On the Role of the Normalization Factors κ_n and of the Pseudo-Metric $\mathcal{P} \neq \mathcal{P}^\dagger$ in Crypto-Hermitian Quantum Models[⋆]

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Abstract. Among $\mathcal{P}$-pseudo-Hermitian Hamiltonians $H = \mathcal{P}^{-1} H^\dagger \mathcal{P}$ with real spectra, the "weakly pseudo-Hermitian" ones (i.e., those employing non-self-adjoint $\mathcal{P} \neq \mathcal{P}^\dagger$) form a remarkable subfamily. We list some reasons why it deserves a special attention. In particular we show that whenever $\mathcal{P} \neq \mathcal{P}^\dagger$, the current involutive operator of charge $\mathcal{C}$ gets complemented by a nonequivalent alternative involutive quasiparity operator $\mathcal{Q}$. We show how, in this language, the standard quantum mechanics is restored via the two alternative inner products in the physical Hilbert space of states, with $\langle \psi_1 | \mathcal{P}\mathcal{Q} | \psi_2 \rangle = \langle \psi_1 | \mathcal{C}\mathcal{P} | \psi_2 \rangle$.

Key words: $\mathcal{P}\mathcal{T}$-symmetry; non-self-adjoint pseudo-metrics; $\mathcal{P}\mathcal{Q}$-crypto-Hermiticity; $\mathcal{C}\mathcal{P}$-crypto-Hermiticity

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1 Introduction

One of the most important keys to the solvability of Schrödinger equations

$$H | \psi \rangle = E | \psi \rangle$$

is often found in the existence of a symmetry S of the Hamiltonian H, i.e., in the commutativity

$$HS - SH = 0. \tag{1}$$

During the development of Quantum Mechanics, the concept of symmetry found various generalizations. For illustration, one could recollect the multiple applications of Lie algebras (where H appears as just one of their generators) or supersymmetries (where one employs both the commutators and anticommutators).

Recently [1], the family of the productive symmetry-related mathematical tools has been enriched by the so called $\mathcal{P}\mathcal{T}$-symmetry where the vanishing commutator (1) contains an *antilinear* operator $S = \mathcal{P}\mathcal{T}$ [2, 3, 4]. In the context of field theory, typically, $\mathcal{P}$ is chosen as parity while the antilinear Hermitian-conjugation factor $\mathcal{T}$ mimics time reversal [1, 5, 6]. More rigorously [2], one replaces equation (1) by the requirement

$$H^\dagger = \mathcal{P}\, H\, \mathcal{P}^{-1}. \tag{2}$$

In an illustrative two-by-two matrix model with

$$H^{(2)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}, \qquad \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

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we get the three constraints $a = a^*$, $d = d^*$, $b = -c^*$. Our $\mathcal{PT}$-symmetric toy Hamiltonian $H^{(2)}$ has four free real parameters (same number as if it were Hermitian) and its four energies E remain real (i.e., in principle, observable) in a specific "physical" subdomain $\mathcal{D}$ of its matrix elements where $(a-d)^2 \geq 4bb^*$. As long as there would be no such a constraint in Hermitian case, new interesting physical as well as mathematical phenomena can be expected to occur along the "exceptional-point" [7] boundary $\partial\mathcal{D}$ where $2|b| = |a-d|$.

Inside $\mathcal{D}$, in the light of the review paper [8], the model $H^{(2)}$ should be called "quasi-Hermitian" since, by construction, all its spectrum is real. This means that our matrix $H^{(2)}$ becomes Hermitian in the (two-dimensional) vector space where the scalar product between elements $|\,a\rangle$ and $|\,b\rangle$ is defined by the overlap $\langle a\,|\,\Theta\,|\,b\rangle$ where $\Theta = \Theta^\dagger > 0$ is a suitable matrix solution of the quasi-Hermiticity condition of [8],

$$H^\dagger = \Theta\, H\, \Theta^{-1}. \tag{4}$$

Mutatis mutandis, all these considerations can be easily transferred to an arbitrary infinite-dimensional Hilbert space $\mathcal{H}$ where the Hamiltonians H have to be assigned a *positive-definite* operator $\Theta = \Theta^\dagger$ exhibiting all the necessary mathematical properties of the metric in $\mathcal{H} = \mathcal{H}^{(\Theta)}$ [8]. Thus, the usual *single* standard Hermiticity condition $H = H^\dagger$ is replaced by the *pair* of the generalized symmetry rules (2) and (4). Also the concept and construction of observables becomes perceivably modified. This definitely opens new horizons in quantum phenomenology [6].

In the related literature (we recommend its long list collected in [6]), it is not always sufficiently emphasized that the proper physical meaning of equations (2) and (4) is in fact perceivably different. Indeed, the latter, quasi-Hermiticity condition (4) is "strong" (it guarantees that Es are real) and "difficult"[1]. In contrast, the former condition (2) (called, usually, $\mathcal{P}$-pseudo-Hermiticity [2]) is just auxiliary (in fact, we need it just for certain technical purposes – see below) and "almost redundant"[2].

From a historical point of view it is a paradox that in spite of the knowledge of the aspects and merits of equation (4) (cf. [8] with examples from nuclear physics), it was just the "naive" parity-pseudo-Hermiticity property (2) of certain models which proved much more inspiring. Anyhow, several aspects of its formal appeal (thoroughly listed in [6]) attracted attention to the whole new class of the models which were often neglected in the past because they happened to be non-Hermitian with respect to the "Dirac's" very special metric $\Theta^{(\mathrm{Dirac})} = I$.

2 Pseudo-Hermitian Hamiltonians

A broad menu of the new, quasi-Hermitian[3] models has been studied in the literature after their pioneering sample has been offered by Bender and Boettcher in 1998 [1, 5]. Among them, paradoxically, some of the most important ones were *not* using the parity operator (3) but rather its two-by-two basis-permutation alternative

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

The related modification of equation (2) is encountered not only in the well known Feshbach's and Villar's version of the Klein–Gordon equation describing relativistic spinless bosons [11, 12]

[1] Technically, the construction of Θ is almost never easy; in the case of our present two-by-two example, [9] could be consulted for an explicit illustration of the form of Θ etc.

[2] It is, in fact, neither necessary nor sufficient for the reality of the energies; sometimes, the concrete choice of $\mathcal{P}$ is even left unspecified [2].

[3] Or, in a terminology coined recently by Andrei Smilga [10], "crypto-Hermitian".

but also in certain equations employed in quantum cosmology [13], in non-Hermitian but $\mathcal{PT}$-symmetric coupled-channel problems [14] and, unexpectedly, even in classical magnetohydrodynamics [15] and electrodynamics [16]. For us, the unexpected and surprisingly widespread applicability of models based on the basis-permutation matrix structure (5) of $\mathcal{P}$ in (2) provided a strong support of our continuing interest in the more complicated non-parity generalizations of the Hermitian pseudometrics[4] $\mathcal{P} = \mathcal{P}^\dagger$ [17].

In the next step of our study of the models with $H \neq H^\dagger$ we were led to an active interest in the weakly pseudo-Hermiticitian cases (introduced by Solombrino [3]) where *non-Hermitian* pseudoparities $\mathcal{P} \neq \mathcal{P}^\dagger$ are admitted. In particular, we contemplated the "first nontrivial" three-dimensional basis-permutations $\mathcal{P}$ of the non-Hermitian form in [18]. To our greatest surprise we revealed that the family of the expectedly more flexible three-dimensional descendants of the above two-dimensional model (3), viz.,

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \left(\mathcal{P}^\dagger\right)^{-1} = \mathcal{P}^{-2}, \qquad H^{(3)} = \begin{pmatrix} a & b & b^* \\ b^* & a & b \\ b & b^* & a \end{pmatrix} \tag{6}$$

is in fact much more constrained as it contains just three free real parameters again, with $a, b \in \mathbb{C}$ but $a = a^*$. For this reason we proposed to rename its "weak" pseudo-Hermiticity feature into a "strengthened $\mathcal{PT}$-symmetry".

Recently, Ali Mostafazadeh re-focused attention on our model (6) in [19], emphasizing that the weak pseudo-Hermiticity and pseudo-Hermiticity specify the same class of operators in finite dimensions (cf. also [20]). As an illustrative example he recalls our equation (6) and argues that our model $H^{(3)}$ proves *also* pseudo-Hermitian with respect to the *Hermitian*

$$\mathcal{P}^{(+)} = \mathcal{P} + \mathcal{P}^\dagger. \tag{7}$$

A similar remark could have been also deduced from the older comment [21] by Bagchi and Quesne who, apparently, did not notice that the trick might lead to a *singular* and, hence, unacceptable $\mathcal{P}^{(+)}$ in general. In this sense one should appreciate that A. Mostafazadeh [19] found an elegant way out of the trap by the mapping of a given $\mathcal{P} \neq \mathcal{P}^\dagger$ on the whole one-parametric set of its eligible Hermitian partners

$$\mathcal{P}^{(AM)}(\theta) = \mathrm{i}\left[\mathcal{P}\exp(\mathrm{i}\theta) - \mathcal{P}^\dagger \exp(-\mathrm{i}\theta)\right] \tag{8}$$

(given by equation (19) of [19]). In what follows we intend to add several further remarks on the specific character and merits of all the models H which are characterized by such an unexpectedly large freedom in the choice between the alternative "pseudo-metrics" given by equation (8).

First of all, we would like to point out that from the purely pragmatic point of view there is an obvious difference between the strongly constrained *three-parametric* $H^{(3)}$-toy-model subfamily (6) and the *much broader*[5] class of the generic $\mathcal{P}^{(+)}$-pseudo-Hermitian three-dimensional Hamiltonians. In this sense, equation (8) only enters the scene as a natural complement and extension of equation (7) and as a very useful tool of a subsumption of some families of the Hamiltonians (in this sense one only has to get accustomed to the fact of life that inside the family of the N-dimensional pseudo-Hermitian Hs there exists just a very small "weakly"-pseudo-Hermitian subfamily).

Secondly, let us add that the flexible recipe (8) would find its applicability in the general N-dimensional context of our systematic coupled-channel study [22] where the Hermitian partner

[4] You could also call it parity, in broader sense.

[5] In fact, seven-parametric.

of the $N = 4$ pseudo-metric

$$\mathcal{P} = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \neq \mathcal{P}^\dagger = \mathcal{P}^{-1}$$

would remain non-invertible whenever represented by the older formula (7), i.e., whenever θ in (8) were chosen as an integer multiple of $\pi/2$.

Thirdly, let us emphasize that in physics, the *only essential* feature of the Hamiltonians $H \neq H^\dagger$ is in fact represented by their quasi-Hermiticity property (4). It is clear that the pseudo-Hermiticity itself is much less relevant because once we get through the *difficult* proof of the necessary reality of the spectrum [5], the pseudo-Hermiticity of a given H becomes in fact *fully equivalent* to its quasi-Hermiticity [2, 4]. In this context we are sure that a more explicit evaluation of some additional practical differences between the more or less purely technical assumptions $\mathcal{P} = \mathcal{P}^\dagger$ and $\mathcal{P} \neq \mathcal{P}^\dagger$ could enhance our understanding of the specific merits of certain specific choices of the non-Hermitian models H with real spectra.

3 Metrics Θ

We are now going to propose a possible comparison between the pseudo-Hermiticity (sampled by equation (2) where $\mathcal{P} = \mathcal{P}^\dagger$) and the weak pseudo-Hermiticity (sampled by equation (2) where $\mathcal{P} \neq \mathcal{P}^\dagger$). Our main idea is twofold. Firstly, we recollect that the simpler the $\mathcal{P}$, the simpler are the explicit formulae for the basis (cf. Subsection 3.1 below). Secondly, in Subsections 4.1 and 4.2 we shall draw some consequences from the fact that in the majority of applications of non-Hermitian Hamiltonians, the most important role played by $\mathcal{P}$ is its occurrence in the factorized metric Θ [6, 23].

For *any* given observable $\mathcal{O}$, the knowledge of the metric is essential for the practical evaluation of its (real) expectation values

$$\langle\, \psi \,|\Theta\, \mathcal{O} \,|\, \psi\rangle.$$

The quantum system can be prepared in a complicated state $|\psi\rangle \in \mathcal{H}^{(\Theta)}$ so that the factorization $\Theta = \mathcal{C}\mathcal{P}$ can be of a key technical significance. It is equally important that this factorization enables us to formulate an important additional postulate $\mathcal{C}^2 = I$ which is often deeply rooted in certain hypothetical physics considerations [6]. Even on a purely formal level, the latter postulate represents one of the most widely accepted ways of getting rid of the well known and highly unpleasant ambiguity [8, 9, 23] of the general solutions Θ of the quasi-Hermiticity constraint (4).

Once we turn our attention to the models where $\mathcal{P} \neq \mathcal{P}^\dagger$, their *different* nature becomes obvious once we interpret them as resulting from an application of a symmetry of the generic form (1). We arrive at the first specific feature of the weak pseudo-Hermiticity which, strictly speaking, replaces equation (1) (containing a *single* antilinear operator $S = \mathcal{P}\mathcal{T}$) by the *triplet of parallel requirements*

$$H^\dagger = \mathcal{P} H \mathcal{P}^{-1}, \qquad H^\dagger = \mathcal{P}^\dagger H \left[\mathcal{P}^{-1}\right]^\dagger, \qquad [H, S] = 0, \qquad S = \mathcal{P}^{-1}\mathcal{P}^\dagger.$$

Although just two of them are independent of course, we already illustrated how they impose *much more stringent* constraints upon H.

3.1 The family of biorthogonal bases

In order to proceed to the technical core of our present message let us first stay in the "usual", auxiliary and non-physical Hilbert space $\mathcal{H}^{(I)}$ and treat a given $H \neq H^\dagger$ with real spectrum $\{E_n\}$ as "non-Hermitian". Using a slightly modified Dirac's notation we may find the respective left and right eigenvectors $|\,E_n\rangle^{(1)}$ and ${}^{(1)}\langle\!\langle E\,|$ of our H from the corresponding doublet of Schrödinger equations,

$$H\,|\,E_n\rangle^{(1)} = E_n\,|\,E_n\rangle^{(1)}, \tag{9}$$

$${}^{(1)}\langle\!\langle E_m\,|\,H = E_m\,{}^{(1)}\langle\!\langle E_m\,|. \tag{10}$$

The reason for our introduction of a superscript ${}^{(1)}$ lies in the fact that even if we impose the standard biorthonormality conditions

$${}^{(1)}\langle\!\langle E_m\,|\,E_n\rangle^{(1)} = \delta_{mn}$$

accompanied by the standard completeness formula in $\mathcal{H}^{(I)}$,

$$\sum_{n=0}^{\infty}|\,E_n\rangle^{(1)}\,{}^{(1)}\langle\!\langle E_n\,| = I$$

we can still redefine our eigenvectors by the formula

$$\begin{aligned} {}^{(\vec{\kappa})}\langle\!\langle E_m\,| &= {}^{(1)}\langle\!\langle E_m\,|\cdot\frac{1}{\kappa_n}\,,\\ |\,E_n\rangle^{(\vec{\kappa})} &= |\,E_n\rangle^{(1)}\cdot\kappa_n \end{aligned} \tag{11}$$

with arbitrary complex $\kappa_0, \kappa_1, \kappa_3, \ldots$ forming an infinite-dimensional vector $\vec{\kappa}$.

4 The role of the set of the normalization factors

We saw that once we change any the normalization constant κ_n we arrive at another, "renormalized" biorthonormal set exhibiting the *same* eigenenergy, orthonormality and completeness properties. Obviously, the freedom of this type would vanish completely whenever one returns to the Hermitian Hamiltonian operators H. In an opposite direction, the specific relevance of the variability of the normalization factors κ_n becomes more important in the scenarios where $\mathcal{P} \neq \mathcal{P}^\dagger$.

In the characteristic latter case one assumes that $\mathcal{P}$ remains extremely elementary. For this reason, even the transition to the Hermitian pseudometric (8) could make some formulae much less transparent. In what follows, we intend to describe a particularly interesting application of such an idea to the specific, very popular models where one constructs the metric Θ in a factorized form.

4.1 The operators $\mathcal{Q}$ of quasiparity

In the generic non-degenerate case with $H^\dagger = \mathcal{P}H\mathcal{P}^{-1}$ and with the non-Hermitian $\mathcal{P} \neq \mathcal{P}^\dagger$, the ${}^{(\vec{\kappa})}$-superscripted versions of equations (9) and (Hermitian conjugate) (10),

$$\begin{aligned} H\,|\,E_n\rangle^{(\vec{\kappa})} &= E_n\,|\,E_n\rangle^{(\vec{\kappa})},\\ H^\dagger\,|\,E_n\rangle\!\rangle^{(\vec{\kappa})} &= E_n\,|\,E_n\rangle\!\rangle^{(\vec{\kappa})} \end{aligned}$$

imply the proportionality of alternative solutions at the same energy, say,

$$|\,E_n\rangle\!\rangle^{(\vec{\kappa})} = \mathcal{P}\,|\,E_n\rangle^{(\vec{\kappa})}\cdot q_n^{(\vec{\kappa})}. \qquad (12)$$

As long as we normalized our basis at all $\vec{\kappa}$, we have

$$1 = {}^{(\vec{\kappa})}\langle E_n\,|\,\mathcal{P}\,|\,E_n\rangle^{(\vec{\kappa})}\cdot q_n^{(\vec{\kappa})}$$

so that, in the light of equation (11), we have $1 = {}^{(1)}\langle E_n\,|\,\mathcal{P}\,|\,E_n\rangle^{(1)}\cdot q_n^{(\vec{\kappa})}\cdot \kappa_n^*\kappa_n$. This leads to the renormalization-dependence formula

$$q_n^{(\vec{\kappa})} = \frac{1}{\kappa_n^*\kappa_n}\,q_n^{(1)}, \qquad q_n^{(1)} = \frac{1}{{}^{(1)}\langle E_n\,|\,\mathcal{P}\,|\,E_n\rangle^{(1)}}.$$

Now we may follow our old preprint [23] and define the *family* of the operators of quasiparity $\mathcal{Q} = \mathcal{Q}^{(\vec{\kappa})}$ by the relation

$$|\,E_n\rangle^{(\vec{\kappa})}\cdot q_n^{(\vec{\kappa})} = \mathcal{Q}^{(\vec{\kappa})}\,|\,E_n\rangle^{(\vec{\kappa})} \qquad (13)$$

inspired by equation (12) and leading to the spectral formula with a simple manifest dependence on normalization,

$$\mathcal{Q}^{(\vec{\kappa})} = \sum_{n=0}^{\infty} |\,E_n\rangle^{(\vec{\kappa})}\,q_n^{(\vec{\kappa})}\;{}^{(\vec{\kappa})}\langle\!\langle E_n\,| = \sum_{n=0}^{\infty} |\,E_n\rangle^{(1)}\,\frac{q_n^{(1)}}{\kappa_n^*\kappa_n}\;{}^{(1)}\langle\!\langle E_n\,|.$$

We may conclude that equations (12) and (13) lead to the correct normalization recipe in the form

$${}^{(\vec{\kappa})}\langle\, E_n\,|\,E_n\rangle\!\rangle^{(\vec{\kappa})} = {}^{(\vec{\kappa})}\langle\, E_n\,|\,\Theta\,|\,E_n\rangle^{(\vec{\kappa})} = {}^{(\vec{\kappa})}\langle\!\langle\, E_n\,|\,E_n\rangle^{(\vec{\kappa})} = 1$$

and, *ipso facto*, to the whole family

$$\begin{aligned}\Theta^{(\vec{\kappa})} = \mathcal{P}\mathcal{Q}^{(\vec{\kappa})} &= \sum_{n=0}^{\infty} \mathcal{P}\,|\,E_n\rangle^{(1)}\,\frac{q_n^{(1)}}{\kappa_n^*\kappa_n}\;{}^{(1)}\langle\!\langle E_n\,| \\ &= \sum_{n=0}^{\infty} |\,E_n\rangle\!\rangle^{(1)}\,\frac{1}{\kappa_n^*\kappa_n}\;{}^{(1)}\langle\!\langle E_n\,| = \sum_{n=0}^{\infty} |\,E_n\rangle\!\rangle^{((\vec{\kappa}))}\;\cdot\;{}^{((\vec{\kappa}))}\langle\!\langle E_n\,|\end{aligned} \qquad (14)$$

of the manifestly renormalization-dependent and factorized, self-adjoint, invertible and positive definite metric operators Θ.

4.2 The operators $\mathcal{C}$ of charge

In a close parallel to our preceding considerations we could have also started from the Hermitian conjugate form of equation (9) accompanied by the original equation (10),

$$\begin{aligned}&{}^{(\vec{\kappa})}\langle E_n\,|\,\mathcal{P}\,H = E_n\;{}^{(\vec{\kappa})}\langle E_n\,|\,\mathcal{P}, \\ &{}^{(\vec{\kappa})}\langle\!\langle E_m\,|\,H = E_m\;{}^{(\vec{\kappa})}\langle\!\langle E_m\,|.\end{aligned}$$

This would *change* the form of our proportionality rule (12), into

$$|\,E_n\rangle\!\rangle^{(\vec{\kappa})} = \mathcal{P}^{\dagger}\,|\,E_n\rangle^{(\vec{\kappa})}\cdot c_n^{(\vec{\kappa})}$$

with an immediate consequence

$$1 = {}^{(\vec{\kappa})}\langle E_n \,|\, \mathcal{P}^\dagger \,|\, E_n\rangle^{(\vec{\kappa})} \cdot c_n^{(\vec{\kappa})} = {}^{(1)}\langle E_n \,|\, \mathcal{P}^\dagger \,|\, E_n\rangle^{(1)} \cdot c_n^{(\vec{\kappa})} \cdot \kappa_n^* \kappa_n$$

i.e.,

$$c_n^{(\vec{\kappa})} = \frac{1}{\kappa_n^* \kappa_n}\, c_n^{(1)}, \qquad c_n^{(1)} = \frac{1}{{}^{(1)}\langle E_n \,|\, \mathcal{P}^\dagger \,|\, E_n\rangle^{(1)}} = \left(q_n^{(1)}\right)^*.$$

Now we may introduce the standard charge operator $\mathcal{C} = \mathcal{C}^{(\vec{\kappa})}$ by setting

$$|\, E_n\rangle^{(\vec{\kappa})} \cdot c_n^{(\vec{\kappa})} = \left[\mathcal{C}^{(\vec{\kappa})}\right]^\dagger |\, E_n\rangle^{(\vec{\kappa})}$$

and

$$\left[\mathcal{C}^{(\vec{\kappa})}\right]^\dagger = \sum_{n=0}^{\infty} |\, E_n\rangle^{(\vec{\kappa})}\, c_n^{(\vec{\kappa})}\, {}^{(\vec{\kappa})}\langle\!\langle E_n \,| = \sum_{n=0}^{\infty} |\, E_n\rangle^{(1)}\, \frac{c_n^{(1)}}{\kappa_n^* \kappa_n}\, {}^{(1)}\langle\!\langle E_n \,|$$

i.e.,

$$\mathcal{C}^{(\vec{\kappa})} = \sum_{n=0}^{\infty} |\, E_n\rangle\!\rangle^{(1)}\, \frac{q_n^{(1)}}{\kappa_n^* \kappa_n}\, {}^{(1)}\langle E_n \,|.$$

This leads to a *non-equivalent* factorization

$$\Theta^{(\vec{\kappa})} = \mathcal{C}^{(\vec{\kappa})}\mathcal{P} = \sum_{n=0}^{\infty} |\, E_n\rangle\!\rangle^{(1)}\, \frac{q_n^{(1)}}{\kappa_n^* \kappa_n}\, {}^{(1)}\langle E_n \,|\, \mathcal{P}$$
$$= \sum_{n=0}^{\infty} |\, E_n\rangle\!\rangle^{(1)}\, \frac{1}{\kappa_n^* \kappa_n}\, {}^{(1)}\langle\!\langle E_n \,| = \sum_{n=0}^{\infty} |\, E_n\rangle\!\rangle^{((\vec{\kappa}))}\, \cdot\, {}^{((\vec{\kappa}))}\langle\!\langle E_n \,|$$

to be compared with formula (14).

5 Summary

Carl Bender [6] lists several reasons why the usual Hermiticity of the quantum Hamiltonians H (i.e., their property $H = H^\dagger$ where the superscript symbolizes the matrix transposition plus complex conjugation) should be replaced by the better motivated rule (2). Although the latter relation offers just a typical sample of a $\mathcal{P}$-pseudo-Hermiticity of H, it is often called, in the context of some older work in this direction [24], "$\mathcal{PT}$-symmetry" of H.

In this context, Mostafazadeh [2] noticed that on a purely formal level, the symbol $\mathcal{P}$ need not coincide with parity at all. He suggested and promoted its "pseudometric" reinterpretation preserving the Hermiticity $\mathcal{P} = \mathcal{P}^\dagger$ but relaxing the involutivity, $\mathcal{P} \neq \mathcal{P}^{-1}$. The first step towards generalizations has been made.

Originally [1] it has been believed that the $\mathcal{PT}$-symmetry of H could possess a deeper physical significance, especially when the operators $\mathcal{P}$ and $\mathcal{T}$ were chosen as representing the physical parity and the time reversal, respectively. Later on, it became clear that this property must be *constructively* complemented by another, independent and much more relevant antilinear symmetry (4) called, mostly, $\mathcal{CPT}$-symmetry of H (where $\mathcal{C}$ is called "charge"). In the light of [2] and [8], just an expectable return to the safe waters of standard quantum mechanics has been accomplished.

In the next step of development, Solombrino [3] and others [18, 21] admitted all $\mathcal{P} \neq \mathcal{P}^\dagger$ which remain invertible. In a way complementing, and inspired by, the related recent remark by Ali Mostafazadeh [19] we have shown here that after transition to non-Hermitian auxiliary operators $\mathcal{P} \neq \mathcal{P}^\dagger$ the concept of charge (defined as a pre-factor in the metric $\Theta = \mathcal{C}\mathcal{P}$) becomes ambiguous (in the sense that we could also have $\Theta = \mathcal{C}'\mathcal{P}^\dagger$ in principle). We clarified this "puzzle" by showing that there really exists *another* auxiliary family of operators $\mathcal{Q}$ such that $\Theta = \mathcal{P}\mathcal{Q} = \mathcal{Q}^\dagger\mathcal{P}^\dagger$. We may note that the unavoidable Hermiticity of the metric Θ implies that the *weak* form of pseudo-Hermiticity leads to the *richer* menu of the alternative forms of the factorization of the metric,

$$\Theta^{(\vec{\kappa})} = \mathcal{P}\mathcal{Q}^{(\vec{\kappa})} = \mathcal{C}^{(\vec{\kappa})}\,\mathcal{P} = \left[\Theta^{(\vec{\kappa})}\right]^\dagger = \left[\mathcal{Q}^{(\vec{\kappa})}\right]^\dagger \mathcal{P}^\dagger = \mathcal{P}^\dagger \left[\mathcal{C}^{(\vec{\kappa})}\right]^\dagger.$$

In the other words, whenever we relax the "usual" constraint $\mathcal{P} = \mathcal{P}^\dagger$, there emerges a certain complementarity between the concepts of the charge $\mathcal{C}$ and quasiparity $\mathcal{Q}$.

It is possible to summarize that on the present level of understanding of the use of $H \neq H^\dagger$ in quantum mechanics, people are aware that in the most relevant cases (when the spectrum $\{E_n\}$ of our H is all real, discrete and, for the sake of brevity of formulae, non-degenerate), the role of $\mathcal{P}$ remains purely auxiliary. Still, a distinct boundary between the "feasible" and "not feasible" applications seems to coincide, more or less, precisely with the boundary between the "sufficiently simple" and "not sufficiently simple" operators $\mathcal{P}$ in (2). For this reason we tried here to draw a few consequences from the use of some "extremely simple" $\mathcal{P} \neq \mathcal{P}^\dagger$. We demonstrated that in both the constructions of the bases and in the factorizations of the metric Θ the use of the non-Hermitian $\mathcal{P}$ could have its merits. Last but not least we also proved that the "natural" requirement $\mathcal{Q}^2 = I$ of the involutivity of the quasiparity (reflecting its usual role in some applications [23, 25]) is fully equivalent to the more standard recipe $\mathcal{C}^2 = I$ which proved, in many models [6], so useful for an efficient suppression of the well known [8] enormous ambiguity of the metric Θ.

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