VARIATION OF PARABOLIC COHOMOLOGY AND POINCARÉ DUALITY

by

Michael Dettweiler & Stefan Wewers

Abstract. — We continue our study of the variation of parabolic cohomology ([DW]) and derive an exact formula for the underlying Poincaré duality. As an illustration of our methods, we compute the monodromy of the Picard-Euler system and its invariant Hermitian form, reproving a classical theorem of Picard.

Résumé (Variation de la cohomologie parabolique et dualité de Poincaré). — On continue l'étude de la variation de la cohomologie parabolique commencée dans [DW]. En particulier, on donne des formules pour l'accouplement de Poincaré sur la cohomologie parabolique, et on calcule la monodromie du système de Picard-Euler, confirmant un résultat classique de Picard.

Introduction

Let $x_1, \ldots, x_r$ be pairwise distinct points on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ and set $U := \mathbb{P}^1(\mathbb{C}) - \{x_1, \ldots, x_r\}$. The Riemann–Hilbert correspondence [Del70] is an equivalence between the category of ordinary differential equations with polynomial coefficients and at most regular singularities at the points $x_i$ and the category of local systems of $\mathbb{C}$-vectorspaces on $U$. The latter are essentially given by an $r$-tuple of matrices $g_1, \ldots, g_r \in \text{GL}_n(\mathbb{C})$ satisfying the relation $\prod_i g_i = 1$. The Riemann–Hilbert correspondence associates to a differential equation the tuple $(g_i)$, where $g_i$ is the monodromy of a full set of solutions at the singular point $x_i$.

In [DW] the authors investigated the following situation. Suppose that the set of points $\{x_1, \ldots, x_r\} \subset \mathbb{P}^1(\mathbb{C})$ and a local system $\mathcal{V}$ with singularities at the $x_i$ depend on a parameter $s$ which varies over the points of a complex manifold $S$. More precisely, we consider a relative divisor $D \subset \mathbb{P}^1_S$ of degree $r$ such that for all $s \in S$ the fibre $D_s \subset \mathbb{P}^1(\mathbb{C})$ consists of $r$ distinct points. Let $U := \mathbb{P}^1_S - D$ denote the complement
and let $V$ be a local system on $U$. We call $V$ a variation of local systems over the base space $S$. The parabolic cohomology of the variation $V$ is the local system on $S$

$$W := R^1 \pi_*(j_* V),$$

where $j : U \hookrightarrow \mathbb{P}^1_S$ denotes the natural injection and $\pi : \mathbb{P}^1_S \to S$ the natural projection. The fibre of $W$ at a point $s_0 \in S$ is the parabolic cohomology of the local system $V_{s_0}$, the restriction of $V$ to the fibre $U_{s_0} = U \cap \pi^{-1}(s_0)$.

A special case of this construction is the middle convolution functor defined by Katz [Kat97]. Here $S = U_0$ and so this functor transforms one local system $V_0$ on $S$ into another one, $W$. Katz shows that all rigid local systems on $S$ arise from one-dimensional systems by successive application of middle convolution. This was further investigated by Dettweiler and Reiter [DR03]. Another special case are the generalized hypergeometric systems studied by Lauricella [Lau93], Terada [Ter73] and Deligne–Mostow [DM86]. Here $S$ is the set of ordered tuples of pairwise distinct points on $\mathbb{P}^1(\mathbb{C})$ of the form $s = (0,1,\infty,x_4,\ldots,x_r)$ and $V$ is a one-dimensional system on $\mathbb{P}^1_S$ with regular singularities at the (moving) points $0,1,\infty,x_4,\ldots,x_r$. In [DW] we gave another example where $S$ is a 17-punctured Riemann sphere and the local system $V$ has finite monodromy. The resulting local system $W$ on $S$ does not have finite monodromy and is highly non-rigid. Still, by the comparison theorem between singular and étale cohomology, $W$ gives rise to $\ell$-adic Galois representations, with interesting applications to the regular inverse Galois problem.

In all these examples, it is a significant fact that the monodromy of the local system $W$ (i.e. the action of $\pi_1(S)$ on a fibre of $W$) can be computed explicitly, i.e. one can write down matrices $g_1,\ldots,g_r \in \text{GL}_n$ which are the images of certain generators $\alpha_1,\ldots,\alpha_r$ of $\pi_1(S)$. In the case of the middle convolution this was discovered by Dettweiler–Reiter [DR00] and Völklein [Vö1]. In [DW] it is extended to the more general situation sketched above. In all earlier papers, the computation of the monodromy is either not explicit (like in [Kat97]) or uses ad hoc methods. In contrast, the method presented in [DW] is very general and can easily be implemented on a computer.

It is one matter to compute the monodromy of $W$ explicitly (i.e. to compute the matrices $g_i$) and another matter to determine its image (i.e. the group generated by the $g_i$). In many cases the image of monodromy is contained in a proper algebraic subgroup of $\text{GL}_n$, because $W$ carries an invariant bilinear form induced from Poincaré duality. To compute the image of monodromy, it is often helpful to know this form explicitly. After a review of the relevant results of [DW] in Section 1, we give a formula for the Poincaré duality pairing on $W$ in Section 2. Finally, in Section 3 we illustrate our method in a very classical example: the Picard–Euler system.
1. Variation of parabolic cohomology revisited

1.1. Let $X$ be a compact Riemann surface of genus 0 and $D \subset X$ a subset of cardinality $r \geq 3$. We set $U := X - D$. There exists a homeomorphism $\kappa : X \to \mathbb{P}^1(\mathbb{C})$ between $X$ and the Riemann sphere which maps the set $D$ to the real line $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$. Such a homeomorphism is called a marking of $(X, D)$.

Having chosen a marking $\kappa$, we may assume that $X = \mathbb{P}^1(\mathbb{C})$ and $D \subset \mathbb{P}^1(\mathbb{R})$. Choose a base point $x_0 \in U$ lying in the upper half plane. Write $D = \{x_1, \ldots, x_r\}$ with $x_1 < x_2 < \cdots < x_r \leq \infty$. For $i = 1, \ldots, r - 1$ we let $\gamma_i$ denote the open interval $(x_i, x_{i+1}) \subset U \cap \mathbb{P}^1(\mathbb{R})$; for $i = r$ we set $\gamma_0 = \gamma_r := (x_r, x_1)$ (which may include $\infty$).

For $i = 1, \ldots, r$, we let $\alpha_i \in \pi_1(U)$ be the element represented by a closed loop based at $x_0$ which first intersects $\gamma_{i-1}$ and then $\gamma_i$. We obtain the following well known presentation

$$\pi_1(U, x_0) = \left\langle \alpha_1, \ldots, \alpha_r \mid \prod_i \alpha_i = 1 \right\rangle,$$

which only depends on the marking $\kappa$.

Let $R$ be a (commutative) ring. A local system of $R$-modules on $U$ is a locally constant sheaf $\mathcal{V}$ on $U$ with values in the category of free $R$-modules of finite rank. Such a local system corresponds to a representation $\rho : \pi_1(U, x_0) \to \text{GL}(V)$, where $V := \mathcal{V}_{x_0}$ is the stalk of $\mathcal{V}$ at $x_0$ (note that $V$ is a free $R$-module of finite rank). For $i = 1, \ldots, r$, set $g_i := \rho(\alpha_i) \in \text{GL}(V)$. Then we have

$$\prod_{i=1}^r g_i = 1,$$

and $\mathcal{V}$ can also be given by a tuple $g = (g_1, \ldots, g_r) \in \text{GL}(V)^r$ satisfying the above product-one-relation.

Convention 1.1. — Let $\alpha, \beta$ be two elements of $\pi_1(U, x_0)$, represented by closed path based at $x_0$. The composition $\alpha \beta$ is (the homotopy class of) the closed path obtained by first walking along $\alpha$ and then along $\beta$. Moreover, we let $\text{GL}(V)$ act on $V$ from the right.

1.2. Fix a local system of $R$-modules $\mathcal{V}$ on $U$ as above. Let $j : U \hookrightarrow X$ denote the inclusion. The parabolic cohomology of $\mathcal{V}$ is defined as the sheaf cohomology of $j_* \mathcal{V}$, and is written as $H^n_p(U, \mathcal{V}) := H^n(X, j_* \mathcal{V})$. We have natural morphisms $H^n_p(U, \mathcal{V}) \to H^n(U, \mathcal{V})$ and $H^0_p(U, \mathcal{V}) \to H^0(U, \mathcal{V})$ ($H_c$ denotes cohomology with compact support). Moreover, the group $H^n(U, \mathcal{V})$ is canonically isomorphic to the group cohomology $H^n(\pi_1(U, x_0), V)$ and $H^1_p(U, \mathcal{V})$ is the image of the cohomology with compact support in $H^1(U, \mathcal{V})$, see [DW, Prop. 1.1]. Thus, there is a natural inclusion

$$H^1_p(U, \mathcal{V}) \hookrightarrow H^1(\pi_1(U, x_0), V).$$

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Let $\delta : \pi_1(U) \to V$ be a cocycle, i.e. we have $\delta(\alpha\beta) = \delta(\alpha) \cdot \rho(\beta) + \delta(\beta)$ (see Convention 1.1). Set $v_i := \delta(\alpha_i)$. It is clear that the tuple $(v_i)$ is subject to the relation

$$v_1 \cdot g_2 \cdots g_r + v_2 \cdot g_3 \cdots g_r + \cdots + v_r = 0.$$ 

By definition, $\delta$ gives rise to an element in $H^1(\pi_1(U,x_0),V)$. We say that $\delta$ is a parabolic cocycle if the class of $\delta$ in $H^1(\pi_1(U),V)$ lies in $H^1_p(U,V)$. By [DW, Lemma 1.2], the cocycle $\delta$ is parabolic if and only if $v_i$ lies in the image of $g_i - 1$, for all $i$. Thus, the assignment $\delta \mapsto (\delta(\alpha_1), \ldots, \delta(\alpha_r))$ yields an isomorphism

$$H^1_p(U,V) \cong W_g := H_g/E_g,$$

where

$$H_g := \{ (v_1, \ldots, v_r) \mid v_i \in \text{Im}(g_i - 1), \text{relation (2) holds} \}$$

and

$$E_g := \{ (v \cdot (g_1 - 1), \ldots, v \cdot (g_r - 1)) \mid v \in V \}.$$ 

1.3. Let $S$ be a connected complex manifold, and $r \geq 3$. An $r$-configuration over $S$ consists of a smooth and proper morphism $\pi : X \to S$ of complex manifolds together with a smooth relative divisor $D \subset X$ such that the following holds. For all $s \in S$ the fiber $X_s := \pi^{-1}(s)$ is a compact Riemann surface of genus 0. Moreover, the natural map $D \to S$ is an unramified covering of degree $r$. Then for all $s \in S$ the divisor $D \cap X_s$ consists of $r$ pairwise distinct points $x_1, \ldots, x_r \in X_s$.

Let us fix an $r$-configuration $(X,D)$ over $S$. We set $U := X - D$ and denote by $j : U \hookrightarrow X$ the natural inclusion. Also, we write $\pi : U \to S$ for the natural projection. Choose a base point $s_0 \in S$ and set $X_0 := \pi^{-1}(s_0)$ and $D_0 := X_0 \cap D$. Set $U_0 := X_0 - D_0 = \pi^{-1}(s_0)$ and choose a base point $x_0 \in U_0$. The projection $\pi : U \to S$ is a topological fibration and yields a short exact sequence

$$1 \to \pi_1(U_0,x_0) \to \pi_1(U,x_0) \to \pi_1(S,s_0) \to 1.$$ 

Let $V_0$ be a local system of $R$-modules on $U_0$. A variation of $V_0$ over $S$ is a local system $V$ of $R$-modules on $U$ whose restriction to $U_0$ is identified with $V_0$. The parabolic cohomology of a variation $V$ is the higher direct image sheaf

$$W := R^1\pi_* (j_* V).$$

By construction, $W$ is a local system with fibre

$$W := H^1_p(U_0,V_0).$$

(Since an $r$-configuration is locally trivial relative to $S$, it follows that the formation of $W$ commutes with arbitrary basechange $S' \to S$.) Thus $W$ corresponds to a representation $\rho : \pi_1(S,s_0) \to \text{GL}(W)$. We call $\rho$ the monodromy representation on the parabolic cohomology of $V_0$ (with respect to the variation $V$).
1.4. Under a mild assumption, the monodromy representation $\eta$ has a very explicit description in terms of the Artin braid group. We first have to introduce some more notation. Define

$$O_{r-1} := \{ D' \subset \mathbb{C} \mid |D'| = r - 1 \} = \{ D \subset \mathbb{P}^1(\mathbb{C}) \mid |D| = r, \infty \in D \}.$$ 

The fundamental group $A_{r-1} := \pi_1(O_{r-1}, D_0)$ is the Artin braid group on $r - 1$ strands. Let $\beta_1, \ldots, \beta_{r-2}$ be the standard generators, see e.g. [DW, § 2.2.] (The element $\beta_i$ switches the position of the two points $x_i$ and $x_{i+1}$; the point $x_i$ walks through the lower half plane and $x_{i+1}$ through the upper half plane.) The generators $\beta_i$ satisfy the following well known relations:

$$\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}, \quad \beta_i \beta_j = \beta_j \beta_i \quad (\text{for } |i - j| > 1). \quad (7)$$

Let $R$ be a commutative ring and $V$ a free $R$-module of finite rank. Set

$$E_r(V) := \{ g = (g_1, \ldots, g_r) \mid g_i \in \text{GL}(V), \prod g_i = 1 \}.$$ 

We define a right action of the Artin braid group $A_{r-1}$ on the set $E_r(V)$ by the following formula:

$$g^{\beta_i} := (g_1, \ldots, g_{i+1}, g_i^{-1} g_i g_{i+1}, \ldots, g_r). \quad (8)$$

One easily checks that this definition is compatible with the relations (7). For $g \in E_r(V)$, let $H_g$ be as in (4). For all $\beta \in A_{r-1}$, we define an $R$-linear isomorphism

$$\Phi(g, \beta) : H_g \longrightarrow H_g^{\beta},$$

as follows. For the generators $\beta_i$ we set

$$v_1, \ldots, v_r)^{\Phi(g, \beta_i)} := (v_1, \ldots, v_{i+1}, v_i (1 - g_i^{-1} g_i g_{i+1}) + v_i g_{i+1}, \ldots, v_r). \quad (9)$$

For an arbitrary word $\beta$ in the generators $\beta_i$, we define $\Phi(g, \beta)$ using (9) and the ‘cocycle rule’

$$\Phi(g, \beta) \cdot \Phi(g, \beta') = \Phi(g, \beta \beta'). \quad (10)$$

(Our convention is to let linear maps act from the right; therefore, the left hand side of (9) is the linear map obtained from first applying $\Phi(g, \beta)$ and then $\Phi(g, \beta')$.) It is easy to see that $\Phi(g, \beta)$ is well defined and respects the submodule $E_g \subset H_g$ defined by (5). Let

$$\Phi(g, \beta) : W_g \longrightarrow W_g^{\beta}$$

denote the induced map on the quotient $W_g = H_g / E_g$.

Given $g \in E_r(V)$ and $h \in \text{GL}(V)$, we define the isomorphism

$$\Psi(g, h) : \begin{cases} H_g^h \longrightarrow H_g \\ (v_1, \ldots, v_r) \longmapsto (v_1 \cdot h, \ldots, v_r \cdot h). \end{cases}$$
where $g^h := (h^{-1}g_1h, \ldots, h^{-1}g_rh)$. It is clear that $\Psi(g, h)$ maps $E_{g^h}$ to $E_g$ and therefore induces an isomorphism $\bar{\Psi}(g, h) : W_{g^h} \sim W_g$.

Note that the computation of the maps $\bar{\Phi}(g, \beta)$ and $\bar{\Psi}(g, h)$ can easily be implemented on a computer.

1.5. Let $S$ be a connected complex manifold, $s_0 \in S$ a base point and $(X, D)$ an $r$-configuration over $S$. As before we set $U := X - D$, $D_0 := D \cap X_{s_0}$ and $U_0 := U \cap X_{s_0}$.

Let $\mathcal{V}_0$ be a local system of $R$-modules on $U_0$ and $\mathcal{V}$ a variation of $\mathcal{V}_0$ over $S$. Let $\mathcal{W}$ be the parabolic cohomology of the variation $\mathcal{V}$ and let $\eta : \pi_1(S, s_0) \to \text{GL}(\mathcal{W})$ be the corresponding monodromy representation. In order to describe $\eta$ explicitly, we find it convenient to make the following assumption on $(X, D)$:

Assumption 1.2

1. $X = \mathbb{P}^1_S$ is the relative projective line over $S$.
2. The divisor $D$ contains the section $\infty \times S \subset \mathbb{P}^1_S$.
3. There exists a point $s_0 \in S$ such that $D_0 := D \cap \pi^{-1}(s_0)$ is contained in the real line $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C}) = \pi^{-1}(s_0)$.

In practise, this assumption is not a big restriction. See [DW] for a more general setup.

By Assumption 1.2, we can consider $D_0$ as an element of $\mathcal{O}_{r-1}$. Moreover, the divisor $D \subset \mathbb{P}^1_S$ gives rise to an analytic map $S \to \mathcal{O}_{r-1}$ which sends $s_0 \in S$ to $D_0 \in \mathcal{O}_{r-1}$. We let $\varphi : \pi_1(S, s_0) \to \text{A}_{r-1}$ denote the induced group homomorphism and call it the braiding map induced by $(X, D)$.

For $t \in \mathbb{R}^+$ let $\Omega_t := \{ z \in \mathbb{C} \mid |z| > t, z \notin (-\infty, 0) \}$. Since $\Omega_t$ is contractible, the fundamental group $\pi_1(U_0, \Omega_t)$ is well defined for $t \gg 0$ and independent of $t$, up to canonical isomorphism. We write $\pi_1(U_0, \infty) := \pi_1(U_0, \Omega_t)$. We can define $\pi_1(U, \infty)$ in a similar fashion, and obtain a short exact sequence

$$1 \to \pi_1(U_0, \infty) \to \pi_1(U, \infty) \to \pi_1(S, s_0) \to 1. \tag{11}$$

It is easy to see that the projection $\pi : U \to S$ has a continuous section $\zeta : S \to U$ with the following property. For all $s \in S$ there exists $t \gg 0$ such that the region $\Omega_t$ is contained in the fibre $U_s := \pi^{-1}(s) \subset \mathbb{P}^1(\mathbb{C})$ and such that $\zeta(s) \in \Omega_t$. The section $\zeta$ induces a splitting of the sequence (11), which is actually independent of $\zeta$. We will use this splitting to consider $\pi_1(S, s_0)$ as a subgroup of $\pi_1(U, \infty)$.

The variation $\mathcal{V}$ corresponds to a group homomorphism $\rho : \pi_1(U, \infty) \to \text{GL}(V)$, where $V$ is a free $R$-module. Let $\rho_0$ denote the restriction of $\rho$ to $\pi_1(U_0, \infty)$ and $\chi$ the restriction to $\pi_1(S, s_0)$. By Part (iii) of Assumption 1.2 and the discussion in § 1.1 we have a natural ordering $x_1 < \cdots < x_r = \infty$ of the points in $D_0$, and a natural choice of a presentation $\pi_1(U_0, \infty) \cong \langle \alpha_1, \ldots, \alpha_r \mid \prod_{i} \alpha_i = 1 \rangle$. Therefore, the local system $\mathcal{V}_0$ corresponds to a tuple $g = (g_1, \ldots, g_r) \in E(V)$, with $g_i := \rho_0(\alpha_i)$. One checks

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that the homomorphism $\chi : \pi_1(S,s_0) \to \text{GL}(V)$ satisfies the condition
\begin{equation}
(12) \quad g^{\varphi(\gamma)} = g^{\chi(\gamma)^{-1}},
\end{equation}
for all $\gamma \in \pi_1(S,s_0)$. Conversely, given $g \in E_r(V)$ and a homomorphism $\chi : \pi_1(S,s_0)$ such that (12) holds then there exists a unique variation $V$ which induces the pair $(g, \chi)$.

With these notations one has the following result (see [DW, Thm. 2.5]):

**Theorem 1.3.** — Let $W$ be the parabolic cohomology of $V$ and $\eta : \pi_1(S,s_0) \to \text{GL}(W_g)$ the corresponding monodromy representation. For all $\gamma \in \pi_1(S,s_0)$ we have
\[
\eta(\gamma) = \bar{\Phi}(g, \varphi(\gamma)) \cdot \bar{\Psi}(g, \chi(\gamma)).
\]

Thus, in order to compute the monodromy action on the parabolic cohomology of a local system $V_0$ corresponding to a tuple $g \in E_r(V)$, we need to know the braiding map $\varphi : \pi_1(S,s_0) \to A_{r-1}$ and the homomorphism $\chi : \pi_1(S,s_0) \to \text{GL}(V)$.

**Remark 1.4.** — Suppose that $R$ is a field and that the local system $V_0$ is irreducible, i.e. the subgroup of $\text{GL}(V)$ generated by the elements $g_i$ acts irreducibly on $V$. Then the homomorphism $\chi$ is determined, modulo the scalar action of $R^\times$ on $V$, by $g$ and $\varphi$ (via (12)). It follows from Theorem 1.3 that the projective representation $\pi_1(S,s_0) \to \text{PGL}(V)$ associated to the monodromy representation $\eta$ is already determined by (and can be computed from) $g$ and the braiding map $\varphi$.

The above result is crucial for recent work of the first author [Det05] on the middle convolution, where the above methods are used to realize special linear groups as Galois groups over $\mathbb{Q}(t)$.

### 2. Poincaré duality

Let $\mathcal{V}$ be a local system of $R$-modules on the punctured Riemann sphere $U$. If $\mathcal{V}$ carries a non-degenerate symmetric (resp. alternating) form, then Poincaré duality induces on the parabolic cohomology group $H^1_p(U, \mathcal{V})$ a non-degenerate alternating (resp. symmetric) form. Similarly, if $R = \mathbb{C}$ and $\mathcal{V}$ carries a Hermitian form, then we get a Hermitian form on $H^1_p(U, \mathcal{V})$. In this section we derive an explicit expression for this induced form.

#### 2.1. Let us briefly recall the definition of singular (co)homology with coefficients in a local system. See e.g. [Spa93] for more details. For $q \geq 0$ let $\Delta^q = |y_0, \ldots, y_q|$ denote the standard $q$-simplex with vertices $y_0, \ldots, y_q$. We will sometimes identify $\Delta^1$ with the closed unit interval $[0,1]$. Let $X$ be a connected and locally contractible topological space and $\mathcal{V}$ a local system of $R$-modules on $X$. For a continuous map $f : Y \to X$ we denote by $\mathcal{V}_f$ the group of global sections of $f^*\mathcal{V}$.

In the following discussion, a $q$-chain will be a function $\varphi$ which assigns to each singular $q$-simplex $\sigma : \Delta^q \to X$ a section $\varphi(\sigma) \in \mathcal{V}_\sigma$. Let $\Delta^q(X, \mathcal{V})$ denote the
set of all $q$-chains, which is made into an $R$-module in the obvious way. A $q$-chain $\varphi$ is said to have compact support if there exists a compact subset $A \subset X$ such that $\varphi|_A = 0$ whenever $\text{supp}(\sigma) \subset X - A$. The corresponding $R$-module is denoted by $\Delta^q(X, V)$. We define coboundary operators $d : \Delta^q(X, V) \to \Delta^{q+1}(X, V)$ and $d : \Delta^f(X, V) \to \Delta^{f+1}(X, V)$ through the formula

$$(d \varphi)(\sigma) := \sum_{0 \leq i \leq q} (-1)^i \cdot \varphi(\sigma(i)).$$

Here $\sigma(i)$ is the $i$th face of $\sigma$ (see [Spa66]) and $\varphi(\sigma(i))$ denotes the unique extension of $\varphi(\sigma)$ to an element of $V_x$. It is proved in [Spa93] that we have canonical isomorphisms

$$(13) \quad H^n(X, V) \cong H^n(\Delta^\bullet(X, V), d), \quad H^n_c(X, V) \cong H^n(\Delta^\bullet_c(X, V), d),$$

i.e. singular cohomology agrees with sheaf cohomology. Let $x_0 \in X$ be a base point and $V$ the fibre of $V$ at $x_0$. Then we also have an isomorphism

$$(14) \quad H^1(X, V) \cong H^1(\pi_1(X, x_0), V).$$

Let $\varphi$ be a 1-chain with $d\varphi = 0$. Let $\alpha : [0, 1] \to X$ be a closed path with base point $x_0$. By definition, $\varphi(\alpha)$ is a global section of $\alpha^* V$. Then $\alpha \mapsto \delta(\alpha) := \varphi(\alpha)(1)$ defines a cocycle $\delta : \pi_1(X, x_0) \to V$, and this cocycle represents the image of $\varphi$ in $H^1(X, V)$.

A $q$-chain $\varphi$ is called finite if $\varphi(\sigma) = 0$ for all but finitely many simplexes $\sigma$. It is called locally finite if every point in $X$ has a neighborhood $U \subset X$ such that $\varphi(\sigma) = 0$ for all but finitely many simplexes $\sigma$ contained in $U$. We denote by $\Delta_q(X, V)$ (resp. by $\Delta_q^f(X, V)$) the $R$-module of all finite (resp. locally finite) $q$-chains. For a fixed $q$-simplex $\sigma$ and a section $v \in V_x$, the symbol $v \otimes \sigma$ will denote the $q$-chain which assigns $v$ to $\sigma$ and $0$ to all $\sigma' \neq \sigma$. Obviously, every finite (resp. locally finite) $q$-chain can be written as a finite (resp. possibly infinite) sum $\sum_p v_p \otimes \sigma_p$. We define boundary operators $\partial : \Delta_q(X, V) \to \Delta_{q-1}(X, V)$ and $\partial : \Delta_q^f(X, V) \to \Delta_{q-1}^f(X, V)$ through the formula

$$(\partial(v \otimes \sigma)) := \sum_{0 \leq i \leq q} (-1)^i \cdot v|_{\sigma(i)} \otimes \sigma(i).$$

We define homology (resp. locally finite homology) with coefficients in $V$ as follows:

$$H_q(X, V) := H_q(\Delta^\bullet(X, V)), \quad H_q^f(X, V) := H_q(\Delta^\bullet_c(X, V)).$$

2.2. Let $X := \mathbb{P}^1(\mathbb{C})$ be the Riemann sphere and $D = \{x_1, \ldots, x_r\} \subset \mathbb{P}^1(\mathbb{R})$ a subset of $r \geq 3$ points lying on the real line, with $x_1 < \cdots < x_r \leq \infty$. Let $V$ be a local system of $R$-modules on $U = X - D$. Choose a base point $x_0$ lying in the upper half plane. Then $V$ corresponds to a tuple $g = (g_1, \ldots, g_r)$ in $\text{GL}(V)$ with $\prod_i g_i = 1$, where $V := V_{x_0}$. See § 1.1. Let $V^* := \text{Hom}(V, R)$ denote the local system dual to $V$. 

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It corresponds to the tuple $g^* = (g_1^*, \ldots, g_r^*)$ in $\text{GL}(V^*)$, where $V^*$ is the dual of $V$ and for each $g \in \text{GL}(V)$ we let $g^* \in \text{GL}(V^*)$ be the unique element such that

$$\langle w \cdot g^*, v \cdot g \rangle = \langle w, v \rangle$$

for all $w \in V^*$ and $v \in V$. Note that $V^{**} = V$ because $V$ is free of finite rank over $R$.

Let $\varphi$ be a 1-chain with compact support and with coefficients in $V^*$. Let $a = \sum_{\mu} v_\mu \otimes a_\mu$ be a locally finite 1-chain with coefficients in $V$. By abuse of notation, we will also write $\varphi$ (resp. $a$) for its class in $H^1_c(U, V^*)$ (resp. in $H^1_c(U, V)$). The cap product

$$\varphi \cap a := \sum_{\mu} \langle \varphi(a_\mu), v_\mu \rangle$$

induces a bilinear pairing

$$\cap : H^1_c(U, V^*) \otimes H^1_c(U, V) \longrightarrow R. \tag{15}$$

It is easy to see from the definition that $H^1_c(U, V) = 0$. Therefore, it follows from the Universal Coefficient Theorem for cohomology (see e.g. [Spa66, Thm. 5.5.3]) that the pairing (15) is nonsingular on the left, i.e. identifies $H^1_c(U, V^*)$ with $\text{Hom}(H^1_c(U, V), R)$. The cap product also induces a pairing

$$\cap : H^1(U, V^*) \otimes H^1(U, V) \longrightarrow R. \tag{16}$$

(This last pairing may not be non-singular on the left. The reason is that

$$H_0(U, V) \cong V/\langle \text{Im}(g_i - 1) \mid i = 1, \ldots, r \rangle$$

may not be a free $R$-module, and so $\text{Ext}^1(H_0(U, V), R)$ may be nontrivial.) Let $f^1 : H^1_c(U, V^*) \to H^1(U, V^*)$ and $f_1 : H^1(U, V) \to H^1_c(U, V)$ denote the canonical maps. Going back to the definition, one can easily verify the rule

$$f^1(\varphi) \cap a = \varphi \cap f_1(a). \tag{17}$$

Let $\varphi \in H^1_c(U, V^*)$ and $\psi \in H^1(U, V)$. The cup product $\varphi \cup \psi$ is defined as an element of $H^2(U, R)$, see [Ste43] or [Spa93]. The standard orientation of $U$ yields an isomorphism $H^2_c(U, R) \cong R$. Using this isomorphism, we shall view the cup product as a bilinear pairing

$$\cup : H^1_c(U, V^*) \otimes H^1(U, V) \longrightarrow R. \tag{18}$$

Similarly, one can define the cup product $\varphi \cup \psi$, where $\varphi \in H^1(U, V^*)$ and $\psi \in H^1_c(U, V)$. Given $\varphi \in H^1_c(U, V^*)$ and $\psi \in H^1_c(U, V)$, one checks that

$$f^1(\varphi) \cup \psi = \varphi \cup f^1(\psi).$$

**Proposition 2.1 (Poincaré duality).** — There exist unique isomorphisms of $R$-modules

$$p : H^1_c(U, V) \xrightarrow{\sim} H^1_c(U, V), \quad p : H^1(U, V) \xrightarrow{\sim} H^1(U, V)$$
such that the following holds. If \( \varphi \in H^1_c(U, V^*) \) and \( a \in H^1_c(U, V) \) or if \( \varphi \in H^1(U, V^*) \) and \( a \in H^1(U, V) \) then we have

\[
\varphi \cap a = \varphi \cup p(a).
\]

These isomorphisms are compatible with the canonical maps \( f_1 \) and \( f^1 \), i.e. we have \( p \circ f_1 = f^1 \circ p \).

**Proof.** — See [Ste43] or [Spa93].

**Corollary 2.2.** — The cup product induces a non-degenerate bilinear pairing

\[
\cup : H^1_p(U, V^*) \otimes H^1_p(U, V) \to R.
\]

**Proof.** — Let \( \varphi \in H^1_p(U, V^*) \) and \( \psi \in H^1_p(U, V) \). Choose \( \varphi' \in H^1_c(U, V^*) \) and \( \psi' \in H^1_c(U, V) \) with \( \varphi = f^1(\varphi') \) and \( \psi = f^1(\psi') \). By (18) we have \( \varphi' \cup \psi = \varphi \cup \psi' \). Therefore, the expression \( \varphi \cup \psi := \varphi' \cup \psi' \) does not depend on the choice of the lift \( \varphi' \) and defines a bilinear pairing between \( H^1_p(U, V^*) \) and \( H^1_p(U, V) \). By Proposition 2.1 and since the cap product (15) is non-degenerate on the left, this pairing is also non-degenerate on the right. But the cup product is alternating (i.e. we have \( \varphi \cup \psi = -\psi \cup \varphi \), where the right hand side is defined using the identification \( V^{**} = V \)), so our pairing is also non-degenerate on the right.

For \( a \in H^1_p(U, V^*) \) and \( b \in H^1(U, V) \), the expression

\[
(a, b) := p(a) \cup p(b)
\]
defines another bilinear pairing \( H^1_p(U, V^*) \otimes H^1_p(U, V) \to R \). It is shown in [Ste43] that this pairing can be computed as an ‘intersection product of loaded cycles’, generalizing the usual intersection product for constant coefficients, as follows. We may assume that \( a \) is represented by a locally finite chain \( \sum \nu_\mu^* \otimes \alpha_\mu \) and that \( b \) is represented by a finite chain \( \sum \nu_\nu \otimes \beta_\nu \) such that for all \( \mu, \nu \) the 1-simplexes \( \alpha_\mu \) and \( \beta_\nu \) are smooth and intersect each other transversally, in at most finitely many points. Suppose \( x \) is a point where \( \alpha_\mu \) intersects \( \beta_\nu \). Then there exists \( t_0 \in [0, 1] \) such that \( x = \alpha(t_0) = \beta(t_0) \) and \( (\frac{\partial \alpha}{\partial t}|_{t_0}, \frac{\partial \beta}{\partial t}|_{t_0}) \) is a basis of the tangent space of \( U \) at \( x \). We set \( v(\alpha, \beta, x) := 1 \) (resp. \( v(\alpha, \beta, x) := -1 \)) if this basis is positively (resp. negatively) oriented. Furthermore, we let \( \alpha_{\mu, x} \) (resp. \( \beta_{\nu, x} \)) be the restriction of \( \alpha \) (resp. of \( \beta \)) to the interval \( [0, t_0] \). Then we have

\[
(\alpha, \beta) = \sum_{\mu, \nu, x} v(\alpha_{\mu, x}, \beta_{\nu, x}) \cdot (v^*)^{\alpha_{\mu, x}, \beta_{\nu, x}}.
\]

**2.3.** Let \( V \otimes V \to R \) be a non-degenerate symmetric (resp. alternating) bilinear form, corresponding to an injective homomorphism \( \kappa : V \hookrightarrow V^* \) with \( \kappa^* = \kappa \) (resp. \( \kappa^* = -\kappa \)). We denote the induced map \( H^1_p(U, V) \to H^1_p(U, V^*) \) by \( \kappa \) as well. Then

\[
(\varphi, \psi) := \kappa(\varphi) \cup \psi
\]
defines a non-degenerate alternating (resp. symmetric) form on \( H^1_p(U, V) \).
Similarly, suppose that \( R = \mathbb{C} \) and let \( \mathcal{V} \) be equipped with a non-degenerate Hermitian form, corresponding to an isomorphism \( \kappa : \mathcal{V} \cong \mathcal{V}^* \). Then the pairing

\[
(\varphi, \psi) := -i \cdot (\kappa(\bar{\varphi}) \cup \psi)
\]

is a nondegenerate Hermitian form on \( H^1_p(U, \mathcal{V}) \) (we identify \( H^1_{\bar{p}}(U, \mathcal{V}) \) with the complex conjugate of the vector space \( H^1_p(U, \mathcal{V}) \) in the obvious way).

Suppose that the Hermitian form on \( \mathcal{V} \) is positive definite. Then we can express the signature of the form (20) in terms of the tuple \( g \), as follows. For \( i = 1, \ldots, r \), let

\[
g_i \sim \begin{pmatrix}
\alpha_{i,1} & & & \\
& \ddots & & \\
& & \alpha_{i,n} & \\
& & & 
\end{pmatrix}
\]

be a diagonalization of \( g_i \in \text{GL}(\mathcal{V}) \). Since the \( g_i \) are unitary, the eigenvalues \( \alpha_{i,j} \) have absolute value one and can be uniquely written in the form \( \alpha_{i,j} = \exp(2\pi i \mu_{i,j}) \), with \( 0 \leq \mu_{i,j} < 1 \). Set \( \bar{\mu}_{i,j} := 1 - \mu_{i,j} \) if \( \mu_{i,j} > 0 \) and \( \bar{\mu}_{i,j} := 0 \) otherwise.

**Theorem 2.3.** — Suppose that \( \mathcal{V} \) is equipped with a positive definite Hermitian form and that \( H^0(U, \mathcal{V}) = 0 \). Then the Hermitian form (20) on \( H^1_p(U, \mathcal{V}) \) has signature

\[
(\sum_{i,j} \mu_{i,j}) - \text{dim}_\mathbb{C} \mathcal{V}, (\sum_{i,j} \bar{\mu}_{i,j}) - \text{dim}_\mathbb{C} \mathcal{V}).
\]

**Proof.** — If \( \text{dim}_\mathbb{C} \mathcal{V} = 1 \), this formula is proved in [DM86, §2]. The general case is proved in a similar manner. We will therefore only sketch the argument.

Let \( \Omega^\bullet(\mathcal{V}) : \mathcal{O}(\mathcal{V}) \to \Omega^1(\mathcal{V}) \) be the holomorphic \( \mathcal{V} \)-valued de Rham complex on \( U \) ([DM86, §2.7]). Let \( j_*^m \Omega^\bullet(\mathcal{V}) \) denote the subcomplex of \( j_* \Omega^\bullet(\mathcal{V}) \) consisting of sections which are meromorphic at all the singular points. Then we have

\[
H^1(U, \mathcal{V}) = \mathbb{H}^1(X, j_*^m \Omega^\bullet(\mathcal{V})) = H^1\Gamma(X, j_*^m \Omega^\bullet(\mathcal{V})).
\]

We define a subbundle \( \mathcal{E} \) of \( j_*^m \mathcal{O}(\mathcal{V}) \) as follows. Fix an index \( i \) and let \( U_i \subset X \) be a disk-like neighborhood of \( x_i \), which does not contain any other singular point. Set \( U_i^* := U_i - \{x_i\} \). We obtain a decomposition

\[
\mathcal{V}|_{U_i^*} = \oplus_j L_j
\]

into local systems of rank one, corresponding to the diagonalization (21) of the monodromy matrix \( g_i \). In the notation of [DM86, §2.11], we set

\[
\mathcal{E}|_{U_i^*} := \oplus_j \mathcal{O}(\mu_{i,j} \cdot x_i)(L_j).
\]

In other words: a holomorphic section of \( \mathcal{E} \) on \( U_i \) can be written as \( \sum_j z^{-\mu_{i,j}} f_j v_j \), where \( z \) is a local parameter on \( U_i \) vanishing at \( x_i \), \( f_j \) is a holomorphic function and \( v_j \) is a (multivalued) section of \( L_j \) on (the universal cover of) \( U_i^* \). It is clear that
$\mathcal{E}$ is a vector bundle of rank $\dim_\mathbb{C} V$. Moreover, it is easy to see (compare [DM86], Proposition 2.11.1) that

\begin{equation}
\deg \mathcal{E} = \sum_{i,j} \mu_{i,j}.
\end{equation}

In the same manner we define a subbundle $\mathcal{E}'$ of $j^m \Omega^1(\bar{V})$. It is clear that

\begin{equation}
\deg \mathcal{E}' = \sum_{i,j} \bar{\mu}_{i,j},
\end{equation}

where $\bar{\mu}_{i,j}$ is defined as above.

We define the subspace $H^{1,0}(U,V)$ of $H^1(U,V)$ as the image of the map

$$H^0(X, \mathcal{E} \otimes \Omega^1_X) \to \mathbb{H}^1(X, j^m \Omega^\bullet(V)) = H^1(U,V).$$

A local computation shows that $H^{1,0}(U,V)$ is actually contained in $H^1(U,V) = H^1(X, j_* V)$. Let $\omega$ be a global section of $\mathcal{E} \otimes \Omega^1_X$ and let $[\omega]$ denote the corresponding class in $H^{1,0}(U,V)$. The pairing (20) applied to $[\omega]$ is then given by the following integral

$$([\omega], [\omega]) = -i \cdot \int_U \omega \wedge \bar{\omega},$$

see [DM86, § 2.18]. Here the integrand is defined as follows: if we write locally $\omega = v \alpha$, where $v$ is a section of $V$ and $\alpha$ is a holomorphic one-form, then $\omega \wedge \bar{\omega} := ||v||^2 \alpha \wedge \bar{\alpha}$.

The definition of $\mathcal{E}$ ensures that the above integral converges. It follows that the pairing (20) is positive definite on $H^{1,0}(U,V)$ and that $H^{1,0}(U,V) = H^0(X, \mathcal{E} \otimes \Omega^1_X)$.

By Riemann–Roch and (22) we have

\begin{equation}
\dim H^{1,0}(U,V) \geq \deg(\mathcal{E} \otimes \Omega^1_X) + \text{rank}(\mathcal{E} \otimes \Omega^1_X)
\end{equation}

\begin{equation}
\geq \sum_{i,j} \mu_{i,j} - \dim V.
\end{equation}

We define $H^{0,1}(U,V)$ as the complex conjugate of $H^{1,0}(U,V)$, considered as a subspace of $H^1(U,V)$. Note that the latter space is the image of $H^0(X, \mathcal{E}' \otimes \Omega^1_X)$, and we can represent an element in $H^{0,1}(U,V)$ as an antiholomorphic form with values in $\mathcal{E}'$. The same reasoning as above shows that the pairing (20) is negative definite on $H^{0,1}(U,V)$ and that $H^{0,1}(U,V)$ is equal to the complex conjugate of $H^0(X, \mathcal{E}' \otimes \Omega^1_X)$.

Furthermore, we have

\begin{equation}
\dim H^{0,1}(U,V) = \deg(\mathcal{E}' \otimes \Omega^1_X) + \text{rank}(\mathcal{E}' \otimes \Omega^1_X) \geq \sum_{i,j} \bar{\mu}_{i,j} - \dim V.
\end{equation}

Together with (24) we get the inequality

\begin{align*}
\dim H^1(U,V) &\geq \dim H^{1,0}(U,V) + \dim H^{0,1}(U,V) \\
&\geq \sum_{i,j} (\mu_{i,j} + \bar{\mu}_{i,j}) - 2 \dim V \\
&= (r - 2) \dim V - \sum_i \dim \ker(g_i - 1).
\end{align*}
But according to [DW, Remark 1.3], this inequality is an equality. It follows that (24) and (25) are equalities as well. The theorem is now a consequence of the fact pointed out before that the pairing (20) is positive definite on $H^{1,0}(U, V)$ and negative definite on $H^{0,1}(U, V)$.

**Remark 2.4.** — The authors expect several applications of the above results, such as the construction of totally real Galois representations of classical groups (in combination with the results of [Det05]). Another possible application would be to find new examples of differential equations with a full set of algebraic solutions, in the spirit of the work of Beukers and Heckman [BH89].

### 2.4.

We are interested in an explicit expression for the pairing of Corollary 2.2. We use the notation introduced at the beginning of § 2.2, with the following modification. By $\gamma_i$, we now denote a homeomorphism between the open unit interval $(0, 1)$ and the open interval $(x_i, x_{i+1})$. We assume that $\gamma_i$ extends to a path $\bar{\gamma_i} : [0, 1] \to \mathbb{P}^1(\mathbb{R})$ from $x_i$ to $x_{i+1}$. We denote by $U^+ \subset \mathbb{P}^1(\mathbb{C})$ (resp. $U^-$) the upper (resp. the lower) half plane and by $U^+$ (resp. $U^-$) its closure inside $U = \mathbb{P}^1(\mathbb{C}) - \{x_1, \ldots, x_r\}$. Since $\bar{U}^+$ is simply connected and contains the base point $x_0$, an element of $V$ extends uniquely to a section of $V$ over $\bar{U}^+$. We may therefore identify $V$ with $V(\bar{U}^+)$ and with the stalk of $V$ at any point $x \in \bar{U}^+$.

Choose a sequence of numbers $\epsilon_n$, $n \in \mathbb{Z}$, with $0 < \epsilon_n < \epsilon_{n+1} < 1$ such that $\epsilon_n \to 0$ for $n \to -\infty$ and $\epsilon_n \to 1$ for $n \to \infty$. Let $\gamma_i^{(n)} : [0, 1] \to U$ be the path $\gamma_i^{(n)}(t) := \gamma_i(\epsilon_n t + \epsilon_{n-1}(1-t))$. Let $w_1, \ldots, w_r \in V$. Since supp($\gamma_i$) $\subset \bar{U}^+$, it makes sense to define

$$w_i \otimes \gamma_i := \sum_n w_i \otimes \gamma_i^{(n)}.$$  

This is a locally finite 1-chain. Set

$$c := \sum_{i=1}^r w_i \otimes \gamma_i.$$  

Note that $\partial(c) = 0$, so $c$ represents a class in $H^1_{lf}(U, V)$.

**Lemma 2.5**

1. The image of $c$ under the Poincaré isomorphism $H^1_{lf}(U, V) \cong H^1(U, V)$ is represented by the unique cocycle $\delta : \pi_1(U, x_0) \to V$ with

$$\delta(\alpha_i) = w_i - w_{i-1} \cdot g_i.$$  

2. The cocycle $\delta$ in (i) is parabolic if and only if there exist elements $u_i \in V$ with $w_i - w_{i-1} = u_i \cdot (g_i - 1)$, for all $i$.

**Proof.** — For a path $\alpha : [0, 1] \to U$ in $U$, consider the following conditions:

(a) The support of $\alpha$ is contained either in $U^+$ or in $U^-$.  

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In Case (b) (resp. in Case (c)) we identify $\delta$ and $\phi$.

Therefore we have

\[ \varphi(\alpha) = \begin{cases} 
0, & \text{if } \alpha \text{ is as in Case (a)} \\
-w_i, & \text{if } \alpha \text{ is as in Case (b)} \\
w_i^{-1}, & \text{if } \alpha \text{ is as in Case (c)}. 
\end{cases} \]

(To show the existence and uniqueness of $\varphi$, choose a triangulation of $U$ in which all edges satisfy Condition (a), (b) or (c). Then use simplicial approximation.) We claim that $\varphi$ represents the image of the cycle $c$ under the Poincaré isomorphism. Indeed, this follows from the definition of the Poincaré isomorphism, as it is given in [Ste43]. Write $\alpha_i = \alpha'_i \alpha''_i$, with $\alpha'_i(1) = \alpha''_i(0) \in U^-$. Using the fact that $\varphi$ is a cocycle we get

\[ \varphi(\alpha_i) = \varphi(\alpha'_i) + \varphi(\alpha''_i) = -w_{i-1} + w_i \cdot g_i^{-1}. \]

Therefore we have $\delta(\alpha_i) = \varphi(\alpha'_i) \cdot g_i = w_i - w_{i-1} \cdot g_i$. See Figure 1. This proves (i).

By Section 1.1, the cocycle $\delta$ is parabolic if and only if $v_i$ lies in the image of $g_i - 1$. So (ii) follows from (i) by a simple manipulation.

**Theorem 2.6.** — Let $\varphi \in H^1_p(U, V^*)$ and $\psi \in H^1_p(U, V)$, represented by cocycles $\delta^* : \pi_1(U, x_0) \to V^*$ and $\delta : \pi_1(U, x_0) \to V$. Set $v_i := \delta(\alpha_i)$ and $v_i^* = \delta^*(\alpha_i)$. If we choose $v'_i \in V$ such that $v'_i \cdot (g_i - 1) = v_i$ (see Lemma 2.5), then we have

\[ \varphi \cup \psi = \sum_{i=1}^r \left( (v_i^*, v'_i) + \sum_{j=1}^{i-1} (v_j^* g_{j+1}^* \cdots g_{i-1}^* (g_i^* - 1), v'_i) \right). \]

**Proof.** — Let $w_1 := v_1$, $w_i^* := v_i^*$, and $w_i := v_i + w_{i-1} \cdot g_i$. Then we have

\[ w_i^* := v_i^* + w_{i-1}^* \cdot g_i^*. \]
for \( i = 2, \ldots, r \). By Lemma 2.5, we can choose \( u_i \in V \) with \( w_i - w_{i-1} = u_i \cdot (g_i - 1) \), for \( i = 1, \ldots, r \). The claim will follow from the following formula:

\[
\varphi \cup \psi = \sum_{i=1}^{r} \langle w_i^* - w_{i-1}^*, u_i - w_{i-1} \rangle.
\]

To prove Equation (26), suppose \( \delta \) is parabolic, and choose \( u_i \in V \) such that \( w_i - w_{i-1} = u_i \cdot (g_i - 1) \). Let \( D_i \subset X \) be a closed disk containing \( x_i \) but none of the other points \( x_j, j \neq i \). We may assume that the boundary of \( D_i \) intersects \( \gamma_{i-1} \) in the point \( \gamma_{i-1}^{(1)} \) but nowhere else, and that \( D_i \) intersects \( \gamma_i \) in the point \( \gamma_i^{(0)} \) but nowhere else. Set \( D_i^+ := D_i \cap \bar{U}^+ \) and \( D_i^- := D_i \cap \bar{U}^- \). Let \( u_i^+ := u_i - w_{i-1} \), considered as a section of \( V \) over \( D_i^+ \) via extension over the whole upper half plane \( U^+ \). It makes sense to define the locally finite chain

\[
u_i^+ \otimes D_i^+ := \sum_{\sigma} u_i^+ \otimes \sigma,
\]

where \( \sigma \) runs over all 2-simplexes of a triangulation of \( D_i^+ \). (Note that \( x_i \not\in D_i^+ \), so this triangulation cannot be finite.) Similarly, let \( u_i^- \in V_{D_i^-} \) denote the section of \( V \) over \( D_i^- \) obtained from \( u_i \in V \) by continuation along a path which enters \( U^- \) from \( U^+ \) by crossing the path \( \gamma_{i-1} \); define \( u_i^- \otimes D_i^- \) as before. Let

\[c' := c + \partial (u_i^+ \otimes D_i^+ + u_i^- \otimes D_i^-)\]

It is easy to check that \( c' \) is homologous to the cocycle

\[
c'' := \sum_i \langle w_i \otimes \gamma_i^{(0)} + u_i^+ \otimes \beta_i^+ + u_i^- \otimes \beta_i^- \rangle,
\]

where \( \beta_i^+ \) (resp. \( \beta_i^- \)) is the path from \( \gamma_i^{(0)} \) to \( \gamma_i^{(1)} \) (resp. from \( \gamma_i^{(1)} \) to \( \gamma_i^{(0)} \)) running along the upper (resp. lower) part of the boundary of \( D_i \). See Figure 2. Note that \( c'' \) is finite and that, by construction, the image of \( c'' \) under the canonical map \( f_1 : H_1(U, V) \to H_1^U(U, V) \) is equal to the class of \( c \). Let \( \psi' \in H_1^U(U, V) \) denote the image of \( c'' \) under the Poincaré isomorphism \( H_1(U, V) \cong H_1^U(U, V) \). The last statement of Proposition 2.1 shows that \( \psi' \) is a lift of \( \psi \in H_1(U, V) \).

Let \( c^* := \sum_i w_i^* \otimes \gamma_i^* \in C_1(U, V^*) \). By (i) and the choice of \( w_i^* \), the image of \( c^* \) under the Poincaré isomorphism \( H_1^U(U, V^*) \cong H_1(U, V^*) \) is equal to \( \varphi \). By definition,
we have \( \varphi \cup \psi = (c^*, c'') \). To compute this intersection number, we have to replace \( c^* \) by a homologous cycle which intersects the support of \( c'' \) at most transversally. For instance, we can deform the open paths \( \gamma_i \) into open paths \( \gamma'_i \) which lie entirely in the upper half plane. See Figure 2. It follows from (19) that

\[
(c^*, c'') = \sum_i \langle w^*_i, u^+_i \rangle - \langle w^*_i, u^+_i \rangle = \sum_i \langle w^*_i - w^*_i, u_i - u_{i-1} \rangle.
\]

This finishes the proof of (26). The formula in (iv) follows from (26) from a straightforward computation, expressing \( u_i \) and \( u_i \) in terms of \( v_i \) and \( v'_i \).

\begin{remark}
In the somewhat different setup, a similar formula as in Theorem 2.6 can be found in [VÖ1, §1.2.3].
\end{remark}

### 3. The monodromy of the Picard–Euler system

Let

\[
S := \{ (s, t) \in \mathbb{C}^2 \mid s, t \neq 0, 1, s \neq t \},
\]

and let \( X := \mathbb{P}^3_\mathbb{C} \) denote the relative projective line over \( S \). The equation

\[
y^3 = x(x-1)(x-s)(x-t)
\]

defines a finite Galois cover \( f : Y \to X \) of smooth projective curves over \( S \), namely ramified along the divisor \( D := \{0, 1, s, t, \infty\} \subset X \). The curve \( Y \) is called the Picard curve. Let \( G \) denote the Galois group of \( f \), which is cyclic of order 3. The equation \( \sigma^* y = \chi(\sigma) \cdot y \) for \( \sigma \in G \) defines an injective character \( \chi : G \hookrightarrow \mathbb{C}^\times \). As we will see below, the \( \chi \)-eigenspace of the cohomology of \( Y \) gives rise to a local system on \( S \) whose associated system of differential equations is known as the Picard–Euler system.

We fix a generator \( \sigma \) of \( G \) and set \( \omega := \chi(\sigma) \). Let \( K := \mathbb{Q}(\omega) \) be the quadratic extension of \( \mathbb{Q} \) generated by \( \omega \) and \( \mathcal{O}_K = \mathbb{Z}[\omega] \) its ring of integers. The family of \( G \)-covers \( f : Y \to X \) together with the character \( \chi \) of \( G \) corresponds to a local system of \( \mathcal{O}_K \)-modules on \( U := X - D \). Set \( s_0 := (2, 3) \in S \) and let \( V_0 \) denote the restriction of \( \mathcal{V} \) to the fibre \( U_0 = \mathbb{A}^1_\mathbb{C} - \{0, 1, 2, 3\} \) of \( U \to S \) over \( s_0 \). We consider \( \mathcal{V} \) as a variation of \( V_0 \) over \( S \). Let \( \mathcal{W} \) denote the parabolic cohomology of this variation; it is a local system of \( \mathcal{O}_K \)-modules of rank three, see [DW, Rem. 1.4]. Let \( \chi' : G \hookrightarrow \mathbb{C}^\times \) denote the conjugate character to \( \chi \) and \( \mathcal{W}' \) the parabolic cohomology of the variation of local systems \( \mathcal{V}' \) corresponding to the \( G \)-cover \( f \) and the character \( \chi' \). We write \( \mathcal{W}_\mathbb{C} \) for the local system of \( \mathbb{C} \)-vectorspaces \( \mathcal{W} \otimes \mathbb{C} \). The maps \( \pi_Y : Y \to S \) and \( \pi_X : X \to S \) denote the natural projections.

\begin{proposition}
We have a canonical isomorphism of local systems

\[
R^1\pi_Y_\ast \mathcal{L} \cong \mathcal{W}_\mathbb{C} \oplus \mathcal{W}'_\mathbb{C}.
\]

This isomorphism identifies the fibres of \( \mathcal{W}_\mathbb{C} \) with the \( \chi \)-eigenspace of the singular cohomology of the Picard curves of the family \( f \).
\end{proposition}
Proof. — The group $G$ has a natural left action on the sheaf $f_*\mathbb{C}$. We have a canonical isomorphism of sheaves on $X$

$$f_*\mathbb{C} \cong \mathbb{C} \oplus j_*\mathcal{V} \oplus j_*\mathcal{V}',$$

which identifies $j_*\mathcal{V}$, fibre by fibre, with the $\chi$-eigenspace of $f_*\mathbb{C}$. Now the Leray spectral sequence for the composition $\pi_Y = \pi_X \circ f$ gives isomorphisms of sheaves on $S$

$$R^1\pi_Y_*\mathbb{C} \cong R^1\pi_{X,*}(f_*\mathbb{C}) \cong W \oplus W'.$$

Note that $R^1\pi_Y_*\mathbb{C} = 0$ because the genus of $X$ is zero. Since the formation of $R^1\pi_Y_*\mathbb{C}$ commutes with the $G$-action, the proposition follows.

The comparison theorem between singular and deRham cohomology identifies $R^1\pi_Y_*\mathbb{C}$ with the local system of horizontal sections of the relative deRham cohomology module $R^1_{\text{dR}}\pi_Y_*\mathcal{O}_Y$, with respect to the Gauss-Manin connection. The $\chi$-eigenspace of $R^1_{\text{dR}}\pi_Y_*\mathcal{O}_Y$ gives rise to a Fuchsian system known as the Picard–Euler system. In more classical terms, the Picard–Euler system is a set of three explicit partial differential equations in $s$ and $t$ of which the period integrals

$$I(s,t; a,b) := \int_a^b \frac{dx}{\sqrt{x(x-1)(x-s)(x-t)}}$$

(with $a,b \in \{0, 1, s, t, \infty\}$) are a solution. See [Pic83], [Hol86], [Hol95]. It follows from Proposition 3.1 that the monodromy of the Picard–Euler system can be identified with the representation $\eta: \pi_1(S) \to \text{GL}_3(\mathcal{O}_K)$ corresponding to the local system $W$.

**Theorem 3.2 (Picard).** — For suitable generators $\gamma_1, \ldots, \gamma_5$ of the fundamental group $\pi_1(S)$, the matrices $\eta(\gamma_1), \ldots, \eta(\gamma_5)$ are equal to

$$\begin{pmatrix} \omega^2 & 0 & 1 - \omega \\ \omega - \omega^2 & 1 & \omega^2 - 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \omega^2 & 0 & 1 - \omega^2 \\ 1 - \omega^2 & 1 & \omega^2 - 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & \omega^2 - 1 \\ 0 & \omega^2 - 1 & -2\omega \end{pmatrix}, \quad \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \omega^2 & \omega - \omega^2 & 0 \\ 0 & 1 & 0 \\ 1 - \omega & \omega^2 - 1 & 1 \end{pmatrix}.$$

The invariant Hermitian form (induced by Poincaré duality, see Corollary 2.2) is given by the matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix}.$$
Figure 3. The braids $\gamma_1, \ldots, \gamma_5$

where $a = \frac{1}{2}(\omega^2 - \omega)$.

Proof. — The divisor $D \subset \mathbb{P}^1_S$ satisfies Assumption 1.2. Let $\varphi : \pi_1(S, s_0) \to A_4$ be the associated braiding map. Using standard methods (see e.g. [Vö1] and [DR00]), or by staring at Figure 3, one can show that the image of $\varphi$ is generated by the five braids

$$\beta_2^2, \beta_3\beta_2^2\beta_3^{-1}, \beta_3\beta_2^2\beta_2^{-1}\beta_3^{-1}, \beta_2^2, \beta_2\beta_2^2\beta_2^{-1}.$$ 

It is clear that these five braids can be realized as the image under the map $\varphi$ of generators $\gamma_1, \ldots, \gamma_5 \in \pi_1(S, s_0)$.

Considering the $\infty$-section as a ‘tangential base point’ for the fibration $U \to S$ as in §1.5, we obtain a section $\pi_1(S) \to \pi_1(U)$. We use this section to identify $\pi_1(S)$ with a subgroup of $\pi_1(U)$. Let $\alpha_1, \ldots, \alpha_5$ be the standard generators of $\pi_1(U_0)$. Let $\rho : \pi_1(U) \to K^\times$ denote the representation corresponding to the $G$-cover $f : Y \to X$ and the character $\chi : G \to K^\times$, and $\rho_0 : \pi_1(U_0) \to G$ its restriction to the fibre above $s_0$. Using (27) one checks that $\rho_0$ corresponds to the tuple $g = (\omega, \omega, \omega, \omega, \omega^2)$, i.e. that $\rho_0(\alpha_i) = g_i$. Also, since the leading coefficient of the right hand side of (27) is one, the restriction of $\rho$ to $\pi_1(S)$ is trivial. Hence, by Theorem 1.3, we have

$$\eta(\gamma_i) = \Phi(g, \varphi(\gamma_i)).$$

A straightforward computation, using (9) and the cocycle rule (10), gives the value of $\eta(\gamma_i)$ (in form of a three-by-three matrix depending on the choice of a basis of $W_k$). For this computation, it is convenient to take the classes of $(1, 0, 0, 0, -\omega^2)$, $(0, 1, 0, 0, -\omega)$ and $(0, 0, 1, 0, -1)$ as a basis. In order to obtain the 5 matrices stated in the theorem, one has to use a different basis, i.e. conjugate with the matrix

$$B = \begin{pmatrix} 0 & -\omega - 1 & -\omega \\ \omega + 1 & \omega + 1 & \omega + 1 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

The claim on the Hermitian form follows from Theorem 2.6 by another straightforward computation. 

\[ \square \]
Remark 3.3. — Theorem 3.2 is due to Picard, see \cite{Pic83, p. 125} and \cite{Pic84, p. 181}. He obtains exactly the matrices given above, but he does not list all of the corresponding braids. A similar list as above is obtained in \cite{Hol86} using different methods.

Remark 3.4. — It is obvious from Theorem 3.2 that the Hermitian form on $\mathcal{W}$ has signature $(1, 2)$ or $(2, 1)$, depending on the choice of the character $\chi$. This confirms Theorem 2.3 in this special case.

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M. DETTWEILER, Universität Heidelberg • E-mail: michael.dettweiler@iwr.uni-heidelberg.de

Stefan Wewers, Universität Bonn • E-mail: wewers@math.uni-bonn.de