Proposition 2 we obtain Theorem 2.

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**References**


