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M. Magno - J.F. Ganghoffer

MECHANICS OF PLANAR BEAMS BY STROBOSCOPY

Abstract. The nonstandard analysis is herewith involved within an engeering point of view, in order to set up continuos planar beam models, starting from a discrete description. The beam is discretized as a set of rigid rods connected with flexional springs ; we thereby intend to describe the mechanical behaviour of a planar inextensible yarn undergoing pure rotations. The potential energy of the yarn is established, and the lemma of stroboscopy leads to a continuous description, considering asumptions regarding order of magnitudes of geometrical and mechanical parameters of the system of rods. The obtained continuous model appears as a planar generalization of the classical Euler's elastica, involving a corrugation term corresponding to a variation of the curvature of the beam.

1. The mathematical frame: Non Standard Analysis

A new way to master the transition from microscopic mechanical models to their macroscopic properties is to use the consistent deal with absolute orders of magnitude which is possible in the frame of Non Standard Analysis. This extension of classical mathematics, first introduced by A. Robinson [3] in the sixties, axiomatized by E. Nelson [2] in 1977, introduces a new concept, standardness, which cannot be defined within classical mathematics, but is ruled by specific axioms called Transfer (T), Idealisation (I) and Standardisation (S). The concept of standardness formalizes the intuitive concept of well defined mathematical objects, like the numbers $0, 1, 2, \ldots, \pi$, exp, ... the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, etc. Indeed, axiom (T) implies the standardness of any object which can be explicitely defined within classical mathematics. Axiom (I) implies that there are integers larger than any *standard* integer. Thus the field of real numbers contains *very large* positive reals (larger than any standard real), *very little* reals (0 or $\frac{1}{|x|}$ very large), and *moderate* reals ($|x|$ is not very large). Write $x \approx y$ whenever $y - x$ is very little. The computational rules are these of Leibnitz's infinitesimal calculus in accordance with engineering intuitions. Axiom (S) insures that every moderate real x is very near an unique standard real ${}^\circ x$, called its *shadow*, and also that a real function f which takes only moderate values has a shadow, that is a standard function ${}^\circ f$ such that for x standard, ${}^\circ f(x) = {}^\circ (f(x))$. Within NSA, all classical limit concepts have a counterpart in terms of the new concepts. For instance, a standard real function $f(x)$ tends to a standard limit l as x tends to a standard point $x_\circ$ if $x \approx x_\circ$, $x \neq x_\circ$, implies $f(x) \approx f(x_\circ)$. The derivative is such that

$$\frac{f(x_\circ + h) - f(x_\circ)}{h} \approx f'(x_\circ)$$

for $h \approx 0$, $h \neq 0$ or

$$f(x_\circ + h) - f(x_\circ) = hf'(x_\circ) + h\epsilon$$

where ϵ means an unknown very little number. From the logical point of view, NSA is a conservative extension of classical mathematics. This means that any theorem in NSA which may be expressed without using the new concepts is also true within classical mathematics. Consistency of NSA follows. NSA is useful to shorten the proof of theorems, but it is also the right context to express the micro-macroscopic duality of models relevant in physics or engineering. To this end an important tool is Stroboscopy for ODE and some PDE (see [4], [6]). We describe this technique and show its use on modelisation of beams.

2. Stroboscopy

In order to get a continuous model from a discrete model, one frequently has a sequence of points which gives the approximate shape of a curve. Stroboscopy uses a differential system as an intermediate object to get this curve. More precisely (see [4], [1]) the following lemma holds:

LEMMA 1. *Let* $f : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ *be a standard continuous function with* p *standard and* $(s_i, x_i)_{0 \leq i \leq n}$ *a finite sequence in* $\mathbb{R} \times \mathbb{R}^p$ *such that:*

- *i)* $s_0 < a < s_1 < \cdots < s_{n-1} < b < s_n$, *where a and b are standard;*
- *ii) for every* $0 \leq i < n$, *we have* $s_{i+1} \sim s_i$ *and* x_i *moderate;*
- *iii) for every* $0 \leq i < n$, *we have* $x_{i+1} - x_i = (s_{i+1} - s_i)(f(x_i) + \eta_i)$ *where* $\|\eta_i\| \approx 0$. *Then there is a standard solution of the ODE:*

$$\begin{cases} \frac{dx}{ds} = f(x(s)) \\ x(0) = {}^{\circ}x_0 \end{cases}$$

defined on the interval $[a, b]$, *such that for every* $0 \leq i \leq n - 1$, *we have* $x_i \approx x(s_i)$.

If f is a function of class C^1, $x(s)$ is clearly the unique solution of the ODE with initial value ${}^{\circ}x_0$.

The stroboscopy lemma has many applications in the asymptotic theory of oscillations, as an alternative to the KBM averaging method (see [4]). The choice of accurate observation times which satisfy condition (i) in a rapidly oscillating system is a crucial point. In the sequel, the values s_i are directly provided by a macroscopic mechanical model. Stroboscopy can be extended to some PDE (see [6]).

3. From microscopic to macroscopic mechanics of planar beams

We study the classical rigid-rods model of planar beams when the number of rods is very large. More precisely, consider n rigid rods of length l linked by $n - 1$ springs which keep the system in a (x, y) plane. Call $\theta_1, \theta_2, \cdot, \theta_n$ the deviation of the rods from the $x - direction$, and α_i, $i = 1, ..., n$ the equilibrium deviation of the rods $n = i$ and $i + 1$ for the spring $n = i$. Thus the couple between successive rods is $k(\theta_i - \theta_{i+1} - \alpha_i)(\theta_i - \theta_{i+1} - \alpha_i)$. Suppose the first rod fixed at the end $(0, 0)$ and the last end constrained on the x-axis is submitted to some constant horizontal force $(-F, 0)$. Moreover, each inner articulation is submitted to a force (f_i, g_i). The energy of the system is then:

$$\begin{aligned} E &= \frac{k}{2}\sum_{i=1}^{n-1}(\theta_i - \theta_{i+1} - \alpha_i)^2 + Fl\sum_{i=1}^{n} cos\theta_i + \\ &- l\sum_{i=1}^{n-1} sin\theta_i(f_i + ... + f_{n-1}) + l\sum_{i=1}^{n-1} cos\theta_i(g_i + ... + g_{n-1}), \end{aligned}$$

with the constraint $\sum_{i=1}^{n} sin\theta_i = 0$. One suppose $l \approx 0$. Then, from the Euler-Lagrange equation (see [5])and the fact that there is no spring at the end $(0, 0)$, we get:

$$\begin{cases} \theta_1 - \theta_2 - \alpha_1 = \frac{l}{k}[(F + g_1 + \cdots + g_{n-1})sin\theta_1 + (f_1 + ... + f_{n-1})cos\theta_1] + \mu cos\theta_1 \\ . \\ . \\ (\theta_i - \theta_{i+1} - \alpha_i) - (\theta_{i-1} - \theta_i - \alpha_{i-1}) = \frac{l}{k}[(F + g_i + \cdots + g_{n-1})sin\theta_i + \\ +(f_i + ... + f_{n-1})cos\theta_i] + \mu cos\theta_i \qquad \text{for } 2 \le i \le n-1 \\ . \\ . \\ -(\theta_{n-1} - \theta_n - \alpha_{n-1}) = \frac{Fl}{k} sin\theta_n + \mu cos\theta_n \end{cases}$$

Adding the lines with $k = \frac{a}{l}$ and a not very little we get:

$$\frac{\mu}{l^2} = -\frac{l\sum_{i=1}^{n-1}(g_i + ... + g_{n-1})sin\theta_i + l\sum_{i=1}^{n-1}(f_i + ... + f_{n-1})cos\theta_i)}{a\sum_{i=1}^{n} l cos\theta_i}$$

Suppose now nl is moderate with $L = {}^\circ(nl)$, n is very large, F is moderate and

$$\frac{\alpha_i}{l} \approx \rho(il),$$

$\frac{f_i}{l} \approx f(il)$, $\frac{g_i}{l} \approx g(il)$ where f, g, are some standard continuous real functions and ρ is a standard derivable one.
Thus:

$$\frac{\mu}{l^2} \approx -\frac{\int_0^L \int_s^L [g(\nu)sin\theta(\xi) + f(\nu)cos\theta(\xi)]d\nu d\xi}{a\int_0^L cos\theta(s)ds} = A(s)$$

with

$$\frac{\theta_1 - \theta_2}{l} \approx \frac{\alpha_1}{l} \approx \rho(0)$$

and

$$\frac{\theta_{n-1} - \theta_n}{l} \approx \frac{\alpha_{n-1}}{l} \approx \rho(L).$$

Put

$$\phi_i = \frac{\theta_{i+1} - \theta_i}{l} = \frac{\theta_{i+1} - \theta_i}{s_{i+1} - s_i}$$

where $s_i = il$. Then we have:

$$\begin{aligned} \frac{\phi_{i-1} - \phi_i}{l} &= \frac{\alpha_i - \alpha_{i-1}}{l^2} + \frac{F}{kl}\sin\theta_i + \sin\theta_i \sum_{j=1}^{n-1} g_j + \cos\theta_i \sum_{j=1}^{n-1} f_j + A(il)cos\theta_i \\ &\approx \frac{d\rho}{ds}(il) + \frac{F}{a} sin\theta_i + \sin\theta_i \sum_{j=1}^{n-1} g_j + \cos\theta_i \sum_{j=1}^{n-1} f_j + A(il)cos\theta_i. \end{aligned}$$

Applying stroboscopy, we get standard functions $\theta(s)$, $\phi(s)$ on $[0, L]$ such that, if

$$m = {}^\circ(\frac{F}{a}),$$

we have:

$$\begin{cases} \frac{d\theta}{ds} = \phi(s) \\ \frac{d\phi}{ds} = \frac{d\rho}{ds}(s_i) + m\sin\theta(s) + \sin\theta\int_s^L g(\xi)d\xi + \cos\theta\int_s^L f(\xi)d\xi + A(s)cos\theta \end{cases}$$

with:

$$\begin{cases} \phi(0) = -\rho(0) \\ \phi(L) = -\rho(L) \end{cases}$$

and for every $1 \leq i \leq n$, we have: $\theta_i \approx \theta(il)$. Moreover, the springs are located at:

$$\begin{cases} x_i = l\sum_{j=1}^{i} cos\theta_j \approx \int_0^{il} cos\theta(s)ds \\ y_i = l\sum_{j=1}^{i} sin\theta_j \approx \int_0^{il} sin\theta(s)ds \end{cases}$$

Thus the curve:

$$\begin{cases} x'(s) = cos\theta(s) \\ y'(s) = sin\theta(s) \end{cases}$$

is the shadow of the discrete beam, s is the arc length, and $\theta(s)$ satisfies the integro-differential equation:

$$\ddot{\theta}(s) + msin\theta(s) + \frac{d\rho}{ds}(s) + sin\theta\int_s^L g(\xi)d\xi + cos\theta\int_s^L f(\xi)d\xi + A(s)cos\theta = 0,$$

with $\dot{\theta}(0) = -\rho(0)$ and $\dot{\theta}(L) = -\rho(L)$, is the equilibrium equation of the planar smooth beam. It is easy to show that $\rho(s)$ is the curvature of the free beam (m, f, g are vanishing).The corrugation term appearing in the equation of equilibrium of a beam, i.e. $\frac{d\rho}{ds}$, is the only component of the director related to the micropolar (local) bending (here called "corrugation") corresponding to a local torque. The shape of the beam can be obtained solving Frênet equations.
Obviously, this model is a planar generalization of the classical Euler's elastica.

The constant k can be identified from the variational formulation of the equilibrium of the beam

Equilibrium equations:

(1)
$$\begin{aligned} \frac{d\underline{N}}{ds} + \underline{p} &= 0 \\ \frac{dM}{ds} + \underline{Nn} &= 0 \end{aligned}$$

projection of (1) on $\underline{n}$:

$$\frac{d\underline{N}}{ds}\underline{n} + \underline{p}\underline{n} = 0$$

The constituve law is:

(2)
$$\begin{aligned} &\underline{N}\cdot\underline{\tau} = EA\frac{\partial\underline{U}}{\partial s}\underline{\tau} = EA\varepsilon(s) \\ &M = EIv_{,ss} \end{aligned}$$

from the displacement $\underline{U} = v\cdot\underline{n}$, we get

$$\begin{cases} \frac{\partial\underline{\underline{U}}}{\partial s} = v_{,s}\,\underline{n} + v\frac{d\underline{\underline{n}}}{ds} \\ \frac{d\underline{\underline{n}}}{ds} = -\frac{\underline{\tau}}{\rho(s)} \\ \frac{\partial\underline{\underline{\vartheta}}}{\partial s} = \frac{1}{\rho(s)} = v_{,ss} \end{cases}$$

thus

$$\begin{gathered} \frac{\partial \underline{U}}{\partial s} = v,_{s}\, \underline{n} - v v,_{s}\, \underline{\tau} \\ \frac{\partial \underline{U}}{\partial s} \underline{\tau} = -v v,_{ss} \end{gathered}$$

which inserted in (2) one further has

$$\left\{ \begin{array}{l} \frac{\underline{N}}{A} \underline{\tau} = -E v v,_{ss} \\ M = E I v,_{ss} \Rightarrow \underline{N} \cdot \underline{n} = -\frac{dM}{ds} = -E I v,_{sss} \end{array} \right. \quad \Rightarrow$$

$$\underline{N} = -E \left(I v,_{sss}\, \underline{n} + A E v v,_{ss}\, \underline{\tau} \right) \Rightarrow \frac{d \left(\underline{N n} \right)}{ds} = \frac{d}{ds} \left(-E I v,_{sss} \right) = -E I v,_{ssss}$$

but

$$\begin{array}{rcl} \dfrac{d \left(\underline{N\ n} \right)}{ds} & = & \dfrac{d \underline{N}}{ds} \underline{n} + \underline{N} \dfrac{d \underline{n}}{ds} = \dfrac{d \underline{n}}{ds} \underline{n} - v,_{ss}\, \underline{\tau\ N} \Rightarrow \\ \dfrac{d \underline{n}}{ds} \underline{n} & = & v,_{ss}\, \underline{\tau N} + \dfrac{d \left(\underline{N\ n} \right)}{ds} = -E \left(I v,_{ssss} + A v \left(v,_{ss} \right)^2 \right) \\ & \approx & -E I v,_{ssss} \text{ with } \left(v \ll v,_{ss} \right) \end{array}$$

then

$$\underline{p} \underline{n} = p_2 = E I v,_{ssss} \tag{3}$$

multiplying (3) by a test function $w \mid w(0) = 0 = w(L)$ gives

$$\int_0^L \left(-p_2 w + E I v,_{ssss}\, w \right) ds = 0$$

integrating by part further gives

$$\int_0^L \left(-p_2 w - E I v,_{sss}\, w,_{s} \right) ds + \left[E I v,_{sss}\, w \right]_0^L = 0$$

We next evaluate

$$\int_0^L \left(-E I v,_{sss}\, w,_{s} \right) ds = \int_0^L \left(-E I v,_{ss}\, w,_{ss} \right) ds - \left[E I v,_{ss}\, w, L s \right]_0^L$$

but

$$\begin{gathered} M = E I v,_{ss} \Rightarrow \left[E I v,_{ss}\, w,_{s} \right]_0^L = \left[M w,_{s} \right]_0^L \Rightarrow \\ \int_0^L \left(-p_2 w - E I v,_{ss}\, w,_{ss} \right) ds = \left[M w,_{s} \right]_0^L \end{gathered}$$

We select for w the real solution v

$$\int_0^L E I \left(v,_{ss} \right)^2 ds = \int_0^L p_2(s)\, v(s)\, ds + \left[M w, s \right]_0^L$$

The choice $M(0) = 0 = M(L)$ gives the continuous formulation and

$$\int_0^L E I \left(v,_{ss} \right)^2 ds = \int_0^L f(\xi)\, d\xi \text{ with } f(s) = p_2(s)\, v(s) \tag{4}$$

in the discrete problem, we have instead:

$$E = \frac{k}{2}\sum_{i=1}^{n-1} l_i^2 \left(\frac{\theta_i - \theta_{i+1} - \alpha_i}{l_i}\right)^2 \tag{5}$$

The identification of (5) and (4) gives

$$\frac{k}{2} = EI \Rightarrow k = 2EI$$

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Massimo MAGNO
Laboratoire de Mathématiques et Applications
4, rue des Frères Lumière
68093 Mulhouse Cédex, FRANCE. e-mail: `m.magno@uha.fr`

J.F. GANGHOFFER Lemta-Ensem 2, Avenue de la Foret de Haye. BP B.P. 160-54504
Vandoeuvre Cedex, FRANCE e-mail:
`Jean-francois.Ganghoffer@ensem.inpl-nancy.fr`

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