

The Structure of Line Bundles over Quantum Teardrops^{*}

Albert Jeu-Liang SHEU

Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA

E-mail: asheu@ku.edu

Received October 07, 2013, in final form March 15, 2014; Published online March 22, 2014

<http://dx.doi.org/10.3842/SIGMA.2014.027>

Abstract. Over the quantum weighted 1-dimensional complex projective spaces, called quantum teardrops, the quantum line bundles associated with the quantum principal $U(1)$ -bundles introduced and studied by Brzezinski and Fairfax are explicitly identified among the finitely generated projective modules which are classified up to isomorphism. The quantum lens space in which these quantum line bundles are embedded is realized as a concrete groupoid C^* -algebra.

Key words: quantum line bundle; quantum principal bundle; quantum teardrop; quantum lens space; groupoid C^* -algebra; finitely generated projective module; quantum group

2010 Mathematics Subject Classification: 46L85; 58B32

Dedicated to Prof. Marc A. Rieffel on the occasion of his 75th birthday

1 Introduction

In the theory of noncommutative topology or geometry [6], a generally noncommutative C^* -algebra \mathcal{A} or a dense “core” $*$ -subalgebra \mathcal{A}^∞ of it is viewed respectively as the algebra $C(X_q)$ of continuous functions or the algebra $\mathcal{O}(X_q)$ of coordinate functions on an imaginary spatial object X_q , called a noncommutative space or a quantum space. In many interesting cases, such an imaginary nonexistent space X_q is closely related to or actually originated from a classical counterpart, a well-defined topological space or manifold X , and we view X_q or its “function algebra” $C(X_q)$ or $\mathcal{O}(X_q)$ as a quantization of the classical spatial object X .

There have been very intriguing discoveries that a lot of topological or geometric concepts or properties of a space X are also carried by (the function algebra of) its quantum counterpart X_q . For example, the concept of a vector bundle E [12] over a compact space X can be reformulated in the noncommutative context as a finitely generated projective left modules $\Gamma(E_q)$ over $C(X_q)$, viewed as the space of continuous cross-sections of some imaginary noncommutative or quantum vector bundle E_q over X_q , as suggested by Swan’s work [25]. Beyond the well-known K -theoretic study of such noncommutative vector bundles up to stable isomorphism, the classification of them up to isomorphism for C^* -algebras was made popular by Rieffel [18, 19] and completed for some interesting quantum spaces by him and others [1, 16, 19, 20, 22].

When the spatial object X is actually a topological group G , the quantization encompasses the group structure by requiring $C(G_q)$ or $\mathcal{O}(G_q)$ to have an additional Hopf $*$ -algebra structure, and we call G_q or its function algebra a quantum group. Generalizing further, we view a surjective Hopf $*$ -algebra homomorphism $\mathcal{O}(G_q) \rightarrow \mathcal{O}(H_q)$ as giving a quantum subgroup H_q of a quantum group G_q , and view the coinvariant $*$ -subalgebra $\mathcal{O}(G_q/H_q)$ of $\mathcal{O}(G_q)$ for the canonical coaction $\mathcal{O}(G_q) \xrightarrow{\Delta^R} \mathcal{O}(G_q) \otimes \mathcal{O}(H_q)$ as defining a “quantum homogeneous space” G_q/H_q . More generally,

^{*}This paper is a contribution to the Special Issue on Noncommutative Geometry and Quantum Groups in honor of Marc A. Rieffel. The full collection is available at <http://www.emis.de/journals/SIGMA/Rieffel.html>

given a coaction $\Delta_R : \mathcal{O}(X_q) \rightarrow \mathcal{O}(X_q) \otimes \mathcal{O}(H_q)$ of a compact quantum group H_q on a compact quantum space X_q , the coinvariant $*$ -subalgebra $\mathcal{O}(X_q/H_q)$ of $\mathcal{O}(X_q)$ defines a “quantum quotient space” X_q/H_q .

Classically some internal structure of a vector bundle E over a space X is often carried by a principal G -bundle P over X for some structure group G represented on some vector space V such that $E = P \times_G V$. The concept of quantum principal bundles has evolved and become well developed through years of study [5, 9]. In a recent work of Brzeziński and Fairfax [3], the quantization of weighted 1-dimensional complex projective spaces $\text{WP}(k, l)$, called teardrops by Thurston, and of principal bundles over them was studied. In particular, the quantum principal $U(1)$ -bundles and the associated quantum line bundles over the quantum teardrops $\text{WP}_q(k, l)$ were introduced and analyzed by Brzeziński and Fairfax. More concretely, they found a family of quantum line bundles $\mathcal{L}[n]$, $n \in \mathbb{Z}$, inside a quantum principal $U(1)$ -bundle $C(L_q(l; 1, l))$ over $\text{WP}_q(k, l)$ and showed that the continuous function C^* -algebra $C(\text{WP}_q(k, l))$ is isomorphic to the unitization $(\mathcal{K}^l)^+$ of l copies of the algebra \mathcal{K} of compact operators.

In this paper, we first show that each of $C(L_q(l; 1, l))$ and $C(\text{WP}_q(1, l))$ can be realized as a concrete groupoid C^* -algebra [17], following the groupoid approach to study C^* -algebras as initiated by Renault [17] and popularized by Curto, Muhly, and Renault [7, 14]. Then we explicitly identify the completed quantum line bundles $\mathcal{L}[n]$ among the well-known classified isomorphism classes of all finitely generated projective left modules over $(\mathcal{K}^l)^+$. This identification exhibits an interesting connection between “winding numbers” and “ranks”.

2 Projective modules

From the analysis point of view, since the category of isomorphism classes of unital commutative C^* -algebras is equivalent to the category of homeomorphism classes of compact Hausdorff spaces, the category of isomorphism classes of C^* -algebras provides a natural context for the development of noncommutative topology or geometry.

In this context, Swan’s theorem [25] makes it legitimate to call an (isomorphism class of) finitely generated projective left module E over a unital C^* -algebra \mathcal{A} an (isomorphism class of) noncommutative vector bundle over \mathcal{A} or more precisely the (generally imaginary, nonexistent) underlying quantum space. On the other hand, a projection p in the C^* -algebra $M_n(\mathcal{A})$ defines a left \mathcal{A} -module endomorphism $\xi \in \mathcal{A}^n \mapsto \xi p \in \mathcal{A}^n$ on the left free \mathcal{A} -module \mathcal{A}^n , and its image is a finitely generated projective left \mathcal{A} -module $E := \mathcal{A}^n p$. It is well-known that this association establishes a bijective correspondence between the unitary equivalence classes of projections p in $M_\infty(\mathcal{A}) := \bigcup_{n=1}^\infty M_n(\mathcal{A})$, where $M_n(\mathcal{A})$ is embedded in $M_{n+1}(\mathcal{A})$ in the canonical way for each n , and the isomorphism classes of finitely generated projective left modules E over \mathcal{A} [2].

Two finitely generated projective left modules E, F over \mathcal{A} are called stably isomorphic if they become isomorphic after being augmented by the same finitely generated free \mathcal{A} -module, i.e. $E \oplus \mathcal{A}^k \cong F \oplus \mathcal{A}^k$ for some $k \in \mathbb{N}$. The K_0 -group of \mathcal{A} classifies such finitely generated projective modules up to stable isomorphism. The cancellation problem dealing with whether two stably isomorphic finitely generated projective left modules are actually isomorphic goes beyond K -theory and is in general an interesting but difficult question. It was Rieffel’s pioneering work [18, 19] that brought the cancellation problem to the attentions and interest of researchers in the theory of C^* -algebras. Over some basic geometrically motivated quantum spaces, the finitely generated projective left modules have been successfully classified [1, 16, 19, 20, 22].

As a simple example, we now describe the classification of finitely generated projective left modules over a fairly elementary C^* -algebra, which is relevant to our main result later.

Let \mathcal{K} be the algebra of all compact operators on a separable infinite-dimensional Hilbert space \mathcal{H} , say, ℓ^2 . Recall that for a C^* -algebra \mathcal{A} , we use \mathcal{A}^+ to denote its unitization, a unital C^* -algebra equal to $\mathcal{A} \oplus \mathbb{C}$ as a vector space and endowed with the algebra multiplication

$(a, s)(b, t) := (ab + sb + ta, st)$ and involution $(a, s)^* := (a^*, \bar{s})$ for $(a, s), (b, t) \in \mathcal{A} \oplus \mathbb{C}$. In particular, $(\mathcal{K}^l)^+ \equiv (\oplus_{s=1}^l \mathcal{K})^+$ for $l \in \mathbb{N}$ denotes the unitization of the direct sum of l copies of \mathcal{K} .

The classification of all isomorphism classes of finitely generated projective left modules over $(\mathcal{K}^l)^+$, or equivalently, all unitary equivalence classes of projections in $M_\infty((\mathcal{K}^l)^+)$ is fairly well understood as summarized below. In the following, we use I to denote the multiplicative unit of the unital C^* -algebra $(\mathcal{K}^l)^+$, and I_r to denote the identity matrix in $M_r((\mathcal{K}^l)^+)$, while

$$P_n := \sum_{i=1}^n e_{ii} \in M_n(\mathbb{C}) \subset \mathcal{K}$$

denotes the standard $n \times n$ identity matrix in $M_n(\mathbb{C}) \subset \mathcal{K}$ for any integer $n \geq 0$ (with $M_0(\mathbb{C}) = 0$ and $P_0 = 0$ understood). In particular, $\oplus_{j=1}^l P_{k_j} \in \mathcal{K}^l$ for integers $k_j \geq 0$.

Proposition 1. *The projections $\oplus_{j=1}^l P_{k_j} \in M_1((\mathcal{K}^l)^+)$ with $k_j \in \mathbb{Z}_{\geq} := \{k \in \mathbb{Z} : k \geq 0\}$ and*

$$I_{r-1} \oplus (I - (\oplus_{j=1}^l P_{n_j})) \oplus (\oplus_{j=1}^l P_{m_j}) \in M_{r+1}((\mathcal{K}^l)^+)$$

with $r \in \mathbb{N}$ and $n_j, m_j \in \mathbb{Z}_{\geq}$ such that $n_j m_j = 0$ for all j represent all unitarily inequivalent classes of projections in $M_\infty((\mathcal{K}^l)^+)$.

3 Quantum spaces and principal bundles

We recall the definition of a compact quantum group by Woronowicz [28] as a unital separable C^* -algebra \mathcal{A} with a comultiplication Δ such that $(\mathcal{A} \otimes 1) \Delta \mathcal{A}$ and $(1 \otimes \mathcal{A}) \Delta \mathcal{A}$ are dense in $\mathcal{A} \otimes \mathcal{A}$. It is known [27, 28] that a compact quantum group \mathcal{A} contains a dense $*$ -subalgebra \mathcal{A}^∞ , forming a Hopf $*$ -algebra $(\mathcal{A}^\infty, \Delta, *, S, \varepsilon)$, and has a Haar state $h \in \mathcal{A}^*$ satisfying $h(1) = 1$ and

$$(\text{id} \otimes h) \Delta a = h(a)1 = (h \otimes \text{id}) \Delta a.$$

We denote \mathcal{A}^∞ by $\mathcal{O}(G_q)$ if \mathcal{A} is denoted as $C(G_q)$.

For a quantum subgroup H_q of a compact quantum group G_q given by a surjective Hopf $*$ -algebra homomorphism $r : \mathcal{O}(G_q) \rightarrow \mathcal{O}(H_q)$, there is a canonical coaction $\mathcal{O}(G_q) \xrightarrow{\Delta_R} \mathcal{O}(G_q) \otimes \mathcal{O}(H_q)$ given by $\Delta_R := (\text{id} \otimes r) \Delta$ for the comultiplication Δ of $\mathcal{O}(G_q)$, and the coinvariant $*$ -subalgebra

$$\mathcal{O}(G_q/H_q) := \{x \in \mathcal{O}(G_q) : \Delta_R(x) = x \otimes 1\}$$

for the coaction Δ_R defines a “quantum homogeneous space” G_q/H_q . A fundamental example is the quantum odd-dimensional sphere $S_q^{2n+1} = \text{SU}_q(n+1)/\text{SU}_q(n)$ [26] with $q \in (0, 1)$ generated by z_0, \dots, z_n subject to the relations $\sum_{m=0}^n z_m z_m^* = 1$, $z_i z_j = q z_j z_i$ for $i < j$, $z_i z_j^* = q z_j^* z_i$ for $i \neq j$, and $z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^n z_m z_m^*$.

More generally, given a coaction $\Delta_R : \mathcal{O}(X_q) \rightarrow \mathcal{O}(X_q) \otimes \mathcal{O}(H_q)$ of a compact quantum group H_q on a compact quantum space X_q , the coinvariant $*$ -subalgebra

$$\mathcal{O}(X_q/H_q) := \{x \in \mathcal{O}(X_q) : \Delta_R(x) = x \otimes 1\}$$

defines a “quantum quotient space” X_q/H_q . An interesting example is the quantum weighted complex projective space $\text{WP}_q(l_0, \dots, l_n)$ with $q \in (0, 1)$ [3], for pairwise coprime numbers

$l_0, \dots, l_n \in \mathbb{N}$, which is the quantum quotient space for the coaction of $\mathcal{O}(U_q(1)) \equiv \mathcal{O}(U(1)) = \mathbb{C}[u, u^*]$ on $\mathcal{O}(S_q^{2n+1})$ defined by

$$z_i \in \mathcal{O}(S_q^{2n+1}) \mapsto z_i \otimes u^{l_i} \in \mathcal{O}(S_q^{2n+1}) \otimes \mathcal{O}(U(1)) \quad \text{for } i = 0, \dots, n.$$

As special cases, this includes the quantum complex projective space $\mathbb{C}\mathbb{P}_q^n$ when $l_0 = \dots = l_n = 1$, and the so-called quantum teardrop $\text{WP}_q(k, l)$ with coprime k, l when $n = 1$.

Brzeziński and Fairfax [3] determined that S_q^3 is a quantum principal $U(1)$ -bundle over $\text{WP}_q(k, l)$, or more precisely, the algebra $\mathcal{O}(S_q^3)$ is a principal $\mathcal{O}(U(1))$ -comodule algebra over $\mathcal{O}(\text{WP}_q(k, l))$, if and only if $k = l = 1$. This result is consistent with the classical $U(1)$ -action $(z, w) \mapsto (u^k z, u^l w)$ for $u \in \mathbb{T}$ on \mathbb{S}^3 . Furthermore they found that the quantum lens space $L_q(l; 1, l)$ [11] provides the total space of a quantum principal $U(1)$ -bundle over $\text{WP}_q(1, l)$, where $L_q(l; 1, l)$ is the quantum quotient space defined by the coaction $\rho : \mathcal{O}(S_q^3) \rightarrow \mathcal{O}(S_q^3) \otimes \mathcal{O}(Z_l)$ with $\rho(\alpha) = \alpha \otimes w$ and $\rho(\beta) = \beta \otimes 1$ where $\alpha := z_0$ and $\beta := z_1^*$ generate $\mathcal{O}(S_q^3) \equiv \mathcal{O}(\text{SU}_q(2))$, and w is the unitary group-like generator of $\mathcal{O}(Z_l)$ with $w^l = 1$. More explicitly, $\mathcal{O}(L_q(l; 1, l))$ is the $*$ -subalgebra of $\mathcal{O}(\text{SU}_q(2))$ generated by $c := \alpha^l$ and $d := \beta$, and a well-defined coaction

$$\rho_l : \mathcal{O}(L_q(l; 1, l)) \rightarrow \mathcal{O}(L_q(l; 1, l)) \otimes \mathcal{O}(U(1))$$

with $\rho_l(c) := c \otimes u$ and $\rho_l(d) := d \otimes u^*$ makes $\mathcal{O}(L_q(l; 1, l))$ a quantum principal $U(1)$ -bundle over $\text{WP}_q(1, l)$.

Corresponding to the irreducible (1-dimensional) representations of $U(1)$ indexed by $n \in \mathbb{Z}$, we have the irreducible corepresentations of $\mathcal{O}(U(1))$ on some left comodules denoted as W_n . Following the general theory of constructing finitely generated projective modules from quantum principal bundles and finite-dimensional corepresentations [4], Brzeziński and Fairfax took the cotensor product of $\mathcal{O}(L_q(l; 1, l))$ with W_n over $\mathcal{O}(U(1))$ to get a finitely generated projective module $\mathcal{L}[n] \subset \mathcal{O}(L_q(l; 1, l))$ over $\mathcal{O}(\text{WP}_q(1, l))$, naturally called a quantum line bundle over $\text{WP}_q(1, l)$, and they computed an idempotent matrix $E[n]$ over $\mathcal{O}(\text{WP}_q(1, l))$ implementing the projective module $\mathcal{L}[n]$ with complicated entries $E[n]_{ij} = \omega(u^n)^{[2]_i} \omega(u^n)^{[1]_j}$, where $\omega(u^n) = \sum_i \omega(u^n)^{[1]_i} \otimes \omega(u^n)^{[2]_i}$ comes from a strong connection

$$\omega : \mathcal{O}(U(1)) \rightarrow \mathcal{O}(L_q(l; 1, l)) \otimes \mathcal{O}(L_q(l; 1, l)),$$

and showed in particular that the $\mathcal{O}(\text{WP}_q(1, l))$ -module $\mathcal{L}[1]$ is not free.

Furthermore Brzeziński and Fairfax found the enveloping C^* -algebra of $\mathcal{O}(\text{WP}_q(k, l))$ as $C(\text{WP}_q(k, l)) \cong (\mathcal{K}^l)^+$ and computed its K -groups from the exact sequence

$$0 \rightarrow \mathcal{K}^l \equiv \bigoplus_{j=1}^l \mathcal{K} \rightarrow (\mathcal{K}^l)^+ \equiv C(\text{WP}_q(k, l)) \rightarrow \mathbb{C} \rightarrow 0.$$

It is then a natural and interesting question to identify explicitly the completed quantum line bundles

$$\overline{\mathcal{L}[n]} \equiv C(\text{WP}_q(1, l)) \otimes_{\mathcal{O}(\text{WP}_q(1, l))} \mathcal{L}[n] = (\mathcal{K}^l)^+ \otimes_{\mathcal{O}(\text{WP}_q(1, l))} \mathcal{L}[n]$$

over $C(\text{WP}_q(1, l))$ for all $n \in \mathbb{Z}$ among the finitely generated projective modules over $(\mathcal{K}^l)^+$ already well classified.

4 Quantum lens space as groupoid C^* -algebra

In the past, there have been successful studies of the structure of some interesting C^* -algebras [7, 14, 21, 23, 24] by realizing them first as a concrete groupoid C^* -algebra, following the groupoid

approach to C^* -algebras initiated by Renault [17] and popularized by the work of Curto, Muhly, and Renault [7, 14]. In this section, we first identify the C^* -algebra $C(L_q(l; 1, l))$ for $q \in (0, 1)$ with a concrete groupoid C^* -algebra, and then find an explicit description of the structure of $C(L_q(l; 1, l))$. We construct the groupoid directly from the irreducible representations of $C(L_q(l; 1, l))$ classified by Brzeziński and Fairfax [3]. Our approach should be compared with the machinery developed by Kumjian, Pask, Raeburn, Renault, and Paterson in [13, 15] that associates groupoid C^* -algebras to graph C^* -algebras.

By Proposition 2.4 of [3], the faithful $*$ -representation $\pi^\oplus \equiv \bigoplus_{s=1}^l \pi_s$ of $\mathcal{O}(\text{WP}_q(1, l))$ on $\bigoplus_{s=1}^l V_s$ factors through the key $*$ -representation π of $\mathcal{O}(\text{SU}_q(2))$ on $V \cong \bigoplus_{s=1}^l V_s$, where each $V_s \cong \ell^2(\mathbb{Z}_{\geq})$ for $\mathbb{Z}_{\geq} := \{k \in \mathbb{Z} : k \geq 0\}$, and by Proposition 5.1 of [3], $\pi^\oplus \equiv \bigoplus_{s=1}^l \pi_s$ extends to a faithful $*$ -representation of $C(\text{WP}_q(1, l))$ identifying $C(\text{WP}_q(1, l))$ with $(\mathcal{K}^l)^+$.

Using the classification [3] of irreducible $*$ -representations of $\mathcal{O}(L_q(l; 1, l)) \subset \mathcal{O}(\text{SU}_q(2))$ as π_s^λ for $s = 0, 1, \dots, l$ and $\lambda \in \mathbb{T}$, we can realize $C(L_q(l; 1, l))$ as a groupoid C^* -algebra as follows.

For $s > 0$ and $\lambda \in \mathbb{T}$, each π_s^λ is an irreducible representation of $\mathcal{O}(L_q(l; 1, l))$ on $\ell^2(\mathbb{Z}_{\geq})$ such that $\pi_s^\lambda(c)$ for any fixed s is the same weighted unilateral shift independent of λ , with strictly positive weights $\prod_{m=1}^l \sqrt{1 - q^{2(pl+s-m)}}$ and different from the (backward) unilateral shift \mathcal{S} on $\ell^2(\mathbb{Z}_{\geq})$, that sends the standard basis vector e_p of $\ell^2(\mathbb{Z}_{\geq})$ to e_{p-1} (with $e_{-1} := 0$), only by a compact operator, while $\bigoplus_{s=1}^l \pi_s^\lambda(d) = \lambda(\bigoplus_{s=1}^l \pi_s^1(d))$ with $\bigoplus_{s=1}^l \pi_s^1(d)$ a compact diagonal operator on $\bigoplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq})$ with distinct nonzero eigenvalues q^{pl+s} , $p \in \mathbb{Z}_{\geq}$. Applying functional calculus to $\bigoplus_{s=1}^l \pi_s^\lambda(d)$ to get scaled diagonal matrix units and then composing with powers of $\bigoplus_{s=1}^l \pi_s^\lambda(c)$ or its adjoint, we can get all matrix units for each component $\ell^2(\mathbb{Z}_{\geq})$ of $\bigoplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq})$ and hence for each $\lambda \in \mathbb{T}$,

$$\bigoplus_{s=1}^l \mathcal{K}(\ell^2(\mathbb{Z}_{\geq})) \subset \overline{\left(\bigoplus_{s=1}^l \pi_s^\lambda\right) \left(\mathcal{O}(L_q(l; 1, l))\right)}.$$

On the other hand, $\left(\bigoplus_{s=1}^l \pi_s^\lambda\right)(c)$ modulo $\bigoplus_{s=1}^l \mathcal{K}(\ell^2(\mathbb{Z}_{\geq}))$ is the direct sum of l copies of the same unilateral shift \mathcal{S} . So the image C^* -algebra $\overline{\pi_s^\lambda(\mathcal{O}(L_q(l; 1, l)))}$ is the standard Toeplitz C^* -algebra \mathcal{T} , with $\sigma(\pi_s^\lambda(c)) = \text{id}_{\mathbb{T}}$ and $\sigma(\pi_s^\lambda(d)) = 0$ for all s , where $\sigma : \mathcal{T} \rightarrow C(\mathbb{T})$ is the standard symbol map of \mathcal{T} , while for the image C^* -algebra of the direct sum $\bigoplus_{s=1}^l \pi_s^\lambda$, we have a short exact sequence

$$0 \rightarrow \bigoplus_{s=1}^l \mathcal{K}(\ell^2(\mathbb{Z}_{\geq})) \rightarrow \overline{\left(\bigoplus_{s=1}^l \pi_s^\lambda\right) \left(\mathcal{O}(L_q(l; 1, l))\right)} \rightarrow C(\mathbb{T}) \rightarrow 0.$$

Note that each of the one-dimensional irreducible representations π_0^μ of $\mathcal{O}(L_q(l; 1, l))$ with $\pi_0^\mu(c) = \mu \in \mathbb{T}$ and $\pi_0^\mu(d) = 0$ factors through each π_s^λ , or more explicitly, $\pi_0^\mu = \eta_\mu \circ \sigma \circ \pi_s^\lambda$ for the evaluation character $\eta_\mu : C(\mathbb{T}) \rightarrow \mathbb{C}$ with $\eta_\mu(f) := f(\mu)$. Hence the \mathbb{T} -parameter family $\{\bigoplus_{s=1}^l \pi_s^\lambda\}_{\lambda \in \mathbb{T}}$ of representations together represent faithfully the enveloping C^* -algebra $C(L_q(l; 1, l))$ of $\mathcal{O}(L_q(l; 1, l))$.

More effectively, we can merge the \mathbb{T} -parameter family $\{\bigoplus_{s=1}^l \pi_s^\lambda\}_{\lambda \in \mathbb{T}}$ of representations of $\mathcal{O}(L_q(l; 1, l))$ into one representation $\bigoplus_{s=1}^l \tilde{\pi}_s$ on the Hilbert space $L^2(\mathbb{T}) \otimes \left(\bigoplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq})\right)$ or equivalently on $\ell^2(\mathbb{Z}) \otimes \left(\bigoplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq})\right)$ via the Fourier transform on \mathbb{T} . More precisely, we have $\bigoplus_{s=1}^l \tilde{\pi}_s(c) = \text{id}_{\ell^2(\mathbb{Z})} \otimes \left(\bigoplus_{s=1}^l \pi_s^1(c)\right)$ and $\tilde{\pi}_s(d) = \mathcal{U} \otimes \left(\bigoplus_{s=1}^l \pi_s^1(d)\right)$ for the (backward) bilateral shift \mathcal{U} on $\ell^2(\mathbb{Z})$. Clearly $\bigoplus_{s=1}^l \tilde{\pi}_s$ is a faithful representation of $\mathcal{O}(L_q(l; 1, l))$ and extends to a faithful representation of $C(L_q(l; 1, l))$. In the following, we denote by $\tilde{\pi}^\oplus := \bigoplus_{s=1}^l \tilde{\pi}_s$ this faithful representation of $C(L_q(l; 1, l))$ on $\ell^2(\mathbb{Z}) \otimes \left(\bigoplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq})\right)$.

Now we consider the (r -discrete) groupoid

$$\mathfrak{G} := \mathbb{Z} \times \left(\left(\mathbb{Z} \times \left(\bigsqcup_{s=1}^l \mathbb{Z} \right)^+ \right) \middle| \left(\bigsqcup_{s=1}^l \mathbb{Z}_{\geq} \right)^+ \right)$$

which is the direct product of the group \mathbb{Z} and the transformation groupoid $\mathbb{Z} \times \left(\bigsqcup_{s=1}^l \mathbb{Z} \right)^+$ restricted to the positive half $\left(\bigsqcup_{s=1}^l \mathbb{Z}_{\geq} \right)^+$, where $\left(\bigsqcup_{s=1}^l \mathbb{Z} \right)^+$ is the one-point compactification of the disjoint union $\bigsqcup_{s=1}^l \mathbb{Z}$ of l copies of \mathbb{Z} , and \mathbb{Z} acts canonically by translation on each component \mathbb{Z} of $\bigsqcup_{s=1}^l \mathbb{Z} \subset \left(\bigsqcup_{s=1}^l \mathbb{Z}_{\geq} \right)^+$ while fixing the point at infinity $\infty \in \left(\bigsqcup_{s=1}^l \mathbb{Z} \right)^+$. More explicitly,

$$(k, m, p)_s (k', m', p')_s = (k + k', m + m', p')_s$$

exactly when $p = p' + m'$ for $k, k', m, m' \in \mathbb{Z}$ and $p, p' \in \mathbb{Z}_{\geq}$, where the subscript s in $(k, m, p)_s$ and $(k', m', p')_s$ indicates that p and p' come from the same s -th component of $\bigsqcup_{s=1}^l \mathbb{Z}_{\geq}$. We remark that with the group \mathbb{Z}^2 being amenable, the full groupoid C^* -algebra of \mathfrak{G} is the same as its reduced groupoid C^* -algebra by Proposition 2.15 of [14].

Before proceeding further, we introduce an open subgroupoid \mathfrak{F} of \mathfrak{G} defined by

$$\mathfrak{F} := \left(\mathbb{Z} \times \left(\left(\mathbb{Z} \times \left(\bigsqcup_{s=1}^l \mathbb{Z} \right)^+ \right) \Big|_{\bigsqcup_{s=1}^l \mathbb{Z}_{\geq}} \right) \right) \cup (\{0\} \times (\mathbb{Z} \times \{\infty\})) \subset \mathfrak{G}.$$

Let $\tilde{\rho}$ be the representation of the groupoid C^* -algebra $C^*(\mathfrak{F})$ induced off the counting measure μ supported on the set $\bigsqcup_{s=1}^l \{0\}$ that generates the dense invariant open subset $\bigsqcup_{s=1}^l \mathbb{Z}_{\geq}$ of the unit space $\left(\bigsqcup_{s=1}^l \mathbb{Z}_{\geq} \right)^+$. By Proposition 2.17 of [14] (or by a direct inspection for this fairly simple r -discrete groupoid), $\tilde{\rho}$ is faithful. We note that the representation space of $\tilde{\rho}$ is isomorphic to

$$\ell^2 \left(\mathbb{Z} \times \left(\bigsqcup_{s=1}^l \mathbb{Z}_{\geq} \right) \right) \cong \ell^2(\mathbb{Z}) \otimes \left(\bigoplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq}) \right),$$

and that

$$\tilde{\pi}^\oplus(c) = \tilde{\rho} \left(\sum_{s=1}^l \left(\sum_{p=1}^{\infty} \left(\prod_{m=1}^l \sqrt{1 - q^{2(pl+s-m)}} \right) \delta_{(0,-1,p)_s} \right) \right),$$

where the argument of $\tilde{\rho}$ is understood as an element of $C_c(\mathfrak{F}) \subset C_c(\mathfrak{G})$ with value equal to

$$\lim_{p \rightarrow \infty} \left(\prod_{m=1}^l \sqrt{1 - q^{2(pl+s-m)}} \right) = 1 \quad (\text{for any } s)$$

at the point $(0, -1, \infty) \in \mathfrak{F} \subset \mathfrak{G}$ while vanishing at $(k, m, \infty) \in \mathfrak{G}$ for all $(k, m) \neq (0, -1)$. Also we have

$$\tilde{\pi}^\oplus(d) = \tilde{\rho} \left(\sum_{s=1}^l \left(\sum_{p=0}^{\infty} q^{pl+s} \delta_{(-1,0,p)_s} \right) \right),$$

where the argument of $\tilde{\rho}$ is an element of $C_c(\mathfrak{F}) \subset C_c(\mathfrak{G})$ with value equal to $\lim_{p \rightarrow \infty} q^{pl+s} = 0$ (for any s) at the point (k, m, ∞) for all (k, m) .

Now via $\tilde{\rho}^{-1} \circ \tilde{\pi}$, we can view c, d as elements of $C_c(\mathfrak{F}) \subset C^*(\mathfrak{F})$ and hence view $C(L_q(l; 1, l))$ as embedded in $C^*(\mathfrak{F})$. Applying functional calculus to d^*d , we can get $\mathbb{C}\delta_{(0,0,p)_s} \subset C(L_q(l; 1, l))$ for all $p \in \mathbb{Z}_{\geq}$ and $1 \leq s \leq l$, and then by composing with c^* and d^* , we get $\mathbb{C}\delta_{(0,1,p)_s}$ and $\mathbb{C}\delta_{(1,0,p)_s}$ contained in $C(L_q(l; 1, l))$ for any $p \in \mathbb{Z}_{\geq}$ and $1 \leq s \leq l$, which generate the convolution $*$ -subalgebra

$$C_c \left(\mathfrak{F} \Big|_{\bigsqcup_{s=1}^l \mathbb{Z}_{\geq}} \right) \subset C^*(\mathfrak{F}) \stackrel{\tilde{\rho}}{\subset} \mathcal{B} \left(\ell^2(\mathbb{Z}) \otimes \left(\bigoplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq}) \right) \right).$$

On the other hand, for any $n \in \mathbb{Z}$, the $|n|$ -th power of c or c^* provides an element of $C_c(\mathfrak{F})$ having a nonvanishing positive value at every point in

$$\{(0, n, p)_s : p \in \mathbb{Z}_{\geq}, 1 \leq s \leq l\} \cup \{(0, n, \infty)\}$$

while vanishing at all other points of \mathfrak{F} . So the C^* -subalgebra $C(L_q(l; 1, l))$ of $C^*(\mathfrak{F})$ contains all elements of $C_c(\mathfrak{F})$ and hence equals $C^*(\mathfrak{F})$.

We summarize:

Theorem 1. $C(L_q(l; 1, l)) \cong C^*(\mathfrak{F})$, where \mathfrak{F} is the topological groupoid

$$\mathfrak{F} = \left[\mathbb{Z} \times \left(\left(\mathbb{Z} \times \left(\bigcup_{s=1}^l \mathbb{Z} \right)^+ \right) \Big|_{\left(\bigcup_{s=1}^l \mathbb{Z}_{\geq} \right)^+} \right) \right] \setminus [(\mathbb{Z} \setminus \{0\}) \times (\mathbb{Z} \times \{\infty\})].$$

In the general theory of groupoid C^* -algebras [17], open invariant subsets and their complements in the unit space of a groupoid give rise respectively to closed ideals and quotients of its groupoid C^* -algebra, and under suitable conditions the association is bijective which broadens a result of Gootman and Rosenberg [8] for transformation groups.

Decomposing the base space $\left(\bigcup_{s=1}^l \mathbb{Z}_{\geq} \right)^+$ of \mathfrak{F} into the open invariant subspace $\bigcup_{s=1}^l \mathbb{Z}_{\geq}$ and its closed invariant complement $\{\infty\}$, we get the closed ideal

$$C^* \left(\mathfrak{F} \Big|_{\bigsqcup_{s=1}^l \mathbb{Z}_{\geq}} \right) = C^* \left(\mathbb{Z} \times \left(\bigcup_{s=1}^l (\mathbb{Z} \times \mathbb{Z}) \Big|_{\mathbb{Z}_{\geq}} \right) \right) \cong C(\mathbb{T}) \otimes \mathcal{K}^l$$

of $C^*(\mathfrak{F})$ and the quotient

$$C^*(\mathfrak{F}) / C^* \left(\mathfrak{F} \Big|_{\bigsqcup_{s=1}^l \mathbb{Z}_{\geq}} \right) \cong C^*(\mathbb{Z} \times \{\infty\}) \cong C(\mathbb{T}),$$

which can be summarized as follows.

Corollary 1. *There is a short exact sequence of C^* -algebras*

$$0 \rightarrow C(\mathbb{T}) \otimes \mathcal{K}^l \rightarrow C(L_q(l; 1, l)) \rightarrow C(\mathbb{T}) \rightarrow 0.$$

In fact, from the above analysis, we actually have the following explicit description

$$C(L_q(l; 1, l)) = \{(a_1, \dots, a_l) \in \bigoplus_{s=1}^l C(\mathbb{T}, \mathcal{T}) : \sigma \circ a_1 = \dots = \sigma \circ a_l \text{ constant on } \mathbb{T}\}$$

in terms of the standard Toeplitz C^* -algebra \mathcal{T} and its symbol map $\sigma : \mathcal{T} \rightarrow C(\mathbb{T})$.

5 Line bundles over quantum teardrops

In this section, we identify concretely the quantum line bundles $\overline{\mathcal{L}[n]}$ over $C(\mathrm{WP}_q(1, l)) \cong (\mathcal{K}^l)^+$ for $q \in (0, 1)$. First we recall that the coaction ρ_l of $\mathcal{O}(\mathrm{U}(1))$ on $\mathcal{O}(L_q(l; 1, l))$ gives a \mathbb{Z} -grading of $\mathcal{O}(L_q(l; 1, l))$ with c of degree 1 and d of degree -1 , such that $\mathcal{O}(\mathrm{WP}_q(1, l))$ generated by $b := cd$ and $a := dd^*$ is the degree-0 component of $\mathcal{O}(L_q(l; 1, l))$, while $\mathcal{L}[n]$ is the degree- n component of $\mathcal{O}(L_q(l; 1, l))$ for general $n \in \mathbb{Z}$ [3].

Now we introduce a compatible \mathbb{Z} -grading on the convolution $*$ -algebra $C_c(\mathfrak{F})$, based on the groupoid structure. We define the homogeneous degree- n component as $C_c(\mathfrak{F})_n := C_c(\mathfrak{F}_n)$ for the open set

$$\mathfrak{F}_n := \bigsqcup_{s=1}^l \{(k, k-n, p)_s : p \in \mathbb{Z}_{\geq}, n-p \leq k \in \mathbb{Z}\} \cup \{(0, -n, \infty)\} \subset \mathfrak{F}.$$

Note that $\mathfrak{F} = \bigsqcup_{n \in \mathbb{Z}} \mathfrak{F}_n$ and $C_c(\mathfrak{F}) = \bigoplus_{n \in \mathbb{Z}} C_c(\mathfrak{F}_n)$ becomes a \mathbb{Z} -graded $*$ -algebra with $\deg(\delta_{(k, m, p)_s}) = k - m$. Furthermore $c \in C_c(\mathfrak{F}_1)$ and $d \in C_c(\mathfrak{F}_{-1})$ for the generators $c, d \in \mathcal{O}(L_q(l; 1, l)) \subset C_c(\mathfrak{F})$ of $\mathcal{O}(L_q(l; 1, l))$. So this groupoid \mathbb{Z} -grading on $C_c(\mathfrak{F})$ when restricted to the $*$ -subalgebra $\mathcal{O}(L_q(l; 1, l)) \subset C_c(\mathfrak{F})$ coincides with the original \mathbb{Z} -grading on $\mathcal{O}(L_q(l; 1, l))$. So when viewed as elements of $C_c(\mathfrak{F})$, the elements of $\mathcal{L}[n] \subset \mathcal{O}(L_q(l; 1, l))$ are homogeneous of degree n . That is

$$\mathcal{L}[n] \subset C_c(\mathfrak{F})_n \equiv C_c(\mathfrak{F}_n).$$

Also note that $C_c(\mathfrak{F})_0 = C_c(\mathfrak{F}_0)$ where $\mathfrak{F}_0 \subset \mathfrak{F}$ consisting of $(0, 0, \infty)$ and elements of the form $(m, m, p)_s$ with $p, m+p \in \mathbb{Z}_{\geq}$ is an open subgroupoid of \mathfrak{F} . It is clear that the $*$ -algebra \mathbb{Z} -grading structure on $C_c(\mathfrak{F})$ makes each $C_c(\mathfrak{F})_n$ a left $C_c(\mathfrak{F})_0$ -module.

By the analysis already done on $\mathcal{L}[0] = \mathcal{O}(\mathrm{WP}_q(1, l))$ in [3] or a direct analysis of the generators a, b of $\mathcal{O}(\mathrm{WP}_q(1, l)) \equiv \mathcal{L}[0] \subset C_c(\mathfrak{F}_0)$, we get

$$C_c(\mathfrak{F}_0) \subset C(\mathrm{WP}_q(1, l)) = C^*(\mathfrak{F}_0) \subset C^*(\mathfrak{F}) \equiv C(L_q(l; 1, l)).$$

In particular, $C(\mathrm{WP}_q(1, l))$ is realized as the groupoid C^* -algebra of the subgroupoid \mathfrak{F}_0 of \mathfrak{F} .

Let $\overline{\mathcal{L}[n]}$ be the completion of $\mathcal{L}[n]$ in $C^*(\mathfrak{F}) = C(L_q(l; 1, l))$. In the following, we show that $\overline{\mathcal{L}[n]}$ is a finitely generated projective left module over $C(\mathrm{WP}_q(1, l)) \subset C^*(\mathfrak{F})$, and hence we can make the canonical identification

$$\overline{\mathcal{L}[n]} \equiv C(\mathrm{WP}_q(1, l)) \otimes_{\mathcal{O}(\mathrm{WP}_q(1, l))} \mathcal{L}[n].$$

It is easy to see that the $\mathcal{O}(\mathrm{WP}_q(1, l))$ -module structure on $\mathcal{L}[n]$ by left multiplication in $C(L_q(l; 1, l))$ is consistent with the $C_c(\mathfrak{F})_0$ -module structure on $C_c(\mathfrak{F})_n$ under the embeddings of $\mathcal{O}(\mathrm{WP}_q(1, l)) \equiv \mathcal{L}[0] \subset C_c(\mathfrak{F})_0$ and $\mathcal{L}[n] \subset C_c(\mathfrak{F})_n$ into $C^*(\mathfrak{F}) = C(L_q(l; 1, l))$.

On the other hand, we have $C_c(\mathfrak{F})_n \subset \overline{\mathcal{L}[n]} \subset C(L_q(l; 1, l)) \equiv C^*(\mathfrak{F})$, using our knowledge of the $|n|$ -th power of c or c^* and that $C_c(\mathfrak{F}_0) \subset \overline{\mathcal{L}[0]}$. So

$$\overline{\mathcal{L}[n]} = \overline{C_c(\mathfrak{F})_n} \subset C^*(\mathfrak{F})$$

for each n .

Let $X_m := \bigsqcup_{s=1}^l \{(p+m, p)_s : p \geq 0\} \subset \mathbb{Z} \times \left(\bigsqcup_{s=1}^l \mathbb{Z}_{\geq} \right)$ for $m \in \mathbb{Z}$, with

$$\ell^2 \left(\mathbb{Z} \times \left(\bigsqcup_{s=1}^l \mathbb{Z}_{\geq} \right) \right) = \bigoplus_{m \in \mathbb{Z}} \ell^2(X_m).$$

Note that for all $m \in \mathbb{Z}$,

$$\tilde{\rho}(c)(\ell^2(X_m)), \tilde{\rho}(d^*)(\ell^2(X_m)) \subset \ell^2(X_{m+1})$$

while

$$\tilde{\rho}(b)(\ell^2(X_m)), \tilde{\rho}(a)(\ell^2(X_m)) \subset \ell^2(X_m).$$

More generally, for all $m \in \mathbb{Z}$,

$$\tilde{\rho}(\overline{\mathcal{L}[n]})(\ell^2(X_m)) = \tilde{\rho}(\overline{C_c(\mathfrak{F})_n})(\ell^2(X_m)) \subset \ell^2(X_{m+n}).$$

Identifying $(p+m, p)_s \in X_m$ with p in the s -th copy of \mathbb{Z}_{\geq} in $\bigsqcup_{s=1}^l \mathbb{Z}_{\geq}$, we get a unitary operator

$$u_m : \ell^2(X_m) \rightarrow \ell^2\left(\bigsqcup_{s=1}^l \mathbb{Z}_{\geq}\right) \cong \oplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq})$$

that intertwines $\tilde{\rho}(b)|_{\ell^2(X_m)}$ and $\tilde{\rho}(a)|_{\ell^2(X_m)}$ with $\pi^{\oplus}(b)$ and $\pi^{\oplus}(a)$ respectively. More generally, the operator $u_m \circ \tilde{\rho}(f) \circ u_{m-n}^{-1} \in \mathcal{B}(\oplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq}))$ for $f \in \overline{C_c(\mathfrak{F})_n} \equiv \overline{C_c(\mathfrak{F}_n)}$ is independent of m , and hence $\overline{\mathcal{L}[n]} = \overline{C_c(\mathfrak{F})_n}$ is embedded isometrically into $\mathcal{B}(\oplus_{s=1}^l \ell^2(\mathbb{Z}_{\geq}))$ by $\rho_{n,m} := u_m \circ \tilde{\rho}(\cdot) \circ u_{m-n}^{-1}$ for any $m \in \mathbb{Z}$. Note that the $\overline{\mathcal{L}[0]}$ -module structure on $\overline{\mathcal{L}[n]}$ is consistent with the $\rho_{0,0}(\overline{\mathcal{L}[0]})$ -module structure on $\rho_{n,0}(\overline{\mathcal{L}[n]})$ under the embedding $\rho_{n,0}$, where

$$\rho_{0,0}(\overline{\mathcal{L}[0]}) = C(\text{WP}_q(1, l)) \cong (\oplus_{s=1}^l \mathcal{K})^+ \equiv (\oplus_{s=1}^l \mathcal{K}(\ell^2(\mathbb{Z}_{\geq})))^+.$$

Furthermore, since $u_m \circ \tilde{\rho}(\chi_{C_n}) \circ u_{m-n}^{-1} = \oplus_{s=1}^l \mathcal{S}^n$ with \mathcal{S} the backward unilateral shift on $\ell^2(\mathbb{Z}_{\geq})$ as defined previously, for the characteristic function $\chi_{C_n} \in C_c(\mathfrak{F}_n)$ of the open and compact set

$$C_n := \{(0, -n, p)_s : n \leq p \in \mathbb{Z}_{\geq}\} \cup \{(0, -n, \infty)\} \subset \mathfrak{F}_n,$$

we have

$$u_m \circ \tilde{\rho}(\overline{\mathcal{L}[n]}) \circ u_{m-n}^{-1} = u_m \circ \tilde{\rho}(\overline{C_c(\mathfrak{F})_n}) \circ u_{m-n}^{-1} = (\oplus_{s=1}^l \mathcal{K}) + \mathbb{C}(\oplus_{s=1}^l \mathcal{S}^n)$$

which is isomorphic, as a left $(\oplus_{s=1}^l \mathcal{K})^+$ -module, to

$$\left((\oplus_{s=1}^l \mathcal{K})^+ \oplus (\oplus_{s=1}^l \mathcal{K})^+ \right) \left(I_1 \oplus (\oplus_{s=1}^l P_n) \right)$$

if $n \geq 0$, and to

$$(\oplus_{s=1}^l \mathcal{K})^+ \left(I - (\oplus_{s=1}^l P_{-n}) \right)$$

if $n < 0$, where we recall that I_1 denotes the identity matrix in $M_1((\oplus_{s=1}^l \mathcal{K})^+)$ while I denotes the identity element of $(\oplus_{s=1}^l \mathcal{K})^+$, and hence $I_1 \oplus (\oplus_{s=1}^l P_n) \in M_2((\oplus_{s=1}^l \mathcal{K})^+)$ while

$$I - (\oplus_{s=1}^l P_{-n}) \in (\oplus_{s=1}^l \mathcal{K})^+ = M_1\left((\oplus_{s=1}^l \mathcal{K})^+\right).$$

As summarized below, we have the modules $\overline{\mathcal{L}[n]}$ identified concretely among the finitely generated projective left modules over $(\mathcal{K}^l)^+$ enumerated earlier in Section 2.

Theorem 2. $\overline{\mathcal{L}[n]}$ is isomorphic to the projective left module over $C(\mathrm{WP}_q(1, l)) \cong (\mathcal{K}^l)^+$ for $q \in (0, 1)$ determined by the projection $I_1 \oplus (\oplus_{j=1}^l P_n) \in M_2((\mathcal{K}^l)^+)$ if $n \geq 0$, and the projection $I - (\oplus_{j=1}^l P_{-n}) \in M_1((\mathcal{K}^l)^+)$ if $n < 0$.

It is interesting to note that this theorem exhibits some kind of an index relation between the “winding number” n of the line bundle $\mathcal{L}[n]$ and the “rank” of its representative projection $I_1 \oplus (\oplus_{j=1}^l P_n)$ or $I - (\oplus_{j=1}^l P_{-n})$.

Finally, we mention the classification of isomorphism classes of finitely generated projective left modules over the quantum 3-sphere by Bach [1] which shows that the projections $1 \otimes P_k$ with $k \geq 0$ and I_r with $r \in \mathbb{N}$ represent all unitarily inequivalent classes of projections in $M_\infty(C(S_q^3))$. In view of this classification, we observe that $C(S_q^3) \otimes_{C(\mathrm{WP}_q(1, l))} \overline{\mathcal{L}[n]}$ for all $n \in \mathbb{Z}$ is the same rank-1 free module over $C(S_q^3)$, showing that these non-isomorphic quantum line bundles $\overline{\mathcal{L}[n]}$ over $\mathrm{WP}_q(1, l)$ pull back to the same quantum line bundles over S_q^3 via the quotient map $S_q^3 \rightarrow \mathrm{WP}_q(1, l)$. This phenomenon resembles the well-known classical result that the pull-back, to the total space P , of a vector bundle $P \times_G V \rightarrow X$ associated with a principal G -bundle $P \rightarrow X$ for some G -vector space V is always trivial. In fact, this classical theorem has a general quantum counterpart [10].

Acknowledgements

The author would like to acknowledge the hospitality and support by the National Center for Theoretical Sciences in Taipei during his 2013 summer visit, and the support by Jonathan Rosenberg’s conference grant, NSF-1266158, to attend the Conference on Noncommutative Geometry and Quantum Groups at the Fields Institute in Toronto, 2013–2014.

References

- [1] Bach K.A., A cancellation problem for quantum spheres, Ph.D. Thesis, University of Kansas, Lawrence, 2003.
- [2] Blackadar B., K -theory for operator algebras, *Mathematical Sciences Research Institute Publications*, Vol. 5, 2nd ed., Cambridge University Press, Cambridge, 1998.
- [3] Brzeziński T., Fairfax S.A., Quantum teardrops, *Comm. Math. Phys.* **316** (2012), 151–170, [arXiv:1107.1417](#).
- [4] Brzeziński T., Hajac P.M., The Chern–Galois character, *C. R. Math. Acad. Sci. Paris* **338** (2004), 113–116, [math.KT/0306436](#).
- [5] Brzeziński T., Majid S., Quantum group gauge theory on quantum spaces, *Comm. Math. Phys.* **157** (1993), 591–638, [hep-th/9208007](#).
- [6] Connes A., *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.
- [7] Curto R.E., Muhly P.S., C^* -algebras of multiplication operators on Bergman spaces, *J. Funct. Anal.* **64** (1985), 315–329.
- [8] Gootman E.C., Rosenberg J., The structure of crossed product C^* -algebras: a proof of the generalized Effros–Hahn conjecture, *Invent. Math.* **52** (1979), 283–298.
- [9] Hajac P.M., Strong connections on quantum principal bundles, *Comm. Math. Phys.* **182** (1996), 579–617, [hep-th/9406129](#).
- [10] Hajac P.M., Private communication.
- [11] Hong J.H., Szymański W., Quantum lens spaces and graph algebras, *Pacific J. Math.* **211** (2003), 249–263.
- [12] Husemoller D., *Fibre bundles*, McGraw-Hill Book Co., New York – London – Sydney, 1966.
- [13] Kumjian A., Pask D., Raeburn I., Renault J., Graphs, groupoids, and Cuntz–Krieger algebras, *J. Funct. Anal.* **144** (1997), 505–541.
- [14] Muhly P.S., Renault J.N., C^* -algebras of multivariable Wiener–Hopf operators, *Trans. Amer. Math. Soc.* **274** (1982), 1–44.

-
- [15] Paterson A.L.T., Graph inverse semigroups, groupoids and their C^* -algebras, *J. Operator Theory* **48** (2002), 645–662, [math.OA/0304355](#).
 - [16] Peterka M.A., Finitely-generated projective modules over the θ -deformed 4-sphere, *Comm. Math. Phys.* **321** (2013), 577–603, [arXiv:1203.6441](#).
 - [17] Renault J., A groupoid approach to C^* -algebras, *Lecture Notes in Mathematics*, Vol. 793, Springer, Berlin, 1980.
 - [18] Rieffel M.A., Dimension and stable rank in the K -theory of C^* -algebras, *Proc. London Math. Soc.* **46** (1983), 301–333.
 - [19] Rieffel M.A., The cancellation theorem for projective modules over irrational rotation C^* -algebras, *Proc. London Math. Soc.* **47** (1983), 285–302.
 - [20] Rieffel M.A., Projective modules over higher-dimensional noncommutative tori, *Canad. J. Math.* **40** (1988), 257–338.
 - [21] Salinas N., Sheu A.J.L., Upmeyer H., Toeplitz operators on pseudoconvex domains and foliation C^* -algebras, *Ann. of Math.* **130** (1989), 531–565.
 - [22] Sheu A.J.L., A cancellation theorem for modules over the group C^* -algebras of certain nilpotent Lie groups, *Canad. J. Math.* **39** (1987), 365–427.
 - [23] Sheu A.J.L., Compact quantum groups and groupoid C^* -algebras, *J. Funct. Anal.* **144** (1997), 371–393.
 - [24] Sheu A.J.L., The structure of quantum spheres, *Proc. Amer. Math. Soc.* **129** (2001), 3307–3311.
 - [25] Swan R.G., Vector bundles and projective modules, *Trans. Amer. Math. Soc.* **105** (1962), 264–277.
 - [26] Vaksman L.L., Soibelman Ya.S., Algebra of functions on the quantum group $SU(n+1)$, and odd-dimensional quantum spheres, *Leningrad Math. J.* **2** (1991), 1023–1042.
 - [27] Woronowicz S.L., Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.
 - [28] Woronowicz S.L., Compact quantum groups, in *Symétries Quantiques (Les Houches, 1995)*, North-Holland, Amsterdam, 1998, 845–884.