Fusion Procedure for Cyclotomic Hecke Algebras

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Abstract. A complete system of primitive pairwise orthogonal idempotents for cyclotomic Hecke algebras is constructed by consecutive evaluations of a rational function in several variables on quantum contents of multi-tableaux. This function is a product of two terms, one of which depends only on the shape of the multi-tableau and is proportional to the inverse of the corresponding Schur element.

Key words: cyclotomic Hecke algebras; fusion formula; idempotents; Young tableaux; Jucys–Murphy elements; Schur element

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1 Introduction

This article is a continuation of the article [14] on the fusion procedure for the complex reflection groups $G(m, 1, n)$. The cyclotomic Hecke algebra $H(m, 1, n)$, introduced in [2, 3, 4], is a natural flat deformation of the group ring of the complex reflection group $G(m, 1, n)$.

In [14], a fusion procedure, in the spirit of [12], for the complex reflection groups $G(m, 1, n)$ is suggested: a complete system of primitive pairwise orthogonal idempotents for the groups $G(m, 1, n)$ is obtained by consecutive evaluations of a rational function in several variables with values in the group ring $\mathbb{C}G(m, 1, n)$. This approach to the fusion procedure relies on the existence of a maximal commutative set of elements of $\mathbb{C}G(m, 1, n)$ formed by the Jucys–Murphy elements.

Jucys–Murphy elements for the cyclotomic Hecke algebra $H(m, 1, n)$ were introduced in [2] and were used in [13] to develop an inductive approach to the representation theory of the chain of the algebras $H(m, 1, n)$. In the generic setting or under certain restrictions on the parameters of the algebra $H(m, 1, n)$ (see Section 2 for precise definitions), the Jucys–Murphy elements form a maximal commutative set in the algebra $H(m, 1, n)$.

A complete system of primitive pairwise orthogonal idempotents of the algebra $H(m, 1, n)$ is indexed by the set of standard $m$-tableaux of size $n$. We formulate here the main result of the article. Let $\lambda$ be an $m$-partition of size $n$ and $T$ be a standard $m$-tableau of shape $\lambda$.

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Theorem. The idempotent $E_{\mathcal{T}}$ of $H(m,1,n)$ corresponding to the standard $m$-tableau $\mathcal{T}$ of shape $\lambda$ can be obtained by the following consecutive evaluations
\[
E_{\mathcal{T}} = F_{\lambda} \Phi(u_1, \ldots, u_n) \bigg|_{u_1 = c_1} \cdots \bigg|_{u_{n-1} = c_{n-1}} \bigg|_{u_n = c_n}.
\]

Here $\Phi(u_1, \ldots, u_n)$ is a rational function with values in the algebra $H(m,1,n)$, $F_{\lambda}$ is an element of the base ring and $c_1, \ldots, c_n$ are the quantum contents of the $m$-nodes of $\mathcal{T}$.

The classical limit of our fusion procedure for algebras $H(m,1,n)$ reproduces the fusion procedure of [14] for the complex reflection groups $CG(m,1,n)$. For $CG(m,1,n)$, the variables of the rational function are split into two parts, one is related to the position of the $m$-node (its place in the $m$-tuple) and the other one – to the classical content of the $m$-node. The position variables can be evaluated simultaneously while the classical content variables have then to be evaluated consequently from 1 to $n$. For the algebra $H(m,1,n)$, the information about positions and classical contents is fully contained in the quantum contents, and now the function $\Phi$ depends on only one set of variables.

Remarkably, the coefficient $F_{\lambda}$ appearing in (1) depends only on the shape $\lambda$ of the standard $m$-tableau $\mathcal{T}$ (cf. with the more delicate fusion procedure for the Birman–Murakami–Wenzl algebra [7]). In the classical limit, this coefficient depends only on the usual hook length, see [14]. However, in the deformed situation, the calculation of $F_{\lambda}$ needs a non-trivial generalization of the hook length. It appears that the coefficient $F_{\lambda}$ is proportional to the inverse of the Schur element (corresponding to the $m$-partition $\lambda$) associated to a specific symmetrizing form on the algebra $H(m,1,n)$ (see [6, 11] for a calculation of these Schur elements and [5] for an expression in terms of generalized hook lengths); for more precise statements, we refer to [15] where we calculate, using the fusion formula presented here, weights of certain central forms and in particular of these Schur elements.

For $m = 1$, the cyclotomic Hecke algebra $H(1,1,n)$ is the Hecke algebra of type A and our fusion procedure reduces to the fusion procedure for the Hecke algebra in [8]. The factors in the rational function are arranged in [8] in such a way that there is a product of “Baxterized” generators on one side and a product of non-Baxterized generators on the other side. For $m > 1$ a rearrangement, as for the type A, of the rational function appearing in (1) is no more possible.

The additional, with respect to $H(1,1,n)$, generator of $H(m,1,n)$ satisfies the reflection equation whose “Baxterization” is known [9]. But – and this is maybe surprising – the full Baxterized form is not used in the construction of the rational function in (1). The rational expression involving the additional generator satisfies only a certain limit of the reflection equation with spectral parameters.

The Hecke algebra of type A is the natural quotient of the Birman–Murakami–Wenzl algebra. The fusion procedure, developed in [7], for the Birman–Murakami–Wenzl algebra provides a one-parameter family of fusion procedures for the Hecke algebra of type A. We think that for $m > 1$ the fusion procedure (1) can be included into a one-parameter family as well.

2 Definitions

2.1 Cyclotomic Hecke algebra and Baxterized elements

Let $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{>0}$. Let $q, v_1, \ldots, v_n$ be complex numbers with $q \neq 0$. The cyclotomic Hecke algebra $H(m,1,n+1)$ is the unital associative algebra over $\mathbb{C}$ generated by $\tau, \sigma_1, \ldots, \sigma_n$ with the defining relations
\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1 \ldots, n - 1,
\]
\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } i, j = 1, \ldots, n \text{ such that } |i - j| > 1,
\]
We define \( H(m,1,0) := \mathbb{C} \). The cyclotomic Hecke algebras \( H(m,1,n) \) form a chain (with respect to \( n \)) of algebras defined by inclusions \( H(m,1,n) \supseteq \tau, \sigma_1, \ldots, \sigma_{n-1} \mapsto \tau, \sigma_1, \ldots, \sigma_{n-1} \in H(m,1,n+1) \) for any \( n \geq 0 \). These inclusions allow to consider (as it will often be done in the article) elements of \( H(m,1,n) \) as elements of \( H(m,1,n+n') \) for any \( n' = 0,1,2,\ldots \).

In the sequel we assume the following restrictions on the parameters \( q,v_1,\ldots,v_m \):

\[
1 + q^2 + \cdots + q^{2N} \neq 0 \quad \text{for } N \text{ such that } N \leq n, \tag{2}
\]

\[
q^{2i}v_j - v_k \neq 0 \quad \text{for } i,j,k \text{ such that } j \neq k \text{ and } -n \leq i \leq n, \tag{3}
\]

\[
v_j \neq 0 \quad \text{for } j = 1,\ldots,m. \tag{4}
\]

The restrictions (2), (3) are necessary and sufficient for the semi-simplicity of the algebra \( H(m,1,n+1) \) [1, main theorem]. The restriction (4) is necessary for the maximality of the commutative set of the Jucys–Murphy elements (as defined in Section 3) [1, Proposition 3.2].

Define the following rational functions in variables \( a,b \) with values in \( H(m,1,n+1) \):

\[
\overline{\sigma}_i(a,b) := \sigma_i + (q-q^{-1}) \frac{b}{a-b}, \quad i = 1,\ldots,n. \tag{5}
\]

The functions \( \overline{\sigma}_i \) are called **Baxterized** elements and the variables \( a \) and \( b \) are called **spectral parameters**. These Baxterized elements satisfy the Yang–Baxter equation with spectral parameters

\[
\overline{\sigma}_i(a,b)\overline{\sigma}_{i+1}(a,c)\overline{\sigma}_{i}(b,c) = \overline{\sigma}_{i+1}(b,c)\overline{\sigma}_i(a,c)\overline{\sigma}_{i+1}(a,b).
\]

The following formula will be used later

\[
\overline{\sigma}_i(a,b)\overline{\sigma}_j(b,a) = \frac{(a-q^2b)(a-q^{-2}b)}{(a-b)^2} \quad \text{for } i = 1,\ldots,n. \tag{6}
\]

Let \( p_i, i = 1,\ldots,m, \) be the eigen-idempotents of \( \tau, \) \( p_i := \prod_{j:j \neq i} (\tau - v_j)/(v_i - v_j), \) so that \( \tau p_i = v_i p_i, p_i p_j = \delta_{ij} p_i, \sum_i p_i = 1 \) and \( \tau = \sum_i v_i p_i. \) Let \( r \) be an indeterminate. The resolvent \( (r-\tau)^{-1} := \sum_i (r-v_i)^{-1} p_i \) of \( \tau \) is an element of \( \mathbb{C}(r) \otimes_{\mathbb{C}} H(m,1,n+1). \) Define a rational function \( \overline{\tau} \) with values in \( H(m,1,n+1) \):

\[
\overline{\tau}(r) := \frac{(r-v_1)(r-v_2)\cdots(r-v_m)}{r-\tau} = \sum_i \left( \prod_{j:j \neq i} (r-v_j) \right) p_i \in \mathbb{C}[r] \otimes_{\mathbb{C}} H(m,1,n+1). \tag{7}
\]

**Remarks.** (i) The function \( \overline{\tau}(r) \) can be expressed in terms of the complex numbers \( a_0, a_1,\ldots,a_m \) defined by

\[
(X - v_1)(X - v_2)\cdots(X - v_m) = a_0 + a_1 X + \cdots + a_m X^m,
\]

where \( X \) is an indeterminate. Let \( a_i(r), i = 0,\ldots,m, \) be the polynomials in \( r \) given by

\[
a_i(r) = a_i + ra_{i+1} + \cdots + r^{m-i}a_m \quad \text{for } i = 0,\ldots,m. \tag{8}
\]
Using that \( r a_{i+1}(r) = a_i(r) - a_i \), for \( i = 0, \ldots, m - 1 \), it is straightforward to verify that

\[
(r - \tau) \sum_{i=0}^{m-1} a_i(r) \tau^i = a_0(r) = (r - v_1)(r - v_2) \cdots (r - v_m).
\]

(9)

It follows from (9) that

\[
\bar{\tau}(r) = a_1(r) + a_2(r) \tau + \cdots + a_m(r) \tau^{m-1} = \sum_{i=0}^{m-1} a_i(r) \tau^i,
\]

(10)

For example, for \( m = 1 \), we have \( \bar{\tau}(r) = 1 \); for \( m = 2 \), we have \( \bar{\tau}(r) = \tau + r - v_1 - v_2 \); for \( m = 3 \), we have \( \bar{\tau}(r) = \tau^2 + (r - v_1 - v_2 - v_3)\tau + r^2 - r(v_1 + v_2 + v_3) + v_1 v_2 + v_1 v_3 + v_2 v_3 \).

(ii) The functions \( \tau \) and \( \bar{\tau} \) satisfy the following equation

\[
\bar{\sigma}_1(a, b) \bar{\tau}(a) \sigma_1^{-1}\tau(b) = \tau(b) \sigma_1^{-1}\tau(a) \bar{\sigma}_1(a, b).
\]

(11)

Indeed, due to (6) and (7), the equality (11) is equivalent to

\[
(\tau - b)\sigma_1(\tau - a) \bar{\sigma}_1(b, a) = \bar{\sigma}_1(b, a)(\tau - a)\sigma_1(\tau - b),
\]

which is proved by a straightforward calculation. The equation (11) is a certain (we leave the details to the reader) limit of the usual reflection equation with spectral parameters (see, for example, [10]).

2.2 \textit{m}-partitions, m-tableaux and generalized hook length

Let \( \lambda \vdash n + 1 \) be a partition of size \( n + 1 \), that is, \( \lambda = (\lambda_1, \ldots, \lambda_l) \), where \( \lambda_j, j = 1, \ldots, l \), are positive integers, \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \) and \( n + 1 = \lambda_1 + \cdots + \lambda_l \). We identify partitions with their Young diagrams: the Young diagram of \( \lambda \) is a left-justified array of rows of nodes containing \( \lambda_j \) nodes in the \( j \)-th row, \( j = 1, \ldots, l \); the rows are numbered from top to bottom. For a node \( \alpha \) in line \( x \) and column \( y \) of a Young diagram, we denote \( \alpha = (x, y) \) and call \( x \) and \( y \) the coordinates of the node.

An \( m \)-partition, or a Young \( m \)-diagram, of size \( n + 1 \) is an \( m \)-tuple of partitions such that the sum of their sizes equals \( n + 1 \); e.g. the Young 3-diagram (\( \square, \square, \square \)) represents the 3-partition ((2), (1), (1)) of size 4.

We shall understand an \( m \)-partition as a set of \( m \)-nodes, where an \( m \)-node \( \alpha \) is a pair \( \{\alpha, k\} \) consisting of a node \( \alpha \) and an integer \( k = 1, \ldots, m \), indicating to which diagram in the \( m \)-tuple the node belongs. The integer \( k \) will be called \textit{position} of the \( m \)-node, and we set \( \text{pos}(\alpha) := k \).

For an \( m \)-partition \( \lambda \), an \( m \)-node \( \alpha \) of \( \lambda \) is called \textit{removable} if the set of \( m \)-nodes obtained from \( \lambda \) by removing \( \alpha \) is still an \( m \)-partition. An \( m \)-node \( \beta \) not in \( \lambda \) is called \textit{addable} if the set of \( m \)-nodes obtained from \( \lambda \) by adding \( \beta \) is still an \( m \)-partition. For an \( m \)-partition \( \lambda \), we denote by \( E_-(\lambda) \) the set of removable \( m \)-nodes of \( \lambda \) and by \( E_+(\lambda) \) the set of addable \( m \)-nodes of \( \lambda \). For example, the removable/addable \( m \)-nodes (marked with \( -/+ \)) for the 3-partition (\( \square, \square, \square \)) are

\[
\begin{pmatrix}
\text{+}, & -\text{+}, & -\text{+} \\
\end{pmatrix}
\]

Let \( \lambda \) be an \( m \)-diagram of size \( n + 1 \). A standard \( m \)-tableau of shape \( \lambda \) is obtained by placing the numbers \( 1, \ldots, n + 1 \) in the \( m \)-nodes of the diagrams of \( \lambda \) in such a way that the numbers in the nodes ascend along rows and down columns in every diagram. The \textit{size} of a standard \( m \)-tableau is the size of its shape.
Let $q, v_1, \ldots, v_m$ be the parameters of the cyclotomic Hecke algebra $H(m, 1, n + 1)$ and let $\alpha = \{ \alpha, k \}$ be an $m$-node with $\alpha = (x, y)$. We denote by $cc(\alpha)$ the classical content of the node $\alpha$, $cc(\alpha) := y - x$, and by $c(\alpha)$ the quantum content of the $m$-node $\alpha$, $c(\alpha) := v_k q^{2cc(\alpha)} = v_k q^{2(y-x)}$.

For a standard $m$-tableau $T$ of shape $\lambda$ let $\alpha_i$ be the $m$-node of $T$ occupied by the number $i$, $i = 1, \ldots, n + 1$; we set $c(T|i) := c(\alpha_i)$, $cc(T|i) := cc(\alpha_i)$ and $pos(T|i) := pos(\alpha_i)$. For example, for the standard 3-tableau $T = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} 4 \end{bmatrix}$ we have

\[
\begin{align*}
c(T|1) &= v_1, & c(T|2) &= v_2, & c(T|3) &= v_1 q^2 \\
cc(T|1) &= 0, & cc(T|2) &= 0, & cc(T|3) &= 1 \\
pos(T|1) &= 1, & pos(T|2) &= 2, & pos(T|3) &= 1
\end{align*}
\]

Generalized hook length. The hook of a node $\alpha$ of a partition $\lambda$ is the set of nodes of $\lambda$ consisting of the node $\alpha$ and the nodes which lie either under $\alpha$ in the same column or to the right of $\alpha$ in the same row; the hook length $h_\lambda(\alpha)$ of $\alpha$ is the cardinality of the hook of $\alpha$. We extend this definition to $m$-nodes. For an $m$-node $\alpha = \{ \alpha, k \}$ of an $m$-partition $\lambda$, the hook length of $\alpha$ in $\lambda$, which we denote by $h_\lambda(\alpha)$, is the hook length of the node $\alpha$ in the $k$-th partition of $\lambda$.

Let $\lambda$ be an $m$-partition. For $j = 1, \ldots, m$, let $l_{\lambda,x,j}$ be the number of nodes in the line $x$ of the $j$-th diagram of $\lambda$, and $c_{\lambda,y,j}$ be the number of nodes in the column $y$ of the $j$-th diagram of $\lambda$. The hook length of an $m$-node $\alpha = \{ (x, y), k \}$ of $\lambda$ can be rewritten as

\[h_\lambda(\alpha) = l_{\lambda,x,k} + c_{\lambda,y,k} - x - y + 1.\]

Define the generalized hook length of $\alpha$ (see also [5]) by

\[h_\lambda^{(j)}(\alpha) := l_{\lambda,x,j} + c_{\lambda,y,k} - x - y + 1 \quad \text{for} \quad j = 1, \ldots, m;
\]

in particular, $h_\lambda^{(k)}(\alpha) = h_\lambda(\alpha)$ is the usual hook length.

For an $m$-partition $\lambda$, we define

\[F_\lambda = \prod_{\alpha \in \lambda} \left( \frac{q^{cc(\alpha)}}{[h_\lambda(\alpha)]_q} \prod_{k = 1, \ldots, m} \frac{q^{cc(\alpha)}}{v_{pos(\alpha)} q^{-h_\lambda^{(k)}(\alpha)} - v_k q^{h_\lambda^{(k)}(\alpha)}} \right),\]

where $[j]_q := q^{j-1} + q^{j-3} + \cdots + q^{-j+1}$ for a non-negative integer $j$. Under the restrictions (2)–(4), the number $F_\lambda$ is well defined for any $m$-partition $\lambda$ of size less or equal to $n + 1$ since $h_\lambda(\alpha) \leq n + 1$ and $h_\lambda^{(k)}(\alpha) \leq n$ if $k \neq pos(\alpha)$ for any $\alpha \in \lambda$.

3 Idempotents and Jucys–Murphy elements of $H(m, 1, n + 1)$

In this section we recall the definition and some properties, from [2], of the Jucys–Murphy elements of the algebra $H(m, 1, n + 1)$, together with some facts about an explicit realization of the irreducible representations of $H(m, 1, n + 1)$. We then derive, in the same spirit as in [12], an inductive formula, that we will use in the next section, for the primitive idempotents corresponding to this realization.

The Jucys–Murphy elements $J_i$, $i = 1, \ldots, n + 1$, of the algebra $H(m, 1, n + 1)$ are defined by the following initial condition and recursion

\[J_1 = \tau \quad \text{and} \quad J_{i+1} = \sigma_i J_i \sigma_i, \quad i = 1, \ldots, n.\]
We recall that, under the restrictions (2)–(4), the elements $J_i$, $i = 1, \ldots, n + 1$, form a maximal commutative set (that is, generate a maximal commutative subalgebra) of $H(m, 1, n + 1)$ [2, Proposition 3.17]. Recall also that

$$J_i \sigma_k = \sigma_k J_i \quad \text{for } k \neq i - 1, i.$$  

The isomorphism classes of irreducible $\mathbb{C}$-representations of $H(m, 1, n + 1)$ are in bijection with the set of $m$-partitions of size $n + 1$. We use the labeling and the explicit realization of the irreducible representations of $H(m, 1, n + 1)$ given in [2]. Namely, for any $m$-partition $\lambda$ of size $n + 1$, the irreducible representation $V_\lambda$ of $H(m, 1, n + 1)$ corresponding to $\lambda$ has a basis $\{v_T\}$ indexed by the set of standard $m$-tableaux of shape $\lambda$, and is characterized (up to a diagonal change of basis) by the fact that the Jucys–Murphy elements act diagonally by

$$J_i(v_T) = c(T|i)v_T, \quad i = 1, \ldots, n + 1.$$  

We will not need the explicit formulas for the action of the generators of $H(m, 1, n + 1)$ on basis elements $v_T$.

The restriction of irreducible representations of $H(m, 1, n + 1)$ to $H(m, 1, n)$ is determined by inclusion of $m$-partitions, that is, for $H(m, 1, n)$-modules, we have

$$V_\lambda \cong \bigoplus_{\mu \subset \lambda, \mu \text{ of size } n} V_\mu. \quad (13)$$

Moreover, in this decomposition, $V_\mu$ is the space spanned by the basis vectors $v_T$, with $T$ such that the standard $m$-tableau (of size $n$) obtained by removing from $T$ the $m$-node containing $n + 1$ is of shape $\mu$.

For a standard $m$-tableau $T$ of size $n + 1$, we denote by $E_T$ the primitive idempotent of $H(m, 1, n + 1)$ corresponding to $v_T$, uniquely defined by $E_T v_T = \delta_{T T'} v_T$. The results recalled above imply that $\{E_T\}$, where $T$ runs through the set of standard $m$-tableaux of size $n + 1$, is a complete set of pairwise orthogonal primitive idempotents of $H(m, 1, n + 1)$. Moreover, we have by construction

$$J_i E_T = E_T J_i = c(T|i) E_T, \quad i = 1, \ldots, n + 1. \quad (14)$$

Due to the maximality of the commutative set formed by the Jucys–Murphy elements, the idempotent $E_T$ can be expressed in terms of the elements $J_i$, $i = 1, \ldots, n + 1$. Let $\gamma$ be the $m$-node of $T$ containing the number $n + 1$. As the $m$-tableau $T$ is standard, the $m$-node $\gamma$ of $\lambda$ is removable. Let $U$ be the standard $m$-tableau obtained from $T$ by removing the $m$-node $\gamma$, and let $\mu$ be the shape of $U$. By (13) and (14), the inductive formula for $E_T$ in terms of the Jucys–Murphy elements reads

$$E_T = E_U \prod_{\substack{\beta \in \mathcal{E}_+(\mu) \\ \beta \neq \gamma}} J_{n + 1} - c(\beta)/c(\gamma) - c(\beta),$$

with the initial condition: $E_{U_0} = 1$ for the unique $m$-tableau $U_0$ of size 0. Here $E_U$ is considered as an element of the algebra $H(m, 1, n + 1)$. Note that, due to the restrictions (2)–(4), we have $c(\beta) \neq c(\gamma)$ for any $\beta \in \mathcal{E}_+(\mu)$ such that $\beta \neq \gamma$.

Let $\{\mathcal{T}_1, \ldots, \mathcal{T}_a\}$ be the set of pairwise different standard $m$-tableaux which can be obtained from $U$ by adding an $m$-node with number $n + 1$. As a consequence of (13), we have the formula

$$E_U = \sum_{i=1}^a E_{\mathcal{T}_i}. \quad (15)$$
The element $J_{n+1}$ satisfies a polynomial equation of finite order so its resolvent is well defined and

$$E_{\mathcal{U}} \frac{u - c(\mathcal{T}|n + 1)}{u - J_{n+1}}$$

is a rational function in an indeterminate $u$ with values in $H(m, 1, n + 1)$. Replacing $E_{\mathcal{U}}$ by the right-hand side of (15) and using (14), we obtain that this function is non-singular at $u = c(\mathcal{T}|n + 1)$ and moreover, due to the restrictions (2)–(4),

$$E_{\mathcal{U}} \frac{u - c(\mathcal{T}|n + 1)}{u - J_{n+1}} \bigg|_{u = c(\mathcal{T}|n + 1)} = E_{\mathcal{T}}. \quad (16)$$

4 Fusion formula for the algebra $H(m, 1, n + 1)$

In this section, we prove, in Theorem 1 below, the fusion formula for the primitive idempotents $E_{\mathcal{T}}$. We use the inductive formula (16) for $E_{\mathcal{T}}$.

Let $\phi_k$, for $k = 1, \ldots, n + 1$, be the rational functions in variables $u_1, \ldots, u_k$ with values in the algebra $H(m, 1, n + 1)$ defined by $\phi_1(u_1) := \tau(u_1)$ and, for $k = 1, \ldots, n$,

$$\phi_{k+1}(u_1, \ldots, u_k, u_{k+1}) := \sigma_k(u_{k+1}, u_k) \phi_k(u_1, \ldots, u_{k-1}, u_{k+1}) \sigma_k^{-1} = \sigma_k(u_{k+1}, u_k) \sigma_{k-1}(u_{k+1}, u_{k-1}) \cdots \sigma_1(u_{k+1}, u_1) \tau(u_{k+1}) \sigma_1^{-1} \cdots \sigma_{k-1}^{-1} \sigma_k^{-1}.$$

Define the following rational function $\Phi$ in variables $u_1, \ldots, u_{n+1}$ with values in $H(m, 1, n + 1)$:

$$\Phi(u_1, \ldots, u_{n+1}) := \phi_{n+1}(u_1, \ldots, u_n, u_{n+1}) \phi_n(u_1, \ldots, u_{n-1}, u_n) \cdots \phi_1(u_1).$$

Let $\lambda$ be an $m$-partition of size $n + 1$ and $\mathcal{T}$ a standard $m$-tableau of shape $\lambda$. For $i = 1, \ldots, n + 1$, we set $c_i := c(\mathcal{T}|i)$.

**Theorem 1.** The idempotent $E_{\mathcal{T}}$ corresponding to the standard $m$-tableau $\mathcal{T}$ of shape $\lambda$ can be obtained by the following consecutive evaluations

$$E_{\mathcal{T}} = F_\lambda \Phi(u_1, \ldots, u_{n+1}) \bigg|_{u_1 = c_1} \cdots \bigg|_{u_n = c_n} \bigg|_{u_{n+1} = c_{n+1}},$$

with $F_\lambda$ defined in (12).

We will prove the theorem in this section in several steps.

Until the end of the text, $\gamma$ and $\delta$ denote the $m$-nodes of $\mathcal{T}$ containing the numbers $n + 1$ and $n$ respectively; $\mathcal{U}$ is the standard $m$-tableau obtained from $\mathcal{T}$ by removing $\gamma$, and $\mu$ is the shape of $\mathcal{U}$; also, $\mathcal{W}$ is the standard $m$-tableau obtained from $\mathcal{U}$ by removing the $m$-node $\delta$ and $\nu$ is the shape of $\mathcal{W}$.

For a standard $m$-tableau $\mathcal{V}$ of size $N$, we define the following rational function in a variable $u$ with complex values

$$F_\mathcal{V}(u) := \frac{u - c(\mathcal{V}|N)}{(u - v_1) \cdots (u - v_m)} \prod_{i=1}^{N-1} \frac{(u - c(\mathcal{V}|i))^2}{(u - q^{2c(\mathcal{V}|i)}) (u - q^{-2c(\mathcal{V}|i)})}; \quad (17)$$

by convention, $F_\mathcal{V}(u) := \frac{u - c(\mathcal{V}|1)}{(u - v_1) \cdots (u - v_m)}$ for $N = 1$.

**Proposition 2.** We have

$$F_{\mathcal{T}}(u) \phi_{n+1}(c_1, \ldots, c_n, u) E_{\mathcal{U}} = \frac{u - c_{n+1}}{u - J_{n+1}} E_{\mathcal{U}}. \quad (18)$$
Proof. We prove (18) by induction on $n$. As $J_1 = \tau$, we have by (7)
\[
\frac{u - c_1}{u - J_1} = \frac{u - c_1}{(u - v_1) \cdots (u - v_m)} \tau(u),
\]
which verifies the basis of induction ($n = 0$).

We have: $E_W E_U = E_U$ and $E_W$ commutes with $\sigma_n$. Rewrite the left-hand side of (18) as
\[
F_T(u) \sigma_n(u, c_n) \cdot \phi_n(c_1, \ldots, c_{n-1}, u) E_W \cdot \sigma_n^{-1} E_U.
\]
By the induction hypothesis we have for the left-hand side of (18)
\[
F_T(u) \left( F_U(u) \right)^{-1} \sigma_n(u, c_n) \frac{u - c_n}{u - J_n} \sigma_n^{-1} E_U.
\]
Since $J_{n+1}$ commutes with $E_U$, the equality (18) is equivalent to
\[
F_T(u) \left( F_U(u) \right)^{-1} (u - c_n) \sigma_n^{-1} (u - J_{n+1}) E_U = \frac{(u - c_{n+1})(u - c_n)^2}{(u - q^2 c_n)(u - q^{-2} c_n)} (u - J_n) \sigma_n(c_n, u) E_U
\]
(19) (the inverse of $\sigma_n(u, c_n)$ is calculated with the help of (6)). By (17),
\[
F_T(u) \left( F_U(u) \right)^{-1} (u - c_n) = (u - c_{n+1}) \frac{(u - c_n)^2}{(u - q^2 c_n)(u - q^{-2} c_n)}.
\]
Therefore, to prove (19), it remains to show that
\[
\sigma_n^{-1} (u - J_{n+1}) E_U = (u - J_n) \sigma_n(c_n, u) E_U.
\]
Replacing $J_{n+1}$ by $\sigma_n J_n \sigma_n$, we write the left-hand side of (20) in the form
\[
(u \sigma_n^{-1} - J_n \sigma_n) E_U.
\]
As $J_n E_U = c_n E_U$, the right-hand side of (20) is
\[
\left( u \sigma_n - J_n \sigma_n + (q - q^{-1})(u - c_n) \frac{u}{c_n - u} \right) E_U
\]
and thus coincides with (21).

To prove Theorem 1, we need the following information about the behavior of the rational function $F_T(u)$ at $u = c_{n+1}$.

Proposition 3. The rational function $F_T(u)$ is non-singular at $u = c_{n+1}$, and moreover
\[
F_T(c_{n+1}) = F_\lambda F_\mu^{-1}.
\]
We will prove this proposition with the help of Lemmas 4 and 5 below, which involve the combinatorics of multi-partitions.

Lemma 4. We have
\[
F_T(u) = (u - c_{n+1}) \prod_{\beta \in \mathcal{E}_-(\mu)} (u - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu)} (u - c(\alpha))^{-1}.
\]

Proof. The proof is by induction on \( n \). For \( n = 0 \), we have

\[
F_T(u) = \frac{u - c_1}{(u - v_1) \cdot (u - v_m)},
\]

which is equal to the right-hand side of (22).

Now, for \( n > 0 \), we rewrite (17) for \( \mathcal{V} = T \) as

\[
F_T(u) = \frac{u - c_{n+1}}{(u - v_1) \cdot (u - v_m)} \cdot \frac{(u - c_n)^2}{(u - q^2 c_n)(u - q^{-2} c_n)} \prod_{i=1}^{n-1} \frac{(u - c_i)^2}{(u - q^2 c_i)(u - q^{-2} c_i)}.
\]

Using the induction hypothesis, we obtain

\[
F_T(u) = \frac{(u - c_{n+1})(u - c_n)^2}{(u - q^2 c_n)(u - q^{-2} c_n)} \prod_{\beta \in \mathcal{E}_-(\nu)} (u - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\nu)} (u - c(\alpha))^{-1}.
\]

Denote by \( \delta_t \) and \( \delta_b \) the \( m \)-nodes which are, respectively, just above and just below \( \delta, \delta_l \) and \( \delta_r \) the \( m \)-nodes which are, respectively, just on the left and just on the right of \( \delta \); it might happen that one of the coordinates of \( \delta_t \) (or \( \delta_l \)) is not positive, and in this situation, by definition, \( \delta_t \notin \mathcal{E}_-(\nu) \) (or \( \delta_l \notin \mathcal{E}_-(\nu) \)). It is straightforward to see that:

- If \( \delta_t, \delta_l \notin \mathcal{E}_-(\nu) \) then
  \[
  \mathcal{E}_-(\mu) = \mathcal{E}_-(\nu) \cup \{\delta\} \quad \text{and} \quad \mathcal{E}_+(\mu) = (\mathcal{E}_+(\nu) \cup \{\delta_b, \delta_r\}) \setminus \{\delta\}.
  \]

- If \( \delta_t \in \mathcal{E}_-(\nu) \) and \( \delta_l \notin \mathcal{E}_-(\nu) \) then
  \[
  \mathcal{E}_-(\mu) = (\mathcal{E}_-(\nu) \cup \{\delta\}) \setminus \{\delta_t\} \quad \text{and} \quad \mathcal{E}_+(\mu) = (\mathcal{E}_+(\nu) \cup \{\delta_b\}) \setminus \{\delta\}.
  \]

- If \( \delta_t \notin \mathcal{E}_-(\nu) \) and \( \delta_l \in \mathcal{E}_-(\nu) \) then
  \[
  \mathcal{E}_-(\mu) = (\mathcal{E}_-(\nu) \cup \{\delta\}) \setminus \{\delta_l\} \quad \text{and} \quad \mathcal{E}_+(\mu) = (\mathcal{E}_+(\nu) \cup \{\delta_r\}) \setminus \{\delta\}.
  \]

- If \( \delta_t, \delta_l \in \mathcal{E}_-(\nu) \) then
  \[
  \mathcal{E}_-(\mu) = (\mathcal{E}_-(\nu) \cup \{\delta\}) \setminus \{\delta_t, \delta_l\} \quad \text{and} \quad \mathcal{E}_+(\mu) = \mathcal{E}_+(\nu) \setminus \{\delta\}.
  \]

In each case, using that \( c(\delta_t) = c(\delta_r) = q^2 c_n \) and \( c(\delta_b) = c(\delta_l) = q^{-2} c_n \), it follows that the right-hand side of (23) is equal to

\[
(u - c_{n+1}) \prod_{\beta \in \mathcal{E}_-(\mu)} (u - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu)} (u - c(\alpha))^{-1},
\]

which establishes the formula (22).

\[\blacksquare\]

Lemma 5. We have

\[
\prod_{\beta \in \mathcal{E}_-(\mu)} (c_{n+1} - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu) \setminus \{\gamma\}} (c_{n+1} - c(\alpha))^{-1} = F_{\lambda\mu}^{-1} F_{\lambda\mu}^{-1}.
\]
Proof. 1. The definition (12), for a partition \( \lambda \), reduces to

\[
F_\lambda := \prod_{\alpha \in \lambda} \frac{q^{cc(\alpha)}}{[h_\lambda(\alpha)]_q}.
\]

The Lemma 5 for a partition \( \lambda \) is established in [8, Lemma 3.2].

2. Set \( k = \text{pos}(\gamma) \). Define, for an \( m \)-partition \( \theta \),

\[
\bar{F}_\theta := \prod_{\alpha \in \theta} \frac{q^{cc(\alpha)}}{[h_\theta(\alpha)]_q},
\]

and, for \( j = 1, \ldots, m \) such that \( j \neq k \),

\[
F^{(j)}_\theta := \prod_{\begin{array}{c} \alpha \in \theta \\ \text{pos}(\alpha) = k \end{array}} \frac{q^{-cc(\alpha)}}{v_k q^{-h_\theta^{(j)}(\alpha)} - v_j q^{h_\theta^{(j)}(\alpha)}}, \quad \prod_{\begin{array}{c} \alpha \in \theta \\ \text{pos}(\alpha) = j \end{array}} \frac{q^{-cc(\alpha)}}{v_j q^{-h_\theta^{(k)}(\alpha)} - v_k q^{h_\theta^{(k)}(\alpha)}}. \tag{24}
\]

By (12), we have

\[
F_\theta = \bar{F}_\theta \prod_{\begin{array}{c} j = 1, \ldots, m \\ j \neq k \end{array}} F^{(j)}_\theta. \tag{25}
\]

Fix \( j \in \{1, \ldots, m\} \) such that \( j \neq k \). We shall show that

\[
\prod_{\beta \in \mathcal{E}_+(\mu) \atop \text{pos}(\beta) = j} (c_{n+1} - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu) \setminus \{\gamma\} \atop \text{pos}(\alpha) = j} (c_{n+1} - c(\alpha))^{-1} = F^{(j)}_\lambda (F^{(j)}_\mu)^{-1}. \tag{26}
\]

Let \( p_1 < p_2 < \cdots < p_s \) be positive integers such that the \( j \)-th partition of \( \mu \) is \( (\mu_1, \ldots, \mu_{p_s}) \) with

\[
\mu_1 = \cdots = \mu_{p_1} > \mu_{p_1 + 1} = \cdots = \mu_{p_2} > \cdots > \mu_{p_{s-1} + 1} = \cdots = \mu_{p_s} > 0.
\]

We set \( p_0 := 0 \), \( p_{s+1} := +\infty \) and \( \mu_{p_{s+1}} := 0 \). Assume that the \( m \)-node \( \gamma \) lies in the line \( x \) and column \( y \). The left-hand side of (26) is equal to

\[
\prod_{b=1}^s \left( v_k q^{2(y-x)} - v_j q^{2(\mu_{p_b}-p_0)} \right) \prod_{b=1}^{s+1} \left( v_k q^{2(y-x)} - v_j q^{2(\mu_{p_{b}}-p_{b-1})} \right)^{-1}. \tag{27}
\]

The factors in the product (24) correspond to \( m \)-nodes of an \( m \)-partition. The \( m \)-nodes lying neither in the column \( y \) of the \( k \)-th diagrams (of \( \lambda \) or \( \mu \)) nor in the line \( x \) of the \( j \)-th diagrams do not contribute to the right-hand side of (26). Let \( t \in \{0, \ldots, s\} \) be such that \( p_t < x \leq p_{t+1} \). The contribution from the \( m \)-nodes in the column \( y \) and lines \( 1, \ldots, p_t \) of the \( k \)-th diagrams is

\[
\prod_{b=1}^t \left( \prod_{a=p_{b-1}+1}^{p_b} \frac{v_k q^{-(\mu_{p_b} - y + a)} - v_j q^{(\mu_{p_b} - y + a)}}{v_k q^{-(\mu_{p_b} - y + a+1)} - v_j q^{(\mu_{p_b} - y + a+1)}} \right);
\]

the contribution from the \( m \)-nodes in the column \( y \) and lines \( p_t + 1, \ldots, x \) of the \( k \)-th diagrams is

\[
\prod_{a=p_t+1}^{x-1} \left( \frac{v_k q^{-(\mu_{p_t+1} - y + a)} - v_j q^{(\mu_{p_t+1} - y + a)}}{v_k q^{-(\mu_{p_t+1} - y + a+1)} - v_j q^{(\mu_{p_t+1} - y + a+1)}} \right) \frac{q^{-cc(\gamma)}}{v_k q^{-(\mu_{p_t+1} - y + 1)} - v_j q^{(\mu_{p_t+1} - y + 1)}}.
\]
The contribution from the \( m \)-nodes lying in the line \( x \) of the \( j \)-th diagrams is

\[
\prod_{b=t+1}^{s} \prod_{a=\mu_{b}+1}^{\mu_{b+1}} \frac{v_{jk}q^{-(y-a+p_{b}-x)}}{v_{jk}q^{-(y-a+p_{b}+1)-x}}.
\]

After straightforward simplifications, we obtain for the right-hand side of (26)

\[
q^{x-y} \prod_{b=1}^{s} \left( v_{jk}q^{-(\mu_{b}-y+x-p_{b})} - v_{jk}q^{(\mu_{b}-y+x-p_{b})} \right)
\times \prod_{b=1}^{s+1} \left( v_{jk}q^{-(\mu_{b}-y+x-p_{b}-1)} - v_{jk}q^{(\mu_{b}-y+x-p_{b}-1)} \right)^{-1}.
\]

(28)

The comparison of (27) and (28) concludes the proof of the formula (26).

**Proof of Proposition 3.** The formula (22) shows that the rational function \( F_{T}(u) \) is non-singular at \( u = c_{n+1} \), and moreover

\[
F_{T}(c_{n+1}) = \prod_{\beta \in \mathcal{E}_{-}(\mu)} (c_{n+1} - c(\beta)) \prod_{\alpha \in \mathcal{E}_{+}(\mu) \setminus \{\gamma\}} (c_{n+1} - c(\alpha))^{-1}.
\]

We use the Lemma 5 to conclude the proof of the proposition.

**Proof of Theorem 1.** The theorem follows, by induction on \( n \), from the formula (16) together with Propositions 2 and 3.

**Example.** Consider, for \( m = 2 \), the standard 2-tableau \( \begin{array}{cc} 1 & 3 \\ 2 \end{array} \). The idempotent of the algebra \( H(2, 1, 3) \) corresponding to this standard 2-tableau reads, by the Theorem 1,

\[
\frac{\sigma_{2}(v_{1}q^{2}, v_{2})\sigma_{1}(v_{1}q^{2}, v_{2})q(v_{1}q^{2})\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{1}(v_{2}, v_{1})q(v_{2})\sigma_{1}^{-1}(v_{1})}{(q + q^{-1})(v_{1}q^{-1} - v_{2}q)(v_{1} - v_{2})(v_{2}q^{-2} - v_{1}q^{2})}.
\]

## 5 Remarks on the classical limit

Recall that the group ring \( \mathbb{C}G(m, 1, n + 1) \) of the complex reflection group \( G(m, 1, n + 1) \) is obtained by taking the classical limit: \( q \mapsto \pm 1 \) and \( v_{i} \mapsto \xi_{i}, \; i = 1, \ldots, m \), where \( \{\xi_{1}, \ldots, \xi_{m}\} \) is the set of distinct \( m \)-th roots of unity. The “classical limit” of the generators \( \tau, \sigma_{1}, \ldots, \sigma_{m} \) of \( H(m, 1, n + 1) \) we denote by \( t, s_{1}, \ldots, s_{n} \).

1. Consider the Baxterized elements (5) with spectral parameters of the form \( v_{p}q^{2a} \) and \( v_{p'}q^{2a'} \) with \( p, p' \in \{1, \ldots, m\} \). One directly finds that

\[
\lim_{q \to 1} \lim_{v_{n} \to \xi_{n}} \overline{\sigma}_{i}(v_{p}q^{2a}, v_{p'}q^{2a'}) = s_{i} + \frac{\delta_{p, p'}}{a - a'}.
\]

(29)

For the Artin generators \( \tilde{s}_{1}, \ldots, \tilde{s}_{n} \) of the symmetric group \( S_{n+1} \), the standard Baxterized elements are given by the rational functions

\[
\tilde{s}_{i} + \frac{1}{a - a'} \quad \text{for} \quad i = 1, \ldots, n.
\]
In view of (29), we define generalized Baxterized elements for the group $G(m,1,n+1)$ as the following functions

$$
\bar{s}_i(p,p',a,a') := s_i + \frac{\delta_{p,p'}}{a - a'}
\quad \text{for } i = 1, \ldots, n.
$$

These elements satisfy the following Yang–Baxter equation with spectral parameters

$$
\bar{s}_i(p,p',a,a') \bar{s}_{i+1}(p,p'',a',a'') \bar{s}_i(p',p'',a,a')
= \bar{s}_{i+1}(p',p'',a',a'') \bar{s}_i(p,p'',a,a') \bar{s}_{i+1}(p,p',a,a').
$$

The Baxterized elements (30) have been used in [14] for a fusion procedure for the complex reflection group $G(m,1,n+1)$.

2. It is immediate that

$$
\lim_{v_i \to \xi_i} a_0(r) = r^{m-1} \quad \text{and} \quad \lim_{v_i \to \xi_i} a_i(r) = r^{m-i}
\quad \text{for } i = 1, \ldots, m,
$$

where $a_i(r)$, $i = 0, \ldots, m$, are defined in (8). It follows from (10) that

$$
\lim_{v_i \to \xi_i} \bar{\tau}(r) = \sum_{i=0}^{m-1} r^{m-1-i} t_i.
$$

The rational function $\bar{\tau}$ defined by $\bar{\tau}(r) := \frac{1}{m} \sum_{i=0}^{m-1} r^{m-1-i} t_i$ with values in $\mathbb{C}G(m,1,n+1)$ was used in [14] for a fusion procedure for the complex reflection group $G(m,1,n+1)$.

3. Define, for an $m$-partition $\lambda$,

$$
f_\lambda := \left( \prod_{\alpha \in \lambda} h_\lambda(\alpha) \right)^{-1}.
$$

The classical limit of $F_\lambda$ is proportional to $f_\lambda$. More precisely, we have

$$
\lim_{q \to 1} \lim_{v_i \to \xi_i} F_\lambda = \tau_\lambda f_\lambda, \quad \text{where } \tau_\lambda = \frac{1}{m^n} \prod_{\alpha \in \lambda} \xi_{\text{pos}(\alpha)}.
$$

The formula (32) is obtained directly from (12) since

$$
\prod_{i=1}^{m} (\xi_k - \xi_i) = m/\xi_k \quad \text{for } k = 1, \ldots, m.
$$

4. Using formulas (29), (31) and (32), it is straightforward to check that the classical limit of the fusion procedure for $H(m,1,n+1)$ given by the Theorem 1 leads to the fusion procedure [14] for the group $G(m,1,n+1)$. Also, for $m = 1$, Theorem 1 coincides with the fusion procedure [8] for the Hecke algebra and, in the classical limit, with the fusion procedure [12] for the symmetric group.

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References


