PERIODIC GROUPS SATURATED BY THE GROUP $U_3(9)$

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Abstract. Let $\mathfrak{M}$ be a set of finite groups. A group $G$ is said to be saturated by $\mathfrak{M}$, if every finite subgroup of $G$ is contained in a subgroup isomorphic to a group from $\mathfrak{M}$. We prove that a periodic group saturated by set consisting of the single finite simple group $U_3(9) = PSU_3(81)$ is isomorphic to $U_3(9)$.

Let $\mathfrak{M}$ be a set of finite groups. A group $G$ is said to be saturated by $\mathfrak{M}$, if every finite subgroup of $G$ is contained in a subgroup isomorphic to a group from $\mathfrak{M}$.

The paper [1] contains a hypothesis that a periodic group saturated by a finite set $\mathfrak{M}$ of finite non-abelian simple groups is finite and also confirms this hypothesis for the case when centralizers of Sylow 2-subgroups of groups from $\mathfrak{M}$ do not contain elements of odd order larger than three.

In this connection, it is interesting to investigate groups saturated by one simple group (precisely, by one-element set containing a finite simple group) in which centralizer of Sylow 2-subgroup contains an element of odd order larger than three. All such groups are listed in [2].

A simple group of the least order in which centralizer of Sylow 2-subgroup contains an element of odd order larger than three is $U_3(9) \simeq SU_3(81)$, and the goal of present article is to prove the following result.

**Theorem.** Periodic group $G$ saturated by group $U_3(9)$ is isomorphic to $U_3(9)$.

The proof uses the following well-known properties of $U_3(9)$ (see for example [3] and [4]).

**Proposition 1.** Let $U \simeq U_3(9)$.

1. Sylow 2-subgroup $T$ in $U$ is semi-diedral group of order 32, i.e. $T = \langle a, b | a^{16} = b^2 = 1, a^b = a^7 \rangle$. 

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2. Every involution of \( U \) is a conjugate of \( b \).

3. The centralizer \( C \) of \( b \) in \( U \) is the direct product of a group of order 5 and a group \( C_0 \) isomorphic to extension of \( C_1 \cong SL_2(9) \) by a group of order 2. All elements of order 4 from \( C \) are contained in \( C_1 \), their squares coincide with an unique involution \( b \) in \( C_1 \). The order of any maximal cyclic subgroup of \( C_1 \) equals 6, 8 or 10.

4. If \( R \) is a Sylow 3-subgroup of \( U \) then \( R \) is a group of exponent 3, which center \( Z \) is an elementary abelian group of order 9, and the factor group by center is an elementary abelian group of order 81, \( N_U(R) \) is a maximal subgroup in \( U \) which is an extension of \( R \) by cyclic group \( D \) of order 80. Here \( |C_D(Z)| = 10 \) and \( D/C_D(Z) \) acts transitively upon conjugation in \( N_U(R) \) on a set of non-trivial elements of \( Z \) and \( D \) acts regularly on \( R/Z \). In particular, \( r^d Z = r^{-1} Z \) for any element \( r \in R \) and any involution \( d \in N_U(R) \). Every 3-local subgroup of \( U \) is contained in a subgroup isomorphic to \( N_U(R) \).

In addition we shall need the following results ([5]).

**Proposition 2.** Periodic group containing an involution with finite centralizer is locally finite.

The proof will also use the following facts which can be easily checked with the help of coset enumeration algorithm (see, for example, [6]).

**Proposition 3.** 1. The order of the group \( \langle x, y, z | x^3 = y^2 = z^2 = (yz)^3 = (yzx)^3 \rangle = 54 \).

2. Suppose \( H = \langle x, y, z | x^3 = y^2 = z^2 = (xyz)^4 = yz = (yzx)^4 = ((x)(yzx))^3 \rangle \), where \( p, q, r \in 3, 4, 5 \) and \( r \neq 4 \). Then \( |H| \leq 24 \). Moreover \( |H| = 24 \) if \( (p, q) = (4, 4) \), and \( |H| = 1 \) in other cases.

3. The order of the group \( \langle x, y, z | x^3 = y^2 = z^2 = (xyz)^4 = (yzx)^4 = (xyz)^4 = (xyz)^4 = (x)(yzx)^4 = 1 \rangle = 14400 \).

**Proof of Theorem.** Obviously the theorem is true for finite group \( G \). Suppose \( G \) is infinite. As shown in [1], every locally finite subgroup of group \( G \) is finite.

By assumption \( G \) contains a subgroup \( U \cong U_3(9) \) and we shall later on use notations from the statement of Proposition 1.

**Lemma 1.** \( N_G(Z) = N_{G(R)} = N_U(R) \).

**Proof.** By Proposition 1.4, \( C_G(Z) \) contains an involution \( b \). If \( c \) is an involution in \( C_G(Z) \) then \( \langle b, c, Z \rangle \) is a finite group contained in a subgroup isomorphic to \( U \). By Proposition 1.4, \( \langle b, c, Z \rangle / Z \) contains an elementary abelian 3-subgroup of index 2. In particular, \( b c \in Z \).

Let us show that every involution \( d \) from \( C_G(Z) \) inverts \( R/Z \). Suppose \( r \) is an element from \( R \). By Proposition 1.4, the coset \( b r Z \) contains an involution \( c \). By Proposition 3.1 and previous paragraph, \( \langle b, c, d, Z \rangle / Z \) is a finite group containing \( r Z \). By Proposition 1.4, \( r^d Z = r^{-1} Z \).

Thus, all involutions from \( C_G(Z) \) generate a subgroup \( N \) which is an extension of \( Z \) by a group containing an elementary abelian 3-subgroup of index 2. In particular, \( N \) is locally finite and hence finite. By Proposition 1.4, a subgroup of index 2 from \( N \) coincides with \( R \). In particular, \( C_G(Z) \leq N_{G(R)} \). This implies that \( N_G(Z) \leq N_{G(R)} \). By Proposition 1.4, \( C_G(R) \leq R \), hence \( N_{G(R)} \) is a finite subgroup isomorphic to a subgroup of \( E \). This implies that \( N_{G(R)} = N_{U(R)} \).

The proof is completed.
Suppose \( b \) is an involution in \( C_U(Z) \), \( H = C_G(b) \) and \( \overline{H} = H/\langle b \rangle \). Denote the image of any element \( h \in H \) and any subset \( M \subseteq H \) by \( \overline{h} \) and \( \overline{M} \), respectively.

**Lemma 2.** Every finite subgroup of \( H \) is isomorphic to some subgroup of \( C = C_U(b) \). If \( \overline{\pi} \) and \( \overline{\tau} \) are involutions conjugated to an involution from \( C_1 \) then \( (\overline{\pi})^m = 1 \) for \( m = 3, 4 \) or 5. If, in addition, \( \overline{\pi} \) is an element of order 4 then \( (\overline{\pi})^2 \) is conjugated to an involution from \( C_1 \).

**Proof.** If \( F \) is a finite subgroup of \( H \) then \( \langle b, F \rangle \) is contained in subgroup \( V \) isomorphic to \( U \) and \( F \leq C_V(b) \simeq C_U(b) \).

If \( \overline{\pi} \) and \( \overline{\tau} \) are conjugated to an involution from \( C_1 \) then, by Proposition 1.3, \( a \) and \( c \) are elements of order 4 and \( a^2 = c^2 = b \). Finite subgroup \( \langle a, c \rangle \) is isomorphic to some subgroup of \( C_1 \) and order of \( ac \) divides one of the numbers 6, 8 or 10. In addition, if this order is even then \( b \in \langle ac \rangle \) and hence equality \( (\overline{\pi})^m = 1 \) holds, where \( m = 3, 4 \) or 5. If in addition the order of \( \overline{\pi} \) equals 4 then \( ac \) is an element of order 8. \( (ac)^2 \) is an element of order 4 and, by Proposition 1.3, \( (ac)^2 \) is conjugated in \( H \) to an element of \( C_1 \). The proof is completed.

**Lemma 3.** If \( \overline{\pi} \) is an involution from \( \overline{H} \) not lying in \( \overline{C_1} \) and conjugated to an involution from \( \overline{C_1} \), and \( \overline{\tau} \) is an involution from \( \overline{C_1} \) then \( (\overline{\pi})^4 = 1 \).

**Proof.** The subgroup \( \overline{C_1} \) is isomorphic to \( A_6 \) and hence contains a subgroup \( \overline{S} \simeq S_4 \). The group \( \overline{S} \) is generated by an element \( x \) of order 3 and an involution \( y \) whose product is an element of order 4. Since all involutions of \( A_6 \) are conjugated in \( A_6 \), without loss of the generality, we can assume that \( \overline{\pi} = (yz)^2 \). By Lemma 2, there exist \( p, q, r \in 3, 4, 5 \) such that \( (yz)^p = (yz)^q = ((yz)^2 z)^r = 1 \). Since \( |\langle x, y, z \rangle| > 24 \), Proposition 3.2 implies that \( r = 4 \). The proof is completed.

**Lemma 4.** \( C_1 \triangleleft H \).

**Proof.** Suppose the contrary. Since \( \overline{C_1} \) is generated by involutions, there exists an involution \( z \in \overline{H} \) not contained in \( \overline{C_1} \) and conjugated to an involution from \( \overline{C_1} \). Suppose again that \( \overline{S} \) is a subgroup of \( \overline{C_1} \) isomorphic to \( S_4 \), and \( x, y \) are elements of \( \overline{S} \) generating \( \overline{S} \) such that \( x^3 = y^2 = (xy)^4 = 1 \). By Lemma 3, \( (yz)^4 = (yz^2)^4 = ((xy)^2 z)^4 = 1 \).

If \( (yz)^2 = 1 \) then supplementary \( (y^2 \cdot yz)^2 = 1 \) and \( \langle x, y, z \rangle \) is finite by Proposition 3.3. Since it is generated by elements of order 2 conjugated with elements of \( \overline{C_1} \), \( \langle x, y, z \rangle \) is isomorphic to a subgroup of \( \overline{C_1} \simeq A_6 \). Since every subgroup of \( A_6 \) isomorphic to \( S_4 \) is maximal in \( A_6 \), \( \overline{C_2} = \langle x, y, z \rangle \simeq A_6 \). Obviously \( \overline{C_1} \cap \overline{C_2} = \overline{S} \).

If now \( z_1 \) is an involution from \( \overline{C_2} \simeq A_6 \) such that the order of \( yz_1 \) equals 5 then \( z_1 \) is conjugated with \( y \) and not lying in \( \overline{S} \), hence \( z_1 \notin C_1 \) and, by Lemma 3, \( (yz_1)^4 = 1 \) which contradicts the choice of \( z_1 \). Thus, \( (yz)^2 \neq 1 \).

Let \( t = (yz)^2 \). Then \( t \) is an involution conjugated by Lemma 2 with an involution from \( \overline{C_1} \). If \( t \notin \overline{C_1} \), then whereas all involutions from \( \overline{C_1} \) are conjugated in \( \overline{C_1} \) we may replace \( y \) with \( y^t \). After this replacing, the equality \( (yz)^2 = 1 \) will hold and equalities \( (yz)^4 = (yz^2)^4 = ((xy)^2 z)^4 = (y^2 \cdot yz)^4 = 1 \) will be kept. As indicated above, these equalities bring us to a contradiction. The proof is completed.

**Lemma 5.** \( H \) is a finite group.

**Proof.** By the choice of the involution \( b \), \( Z \) is contained in \( C_1 \) as a Sylow 3-subgroup. Hence \( H = C_1 N_G(Z) \). By Lemma 1, \( N_G(Z) \) is a finite group. The proof is completed.
Now, by Proposition 2, \( G \) is a locally finite and hence finite group. This implies that \( G \) is isomorphic to \( U \). Theorem is proved.

References


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